QCD amplitudes with external massive quarks in the soft limit

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in collaboration with Michał Czakon

66 Cracow School of Theoretical Physics, Kraków, 14-19 June 2026

A long time ago in a galaxy far, far away...

Cracow School of Theoretical Physics, XLV Course, 2005

12	T. Romanczukiewicz	Krakow	Interaction between radiation and topological defects
13	Ch. Royon	Saclay/Fermilab	New results and prospects from the Tevatron
14	S. Sapeta	Krakow	Diffraction at Tevatron and LHC in the Miettinen-Pumplin model
15	M Shifman	Minnesota	1. Nonperturbative Yang-Mills from SUSY and strings 2. QCD k-strings

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[Phys.Lett.B 597 (2004) 352-355]

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In conclusion, we have shown that the Miettinen and Pumplin model correctly describes diffraction dissociation in hadron-hadron collisions with the energies of the order of TeV. Calculated values of σ_{SD} are in reasonable agreement with experimental data. Moreover, the dependence on energy is similar to that calculated by Goulianos within his model of renormalized Pomeron flux.

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- Each collision at the LHC involves interactions of quarks and gluons
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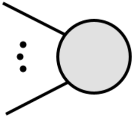
Predictions in perturbative QCD

- In the region where the strong coupling $\alpha_s \ll 1$, fixed-order perturbative expansions is expected to work well

$$\sigma = \underbrace{\sigma_0}_{\text{LO}} + \underbrace{\alpha_s \sigma_1}_{\text{NLO}} + \underbrace{\alpha_s^2 \sigma_2}_{\text{NNLO}} + \underbrace{\alpha_s^3 \sigma_3}_{\text{N}^3\text{LO}} + \cdots$$

Building blocks of N3LO amplitudes

- Born level



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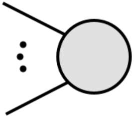

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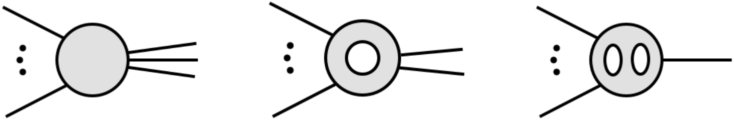

Kinematic regions of gluon emissions

Gluons' momenta in light-cone coordinates

$$k_i^\mu = \left(k_i^+, k_i^-, \boldsymbol{k}_i^\perp\right) \qquad \text{where} \qquad k^\pm = k^0 \pm k^3$$

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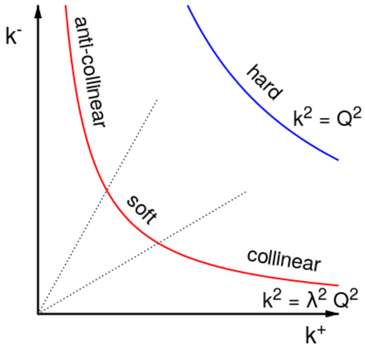

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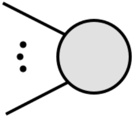

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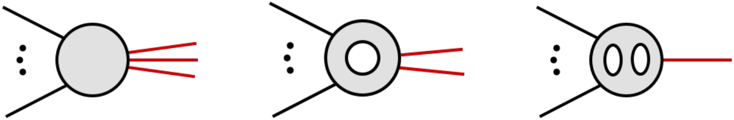

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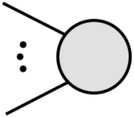

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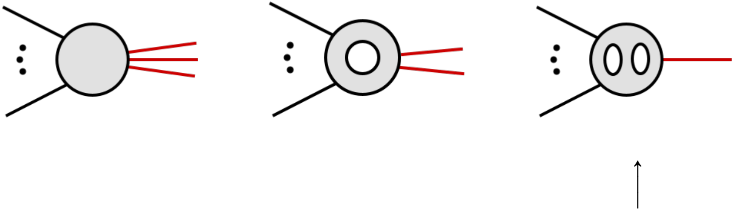
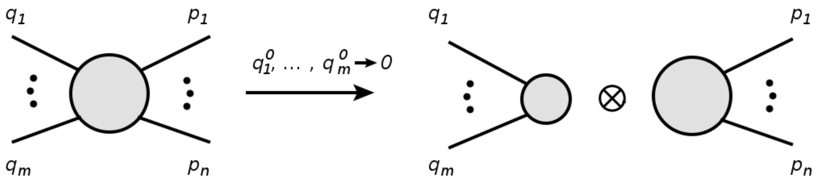


single soft limit at two loops

Soft factorization in QCD: tree level



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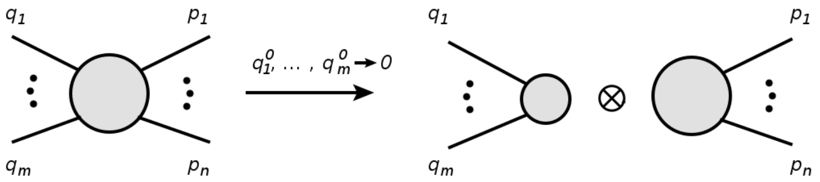

$$|\mathcal{M}^{(0)}(q_1, \ldots, q_m, p_1, \ldots, p_n)\rangle \xrightarrow{q_1^0,\ldots,q_m^0 \to 0} \boldsymbol{J}^{(0)}(q_1, \ldots, q_m)\, |\mathcal{M}^{(0)}(p_1 \ldots, p_n)\rangle$$

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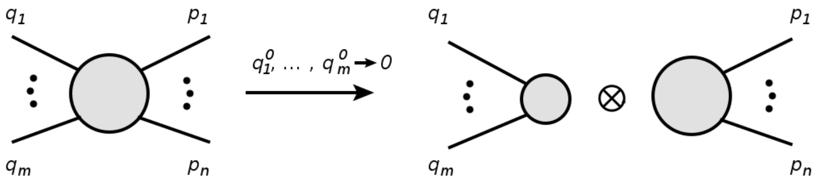

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- $\boldsymbol{J}^{(0)}(q_1,\ldots,q_m)$ is the soft current at tree level

Soft factorization in QCD: higher orders

One loop

$$|\mathcal{M}^{(1)}(q_1,\ldots,q_m,p_1,\ldots,p_n)\rangle \xrightarrow{q_1^0,\ldots,q_m^0\to 0} \boldsymbol{J}^{(1)}(q_1,\ldots,q_m)\,|\mathcal{M}^{(0)}(p_1\ldots,p_n)\rangle + \boldsymbol{J}^{(0)}(q_1,\ldots,q_m)\,|\mathcal{M}^{(1)}(p_1\ldots,p_n)\rangle$$

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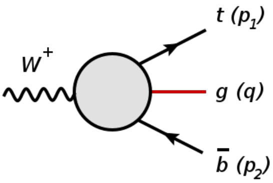
Two loops

$$|\mathcal{M}^{(2)}(q_1,\ldots,q_m,p_1,\ldots,p_n)\rangle \xrightarrow{q_1^0,\ldots,q_m^0\to 0} \boldsymbol{J}^{(2)}(q_1,\ldots,q_m)\,|\mathcal{M}^{(0)}(p_1\ldots,p_n)\rangle + \boldsymbol{J}^{(1)}(q_1,\ldots,q_m)\,|\mathcal{M}^{(1)}(p_1\ldots,p_n)\rangle + \boldsymbol{J}^{(0)}(q_1,\ldots,q_m)\,|\mathcal{M}^{(2)}(p_1\ldots,p_n)\rangle$$

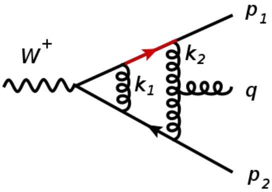
Kinematics

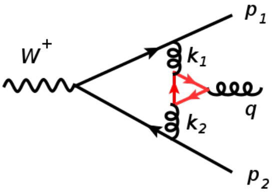


- Five invariants:

$$
\begin{aligned}
s_{1q} &= (p_1 + q)^2 \\
s_{2q} &= (p_2 + q)^2 \\
s_{12} &= (p_1 + p_2)^2 \\
m_t^2 &= p_1^2 \\
m_b^2 &= p_2^2
\end{aligned}
$$

What happened to the propagators (and vertices)?





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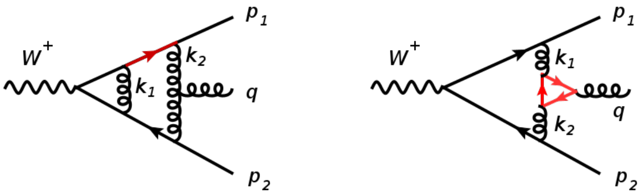

- External quarks:

$$\frac{\not{p}_1 - \lambda \not{k}_2}{(p_1 - \lambda k_2)^2 - m_t^2} = \simeq \frac{-\not{p}_1}{\lambda\, p_1 \cdot k_2} \quad \text{(eikonal)}$$

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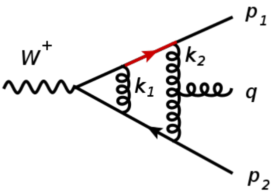

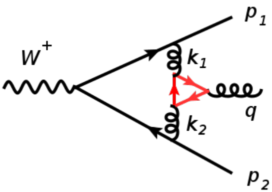

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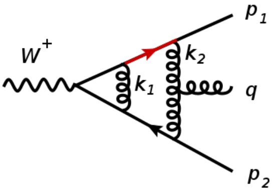

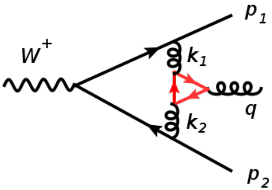

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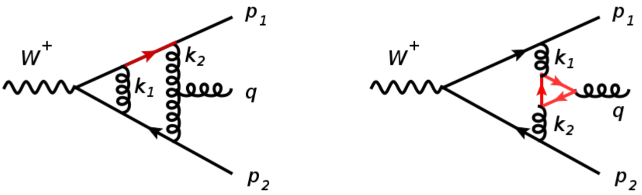

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- Tipple-gluon vertex: exact

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In dimensional regularization, the integral over total derivative is zero

$$\int d^d k_1 \ldots d^d k_L \frac{\partial}{\partial k_i^\mu} \left(\frac{q^\mu}{P_1^{a_1} \cdots P_N^{a_N}} \right) = 0 \, ,$$

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- This leads to algebraic relations between integrals
- We used the KIRA package [Maierhöfer, Usovitsch, Uwer '17; Klappert, Lange, Maierhöfer, Usovitsch '20] and were able to reduce the original set of 928 integrals to 65 master integrals

Variables

As mentioned earlier, the process is characterized by five invariants, or, equivalently, by five scalar products: $p_1 \cdot q,\ p_2 \cdot q,\ p_1 \cdot p_2,\ p_1^2,\ p_2^2$

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Hence, our integrals will also be invariant under rescaling of the heavy quark momenta, p_1 and p_2. This can be achieved only by the three ratios

$$\frac{(p_1 \cdot p_2)}{(p_1 \cdot 1)(p_2 \cdot q)}\,, \qquad \frac{(p_1 \cdot p_1)\,(p_2 \cdot q)}{(p_1 \cdot p_2)(p_1 \cdot q)}\,, \qquad \frac{(p_2 \cdot p_2)\,(p_1 \cdot q)}{(p_1 \cdot p_2)(p_2 \cdot q)}$$

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Loop integrals:

$$\int \prod dk_i^4 \to \text{ dimensionless}$$

$$\int \prod dk_i^{4-2\epsilon} \to\ m^{d-4} \text{ per loop}$$

Variables

Hence, our integrals will evaluate to the following functions:

$$I_i(p_1 \cdot q, p_2 \cdot q, p_1 \cdot p_2, p_1^2, p_2^2) = q_\epsilon^{-2\epsilon} M_i(\alpha_1, \alpha_2)$$

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And we can write differential equations our masters in terms of dimensionless functions M_i of dimensionless variables α_1, α_2:

$$\frac{\partial}{\partial\alpha_1}\vec{M}(\alpha_1,\alpha_2) = \boldsymbol{a}_1(\epsilon,\alpha_1,\alpha_2)\,\vec{M}(\alpha_1,\alpha_2)$$
$$\frac{\partial}{\partial\alpha_2}\vec{M}(\alpha_1,\alpha_2) = \boldsymbol{b}_2(\epsilon,\alpha_1,\alpha_2)\,\vec{M}(\alpha_1,\alpha_2)$$

System of differential equations

$$\frac{\partial}{\partial \alpha_1} \begin{bmatrix} M_1 \\ M_2 \\ M_3 \\ \vdots \\ M_{65} \end{bmatrix} = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,65} \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,65} \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots & a_{3,65} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{65,1} & a_{65,2} & a_{65,3} & \cdots & a_{65,65} \end{bmatrix} \begin{bmatrix} M_1 \\ M_2 \\ M_3 \\ \vdots \\ M_{65} \end{bmatrix}$$

$$\frac{\partial}{\partial \alpha_2} \begin{bmatrix} M_1 \\ M_2 \\ M_3 \\ \vdots \\ M_{65} \end{bmatrix} = \begin{bmatrix} b_{1,1} & b_{1,2} & b_{1,3} & \cdots & b_{1,65} \\ b_{2,1} & b_{2,2} & b_{2,3} & \cdots & b_{2,65} \\ b_{3,1} & b_{3,2} & b_{3,3} & \cdots & b_{3,65} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{65,1} & b_{65,2} & b_{65,3} & \cdots & b_{65,65} \end{bmatrix} \begin{bmatrix} M_1 \\ M_2 \\ M_3 \\ \vdots \\ M_{65} \end{bmatrix}$$

Closed subsystems

No.	Size homogeneous	Size inhomogeneous
1-14	1	1
15-26	2	1
27	2	2
28	2	2
29	3	1
30	3	1
31	4	2
32	4	2
33	5	1
34	5	2
35	5	1
36	5	2
37	5	1
38	6	2
39	6	2
40	8	4
41	10	1
42	10	1
43	12	2
44	13	1
45	13	1
46	16	2
47	29	3
48	29	3

Canonical form

All our differential systems, $s \in \{1, \ldots, 48\}$, have the form

$$\frac{\partial}{\partial \alpha_i} \vec{M}_s = \boldsymbol{A}_{si}(\alpha_i, \epsilon)\, \vec{M}_s$$

where $\vec{M}_s = \{M_1, \ldots, M_n\} \subset \{M_1, \ldots, M_{65}\}$.

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the system of differential equations can be often written in the form

$$\frac{\partial}{\partial \alpha_i} \vec{J}_s = \epsilon\, \boldsymbol{S}_{si}(\alpha_i)\, \vec{J}_s$$

Canonical form

All our differential systems, $s \in \{1, \ldots, 48\}$, have the form

$$\frac{\partial}{\partial \alpha_i} \vec{M}_s = \boldsymbol{A}_{si}(\alpha_i, \epsilon)\, \vec{M}_s$$

where $\vec{M}_s = \{M_1, \ldots, M_n\} \subset \{M_1, \ldots, M_{65}\}$.

- But the set of masters $\{M_1, \ldots, M_{65}\}$ corresponds just to a particular choice of basis in the space of integrals.

As observed in [Henn '13], by a proper change of basis of masters

$$\vec{M}_s = \boldsymbol{T}_s \vec{J}_s\,,$$

the system of differential equations can be often written in the form

$$\frac{\partial}{\partial \alpha_i} \vec{J}_s = \epsilon\, \boldsymbol{S}_{si}(\alpha_i)\, \vec{J}_s$$

- The dependence on ϵ factorizes! This is the so-called canonical form.

Evaluating integrals

The problem is the fully solved:

$$J^{(0)}(x_1, x_2) = B^{(0)}$$

$$J^{(i)}(x_1, x_2) = \int_{(a_1,a_2)}^{(x_1,x_2)} (\boldsymbol{S}_1 dx_1' + \boldsymbol{S}_2 dx_2') J^{(i-1)}(x_1', x_2') + B^{(i)}$$

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  $\hookrightarrow$ AMFlow [Liu, Ma, Wang '18-'23]

AMFlow can also be used to numerically compute $J^{(i)}(x_1,x_2)$ outside of the boundary and this can serve an ultimate validation of our solutions!

The only "but" is...

This all works provided that we can find the canonical form!

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This is true for all systems $\{1, \ldots, 48\}$ except 43 and 46

So let's see what we have got

No.	Size homogeneous	Size inhomogeneous	Canonical form
1-42	1-10	1-4	✓
43	12	2	X
44	13	1	✓
45	13	1	✓
46	16	2	X
47	29	3	✓
48	29	3	✓

System 43

$$\frac{\partial}{\partial t_1}\begin{bmatrix}\boldsymbol{m}\\ M_{44}\\ M_{61}\end{bmatrix} = \begin{bmatrix}\epsilon\boldsymbol{S}_a & 0 & 0\\ \boldsymbol{R}_{1,1} & a_{1,1} & a_{1,2}\\ \boldsymbol{R}_{1,2} & a_{2,1} & a_{2,2}\end{bmatrix}\begin{bmatrix}\boldsymbol{m}\\ M_{44}\\ M_{61}\end{bmatrix}$$

$$\frac{\partial}{\partial t_2}\begin{bmatrix}\boldsymbol{m}\\ M_{44}\\ M_{61}\end{bmatrix} = \begin{bmatrix}\epsilon\boldsymbol{S}_b & 0 & 0\\ \boldsymbol{R}_{2,1} & b_{1,1} & b_{1,2}\\ \boldsymbol{R}_{2,2} & b_{2,1} & b_{2,2}\end{bmatrix}\begin{bmatrix}\boldsymbol{m}\\ M_{44}\\ M_{61}\end{bmatrix}$$

where $\boldsymbol{S}_a$, $\boldsymbol{S}_b$ are canonical submatrices for 10 out of 12 masters

$$\boldsymbol{m} = [M_1, M_2, M_{15}, M_{18}, M_{20}, M_{26}, M_{32}, M_{53}, M_{54}, M_{55}]$$

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$$\frac{\partial}{\partial t_1}\begin{bmatrix}\boldsymbol{m}\\ M_{44}\\ M_{61}\end{bmatrix}=\begin{bmatrix}\epsilon\boldsymbol{S}_a & 0 & 0\\ \boldsymbol{R}_{1,1} & a_{1,1} & a_{1,2}\\ \boldsymbol{R}_{1,2} & a_{2,1} & a_{2,2}\end{bmatrix}\begin{bmatrix}\boldsymbol{m}\\ M_{44}\\ M_{61}\end{bmatrix}$$

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The question is: can we find a canonical of form the matrices:

$$\begin{bmatrix}a_{1,1} & a_{1,2}\\ a_{2,1} & a_{2,2}\end{bmatrix} \qquad \begin{bmatrix}b_{1,1} & b_{1,2}\\ b_{2,1} & b_{2,2}\end{bmatrix}$$

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- Standard algorithms of CANONICA and LIBRA do not find a rational transformation. No surprise.

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- Standard algorithms of CANONICA and LIBRA do not find a rational transformation. No surprise.
- We could however try to find it manually!

System 43

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By definition, canonical form is achieved though the following transformation

$$\epsilon\,\boldsymbol{S} = \boldsymbol{T}^{-1}\boldsymbol{A}\boldsymbol{T} - \boldsymbol{T}^{-1}d\,\boldsymbol{T} \qquad (❀)$$

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where $\boldsymbol{A}$ is our original matrix and $\boldsymbol{T}$ is the transformation matrix we are looking for

$$\boldsymbol{T} = \begin{bmatrix} v_{11}(t_1, t_2) & v_{12}(t_1, t_2) \\ v_{21}(t_1, t_2) & v_{22}(t_1, t_2) \end{bmatrix}$$

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This leads to four partial differential equations for $v_{ij}(t_1, t_2)$

System 43

$$\boldsymbol{T} = \frac{1}{\sqrt{t_1 t_2}} \begin{bmatrix} Q_{-\frac{1}{2}}(x) & P_{-\frac{1}{2}}(x) \\ \frac{1}{t_2} Q_{\frac{1}{2}}(x) & \frac{1}{t_2} P_{\frac{1}{2}}(x) \end{bmatrix},$$

where $x = \frac{1-t_1-t_2^2}{t_1 t_2}$ and $P_n(x)$ and $Q_n(x)$ are Legendre polynomials

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$$P_{\frac{1}{2}}(x) = \frac{2}{\pi}\left[2E\left(\frac{1-x}{2}\right) - K\left(\frac{1-x}{2}\right)\right]$$

$$P_{-\frac{1}{2}}(x) = \frac{2}{\pi}E\left(\frac{1-x}{2}\right)$$

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- Hence, we found transformation to canonical form! We checked that it's invertible and it works.

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$$Q_{-\frac{1}{2}}(x) = E\left(\frac{1+x}{2}\right)$$

- Hence, we found transformation to canonical form! We checked that it's invertible and it works.
- The transformation is not rational and it involves elliptic integrals.

Solutions for systems 43 and 46

Just like for other systems

$$J^{(i)}(x_1, x_2) = \int_{(a_1,a_2)}^{(x_1,x_2)} (\boldsymbol{S}_1 dt_1 + \boldsymbol{S}_2 dt_2) J^{(i-1)}(t_1, t_2) + B^{(i)}$$

Solutions for systems 43 and 46

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Only that, now, some pieces of $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ depend on elliptic integrals and irreducible square roots, *i.e.* they contain terms of the form

$$\int\limits_{(a_1,a_2)}^{(x_1,x_2)} dt_2 \sqrt{t_1}\, t_2^{5/2} E\left[\frac{(1+t_2)(1-t_1-t_2)}{2t_1 t_2}\right] \frac{G(-1+t_1,-1+t_1,-1+t_1,t_2)}{(1-t_2)(1-t_1+t_2)(1-t_1-t_2)}$$

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Solutions in terms of iterated integrals

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Solutions in terms of iterated integrals

And those, and only those terms need to be integrated numerically.

All masters found

No.	Size homogeneous	Size inhomogeneous	Canonical form	Solved and validated with AMFlow
1-42	1-10	1-4	✓	✓
43	12	2	✗	
44	13	1	✓	✓
45	13	1	✓	✓
46	16	2	✗	
47	29	3	✓	
48	29	3	✓	

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44	13	1	✓	✓
45	13	1	✓	✓
46	16	2	✓	✓ (2 iter)
47	29	3	✓	
48	29	3	✓	

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46	16	2	✓	✓ (2 iter)
47	29	3	✓	✓ (3 iter)
48	29	3	✓	✓ (3 iter)

(Bare) soft current

$$\boldsymbol{J} = \boldsymbol{J}^{(0)} + \boldsymbol{J}^{(1)} + \boldsymbol{J}^{(2)} + \cdots$$

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Using all the ingredients discussed above, we can get $\boldsymbol{J}^{(2)}$. At a specific phase-space point

$$\left(s_{1q} = 5/3\,, \quad s_{2q} = 7/3\,, \quad s_{12} = 15/2\,, \quad m_t^2 = 1\,, \quad m_b^2 = 3/2\right),$$

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we have

$$\boldsymbol{J}^{(2)} = \frac{0.50000000000\,C_A^2}{\epsilon^4} + \frac{0.16666666667 n_f C_A - (0.785772815 - 2.679531386i)\,C_A^2}{\epsilon^3} + \frac{(0.321409062 + 0.893177129i)\,n_f C_A - (6.72508425 + 7.65302853i)\,C_A^2}{\epsilon^2} + \frac{(-1.09297868 - 0.57223045i)\,n_f C_A + (24.9117808 - 5.2923755i)\,C_A^2}{\epsilon} + (-1.4332049 - 1.2557746i)\,n_f C_A + (33.001749 - 0.322956i)\,C_A^2$$

Universal form of IR-singular part of two-loop amplitudes

[Catani '98; ...; Becher, Neubert '09]

$$|\mathcal{M}_n^{(2)}\rangle = \left(\mathbf{Z}^{(2)} - \mathbf{Z}^{(1)}\mathbf{Z}^{(1)}\right)|\mathcal{M}_n^{(0)}\rangle + \mathbf{Z}^{(1)}|\mathcal{M}_n^{(1)}\rangle + |\mathcal{F}_n^{(2)}\rangle\,,$$

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where

$$\mathbf{Z}^{(1)} = \frac{\alpha_s}{4\pi}\left(\frac{\Gamma_0'}{4\epsilon^2} + \frac{\boldsymbol{\Gamma}_0}{2\epsilon}\right)\,,$$

$$\mathbf{Z}^{(2)} = \left(\frac{\alpha_s}{4\pi}\right)^2\left[\frac{(\Gamma_0')^2}{32\epsilon^4} + \frac{\Gamma_0'}{8\epsilon^3}\left(\boldsymbol{\Gamma}_0 - \frac{3}{2}\,\beta_0\right) + \frac{\boldsymbol{\Gamma}_0}{8\epsilon^2}\,(\boldsymbol{\Gamma}_0 - 2\beta_0) + \frac{\Gamma_1'}{16\epsilon^2} + \frac{\boldsymbol{\Gamma}_1}{4\epsilon}\right]$$

Perfect agreement of the pole part

$$\boldsymbol{J}^{(2)}_{\text{direct}} = \frac{0.50000000000\,C_A^2}{\epsilon^4} + \frac{0.16666666667 n_f C_A - (0.785772815 - 2.679531386 i) C_A^2}{\epsilon^3} + \frac{(0.321409062 + 0.893177129 i) n_f C_A - (6.72508425 + 7.65302853 i) C_A^2}{\epsilon^2} + \frac{(-1.09297868 - 0.57223045 i) n_f C_A + (24.9117808 - 5.2923755 i) C_A^2}{\epsilon}$$

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BACKUP

Soft current

In general

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The tree level result, for massive ($p_i^2 > 0$) and massless ($p_i^2 = 0$) hard partons, takes the simple form

$$\boldsymbol{J}_a^{\mu(0)} = \sum_{i=1}^{n} \boldsymbol{T}_i^a \frac{p_i^\mu}{p_i^\mu \cdot q} \,,$$

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while at two loops both from dipole and tripole emissions

$$\boldsymbol{J}_a^{\mu(1)} = \sum_{i \neq j} \boldsymbol{T}_i^{a_i} \boldsymbol{T}_j^{a_j} S_{a,ij}(p_i, p_j, \{q_m\}) + \sum_{i \neq j \neq k} \boldsymbol{T}_i^{a_i} \boldsymbol{T}_j^{a_j} \boldsymbol{T}_k^{a_k} S_{a,ijk}(p_i, p_j, p_k, \{q_m\}) \, .$$

Kinematic regions of gluon emissions

Gluons' momenta in light-cone coordinates

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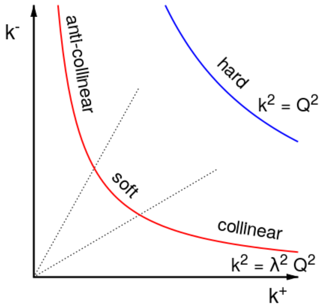
collinear	$k_i^\mu \sim (1, \lambda^2, \lambda)\, Q^2$
anti-collinear	$k_i^\mu \sim (\lambda^2, 1, \lambda)\, Q^2$
hard	$k_i^\mu \sim (1, 1, 1)\, Q^2$
soft	$k_i^\mu \sim (\lambda, \lambda, \lambda)\, Q^2$



where $\lambda \ll 1$ and $Q^2 \sim \mathcal{O}(1)$

Soft current - state of the art

Massless fermions

- one loop

  [Catani, Grazzini '00] *exact in* ϵ

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  [Li and Zhu '13] *dipole* $\mathcal{O}\left(\epsilon^2\right)$

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  [Bierenbaum, Czakon, Mitov '12, Czakon, Mitov '18] *dipole* $\mathcal{O}(\epsilon)$

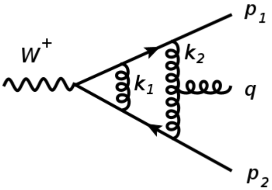
Our aim is to get the massive soft current at two loops to $\mathcal{O}(\epsilon)$

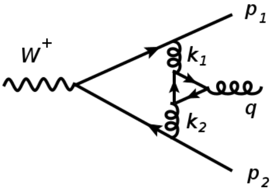
Details of the calculation

- Generate two-loop diagrams (196 in total) for the process:

$$W^+ \to t + \bar{b} + g$$

in Feynman gauge, with FeynArts





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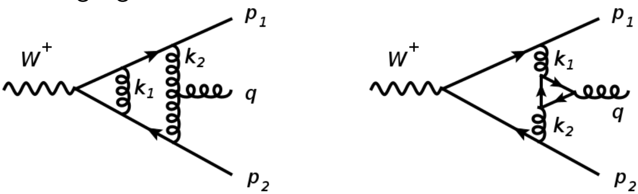

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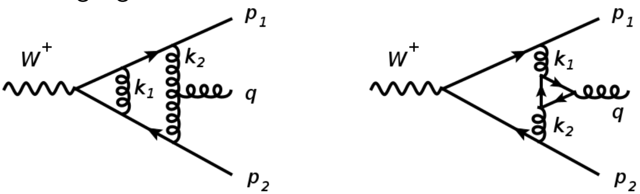

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- Parameterize the gluon momenta

$$k_1 \to \lambda k_1 \,,$$
$$k_2 \to \lambda k_2 \,,$$
$$q \to \lambda k_1 \,,$$

expand the amplitude in λ and take the leading (most singular) term.
This is the soft limit of $\mathcal{A}^{(2)}_{W^+\to t\bar{b}g}$