Gauge Fields on the (real-time) Lattice

Entanglement, hydrodynamics and "magic"

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- Xiaojun Yao (IQuS Seattle, assistant professor)
- BM

- Main articles/preprints:
  - *Commun. Phys.* 8 (2025) 368 [2411.04550]
  - 2510.11681 [quant-ph]
  - 2601.10065 [hep-lat]

Overview

- Motivation
- Hamiltonian Lattice Gauge Theory
- Spectral form factor and energy diffusion
- Two-step thermalization
- The “Magic Barrier”
- SU(3) lattice gauge theory
  - Minimal truncation scheme
  - Quantum chaotic properties
  - $Q\bar{Q}$ and QQQ potential

Motivation

How do gauge fields thermalize?

- General consensus: Thermalization RHI collisions is driven by gluons
- We have descriptions at
  - Weak coupling: Effective kinetic theory
  - Strong coupling: AdS/CFT
- Real-world QCD coupling is in the range of coupling constants where neither approach is reliable
- Possible universal approach: Real-time lattice gauge theory (RT-LGT)
- RT-LGT is much harder than euclidean LGT because there is no classical (stochastic) algorithm that can evaluate the path integral
- Ultimately, quantum computers promise to be the solution, but in the meantime let's start and push classical digital computer to their limit
- Issue: Exponential growth of the Hilbert space dimension (current limit: $D_H \sim 10^5$)

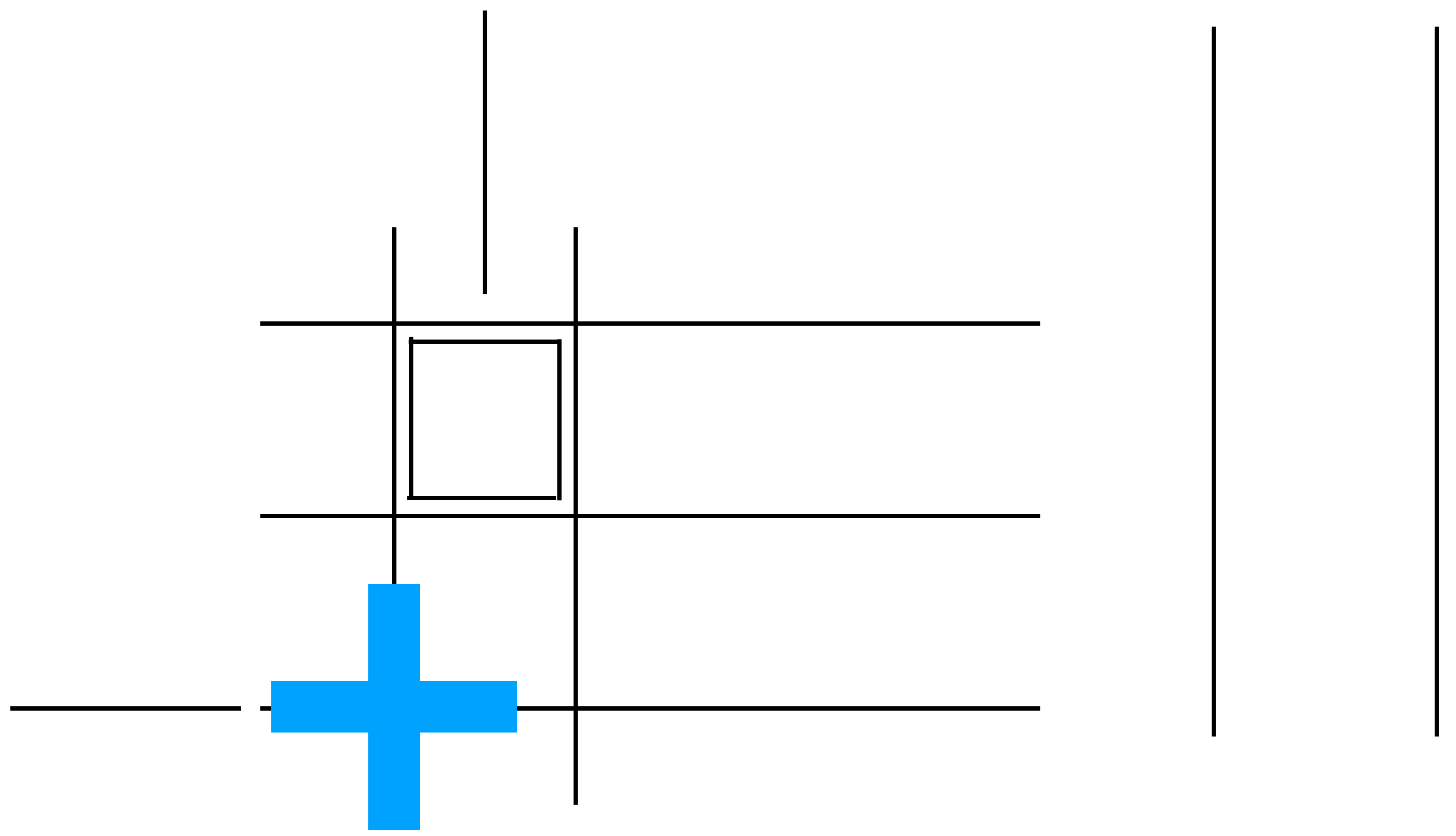
(2+1)-D SU(2) Lattice Gauge Theory

Kogut-Susskind Hamiltonian: $$H = \frac{g^2}{2}\sum_{\text{links}}(E_i^a)^2 - \frac{2}{a^2 g^2}\sum_{\text{plaquettes}}\Box(\boldsymbol{n})$$

$$\Box(\boldsymbol{n}) = \mathrm{Tr}[U^\dagger(\boldsymbol{n},\hat{y})U^\dagger(\boldsymbol{n}+\hat{y},\hat{x})U(\boldsymbol{n}+\hat{x},\hat{y})U(\boldsymbol{n},\hat{x})]$$

$$[E_i^a, U(\boldsymbol{n},\hat{j})] = -\delta_{ij}T^a U(\boldsymbol{n},\hat{j})$$

$$[E_i^a, E_i^b] = i f^{abc} E_i^c$$



Gauss's law: Every vertex transforms as a singlet for a state to be physical

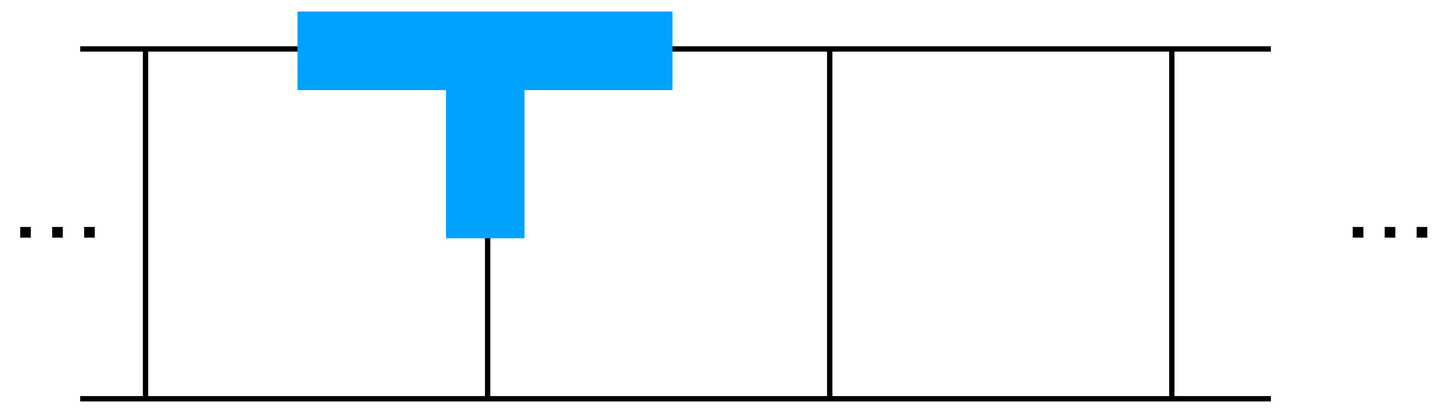
Electric basis on links: $|j\, m_L\, m_R\rangle$

$$E^2|j\, m_L\, m_R\rangle = j(j+1)|j\, m_L\, m_R\rangle$$

Byrnes, Yamamoto, quant-ph/0510027

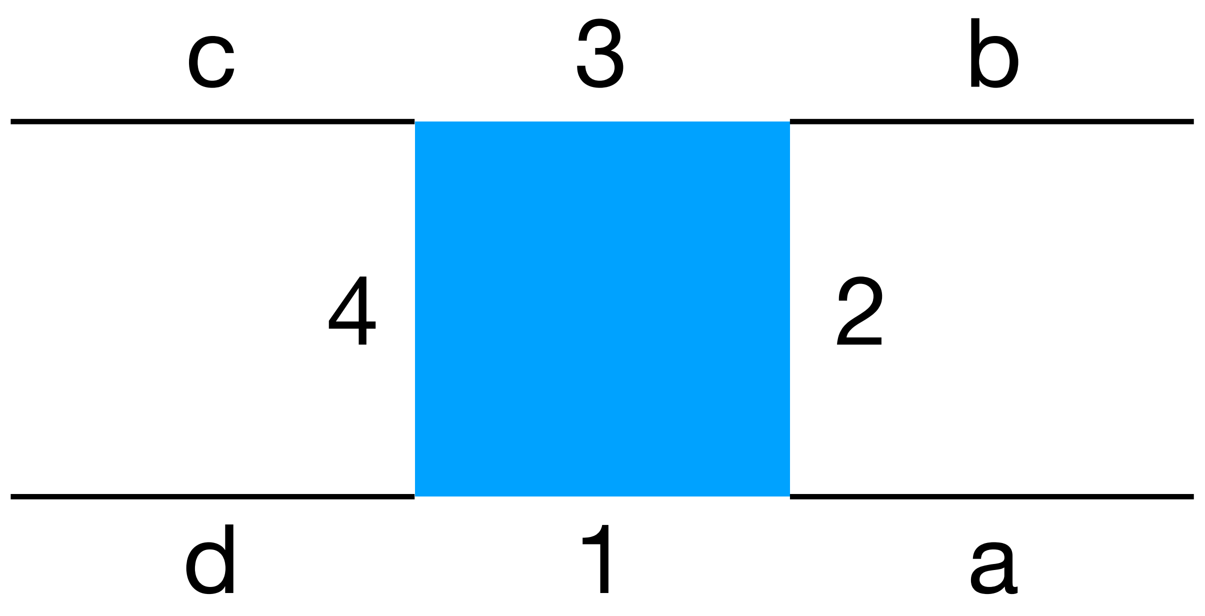
(2+1)-D SU(2) on Periodic Plaquette Chain

Each vertex has three links: singlet is uniquely defined by the *j* values on the three links



Klco, Stryker, Savage, 1908.06935

Matrix elements between physical states (singlets) expressed in 6*j* symbols



j: initial

J: final

$$\langle J_1 J_2 J_3 J_4 | \Box | j_1 j_2 j_3 j_4 \rangle = \prod_{\alpha=a,b,c,d} (-1)^{j_\alpha} \prod_{\alpha=1,2,3,4} \left[(-1)^{j_\alpha + J_\alpha} \sqrt{(2j_\alpha + 1)(2J_\alpha + 1)} \right]$$

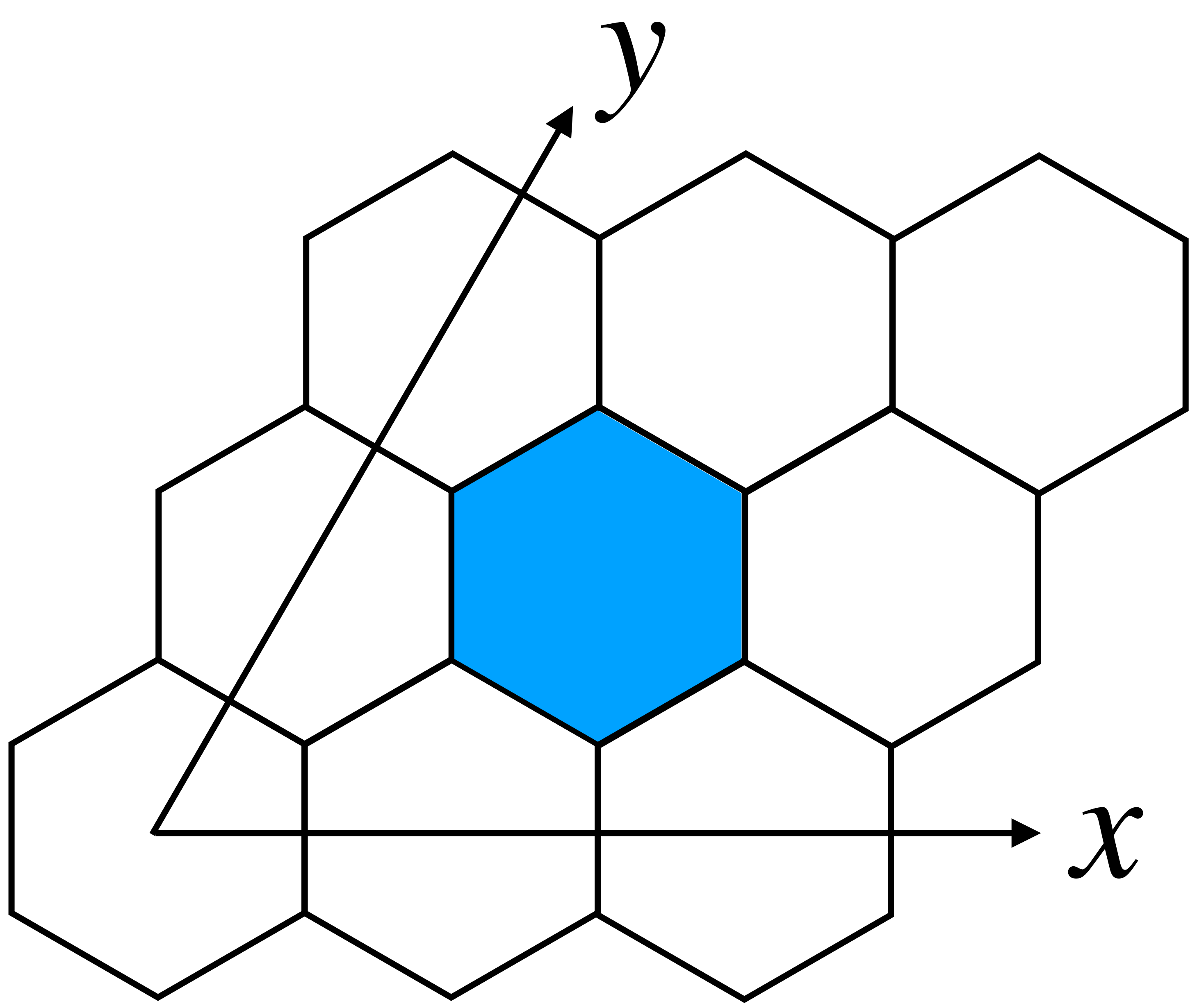
$$\begin{Bmatrix} j_a & j_1 & j_2 \\ 1/2 & J_2 & J_1 \end{Bmatrix} \begin{Bmatrix} j_b & j_2 & j_3 \\ 1/2 & J_3 & J_2 \end{Bmatrix} \begin{Bmatrix} j_c & j_3 & j_4 \\ 1/2 & J_4 & J_3 \end{Bmatrix} \begin{Bmatrix} j_d & j_4 & j_1 \\ 1/2 & J_1 & J_4 \end{Bmatrix}$$

Also for honeycomb lattice (2307.00045)

(2+1)-D SU(2) on Honeycomb Lattice

On square lattice each vertex has four links and singlet is not unique

Solution: use honeycomb lattice



$$H_{\rm el} = \frac{g^2}{2}\frac{3\sqrt{3}}{2}\sum_{\boldsymbol{n}}\sum_{i=1}^{3} E_i^2(\boldsymbol{n})$$

$$H_{\rm mag} = -\frac{4\sqrt{3}}{9a^2g^2}\sum_{\boldsymbol{n}} \hexagon(\boldsymbol{n})$$

$\langle J_i | \hexagon | j_i \rangle$ between physical states

$= \text{product of six } 6j \text{ symbols}$

BM, X. Yao, PRD 108 (2023) 094505

Non-abelian gauge theories are chaotic

SU(2)/SU(3) is classically chaotic

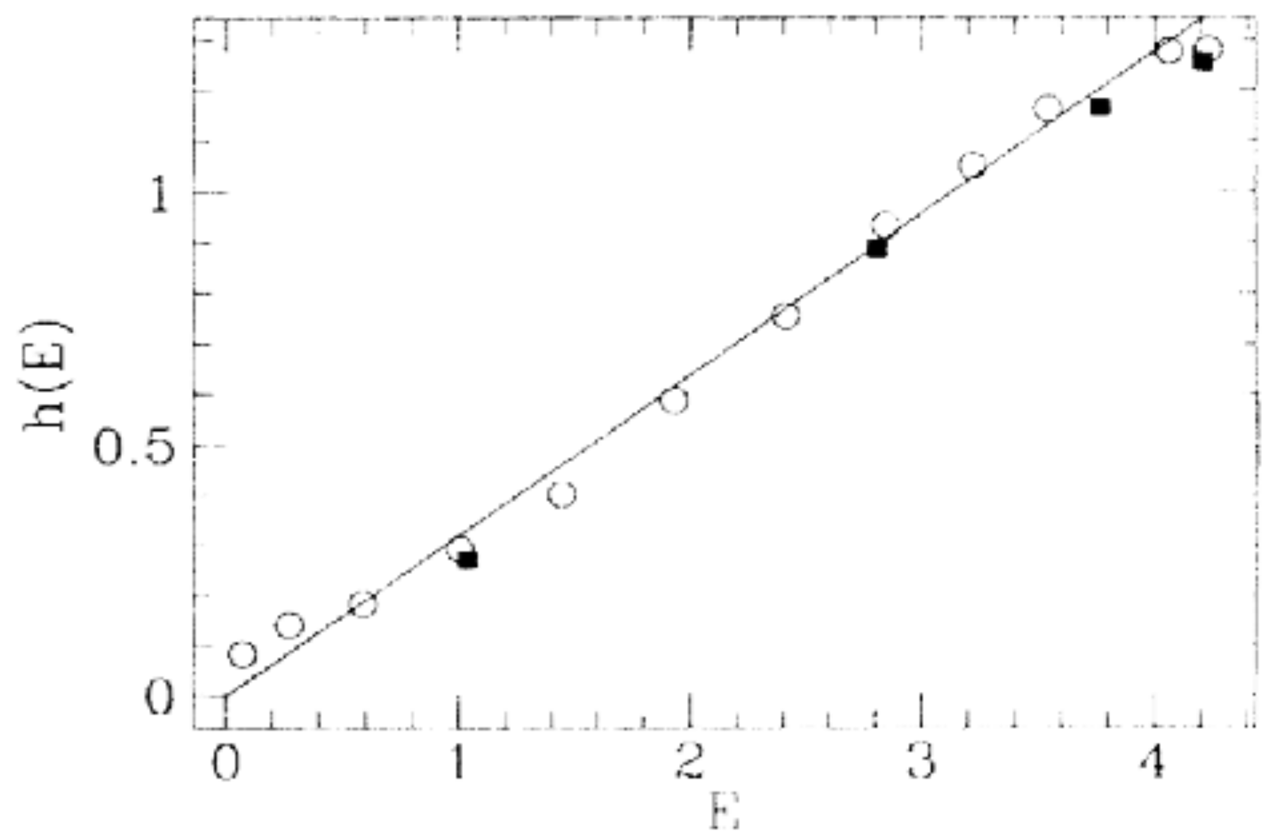
VOLUME 68, NUMBER 23 PHYSICAL REVIEW LETTERS 8 JUNE 1992

Deterministic Chaos in Non-Abelian Lattice Gauge Theory

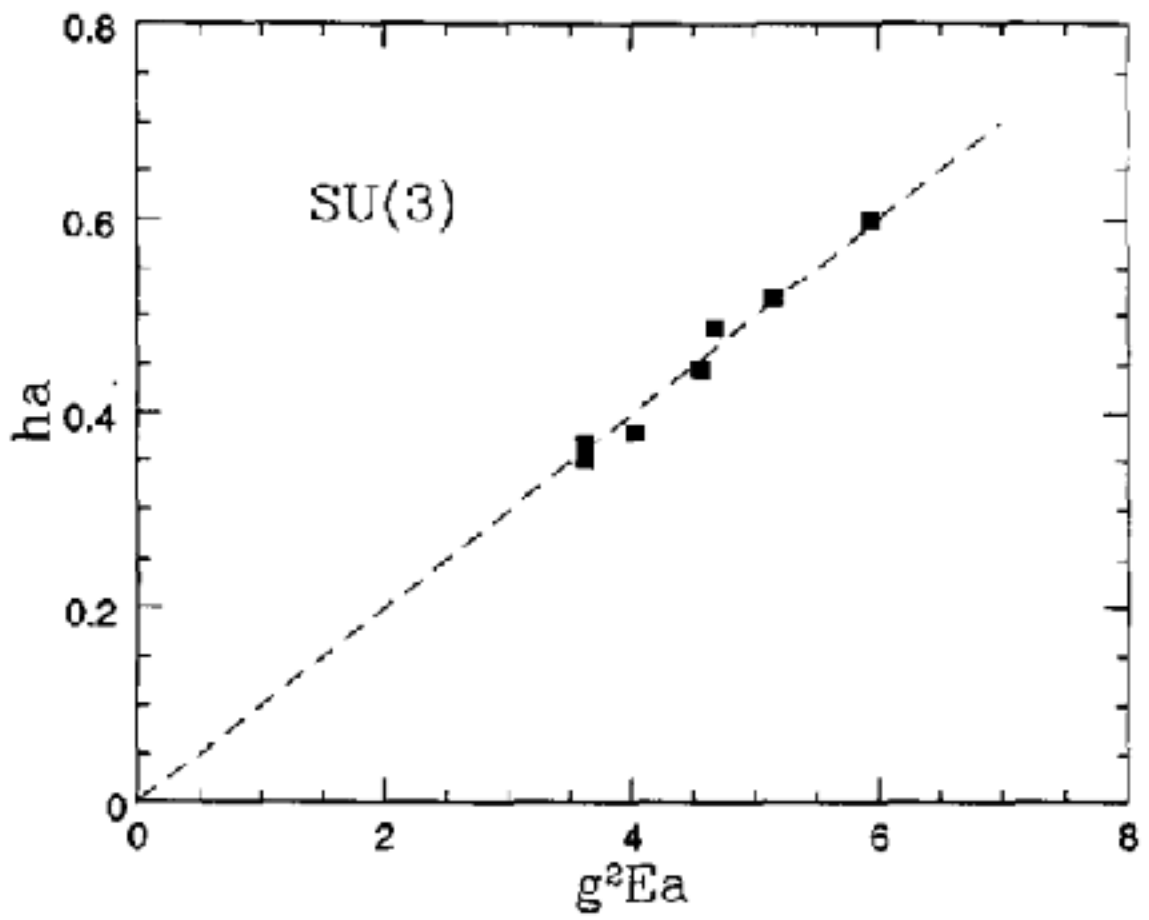
Berndt Müller[(1),(a)] and Atanas Trayanov[(1),(2)]
[(1)] *Department of Physics, Duke University, Durham, North Carolina 27706*
[(2)] *North Carolina Supercomputing Center, Research Triangle Park, North Carolina 27709*
(Received 18 February 1992; revised manuscript received 10 April 1992)

We present numerical evidence that the real-time Hamiltonian dynamics of SU(2) gauge theory on a spatial lattice exhibits deterministic chaos in the semiclassical limit. We determine the largest Lyapunov exponent of the gauge field as a function of energy density, and derive a nonperturbative expression for the thermalization time.

$h_{\mathrm{SU}(2)}(E)\,a \approx 0.17 g^2 E\,a$



$h_{\mathrm{SU}(3)}(E)\,a \approx 0.10 g^2 E\,a$



Physics Letters B 298 (1993) 257–262
North-Holland

Lyapunov exponent of classical SU(3) gauge theory

C. Gong
Physics Department, P.O. Box 90305, Duke University, Durham, NC 27708-0305, USA

Received 28 September 1992

The classical SU(3) gauge theory is shown to be deterministic chaotic. Its largest Lyapunov exponent is determined, from which a short time scale of thermalization of a pure gluon system is estimated. The connection to gluon damping rate is discussed.

K-S entropy rate

$$h = \sum_i \lambda_i \theta(\lambda_i)$$

Eigenstate Thermalization Hypothesis (ETH)

For most non-integrable systems, matrix elements of “typical” local operators for “typical” energy eigenstates can be represented as

$$\langle n|O|m\rangle = \langle O\rangle_{\rm mc}(E)\delta_{nm} + e^{-S(E)/2} f(E,\omega) R_{nm}$$

$$E = (E_n + E_m)/2$$

$$\omega = E_n - E_m$$

- $\langle O\rangle_{\rm mc}(E)\delta_{nm}$: Diagonal part close to microcanonical ensemble average
- $e^{-S(E)/2}$: Correction suppressed exponentially by system size
- $f(E,\omega)$: Spectral function decays with ω
- R_{nm}: Gaussian (?) random matrix

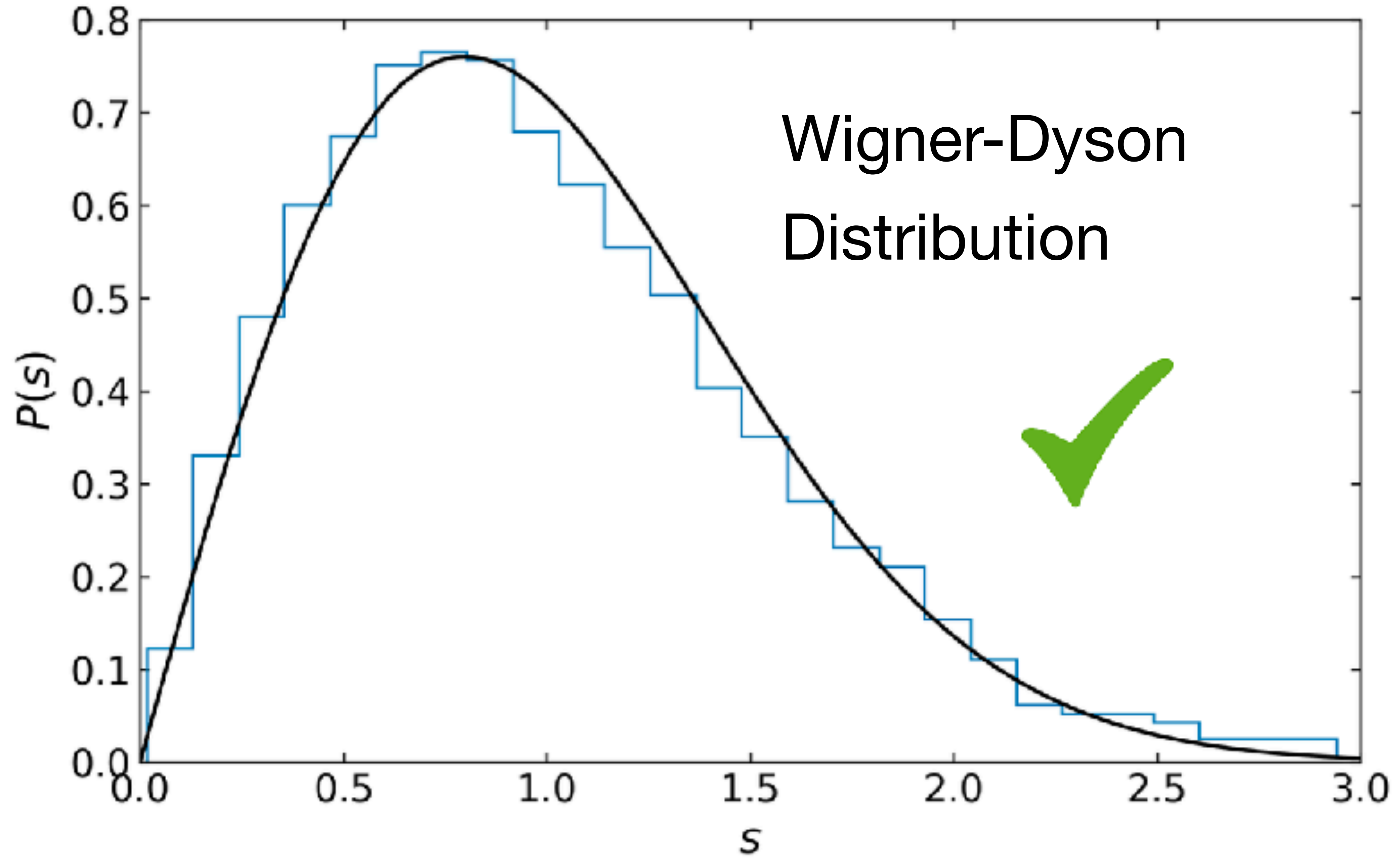
Deutsch, PRA 43, 2046 (1991)

Srednicki, PRE 50, 888 (1994)

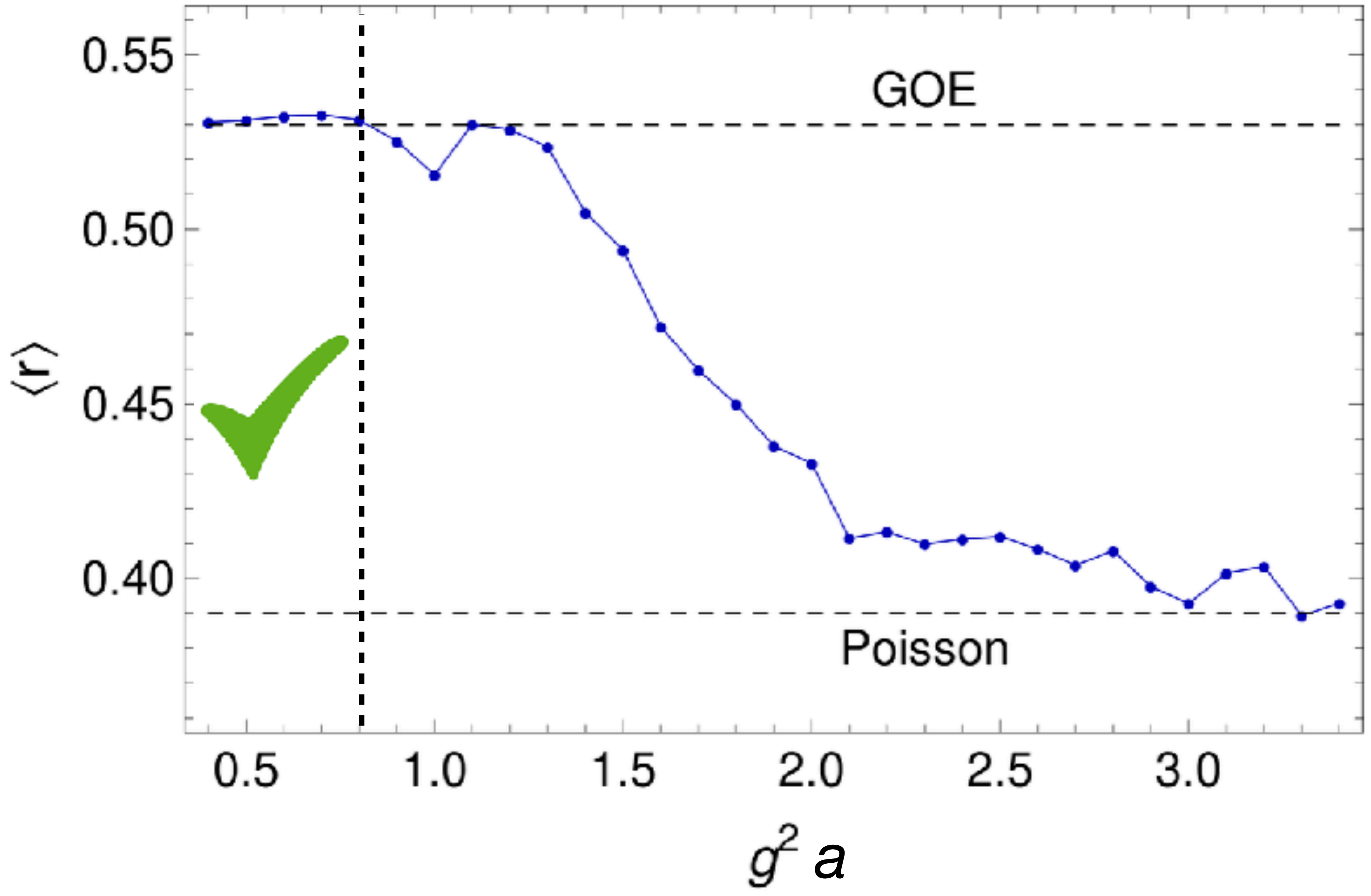
L. D’Alessio, Y. Kafri, A. Polkovnikov, M. Rigol, Adv. Phys. 65 (2016) 239 [1509.06411]

SU(2) gauge theory is quantum chaotic

Nearest-neighbor level statistics exhibits GOE characteristics at $g^2a = 0.8$

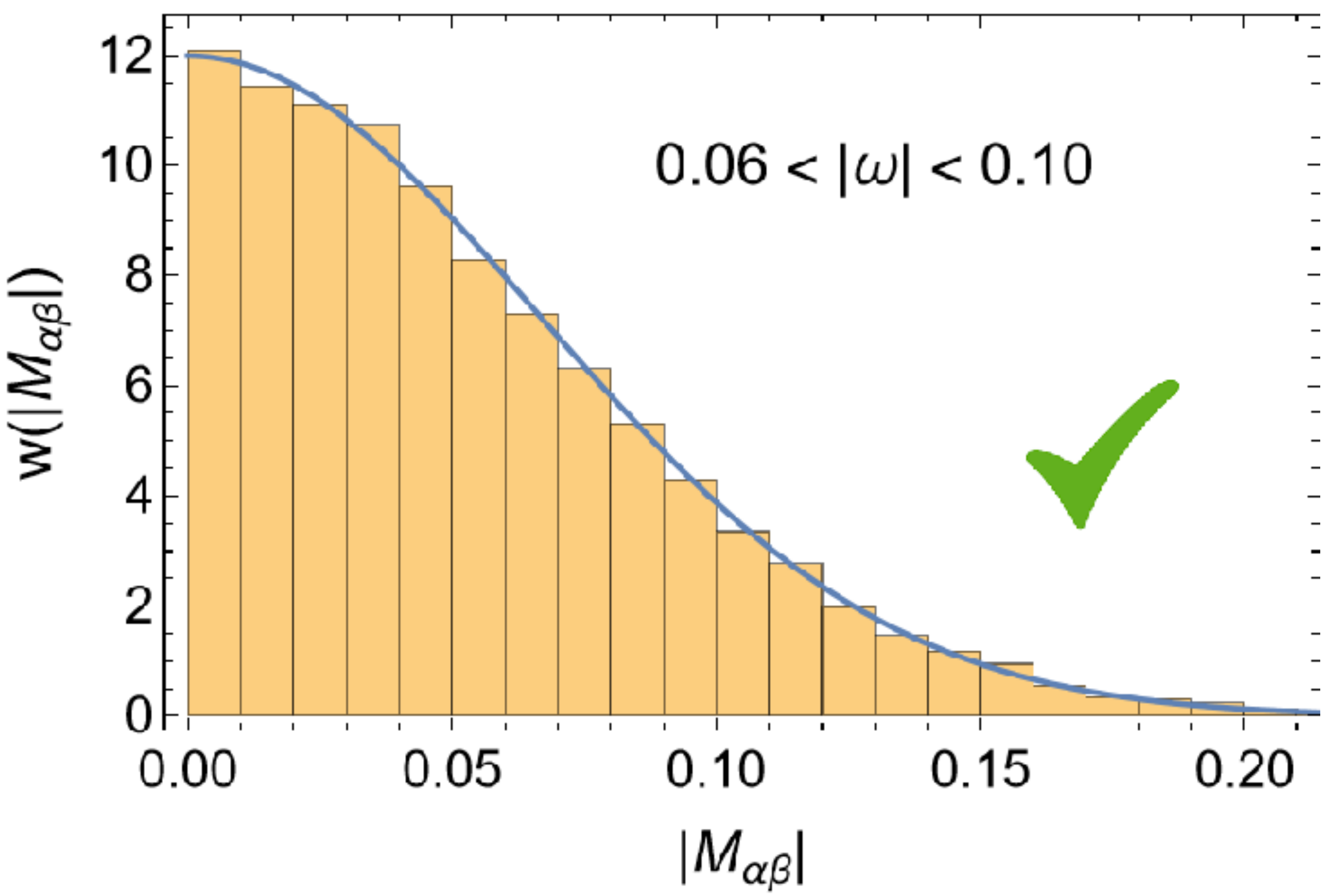


Mean restricted gap ratio shows GOE behavior at weak coupling and Poisson at strong coupling

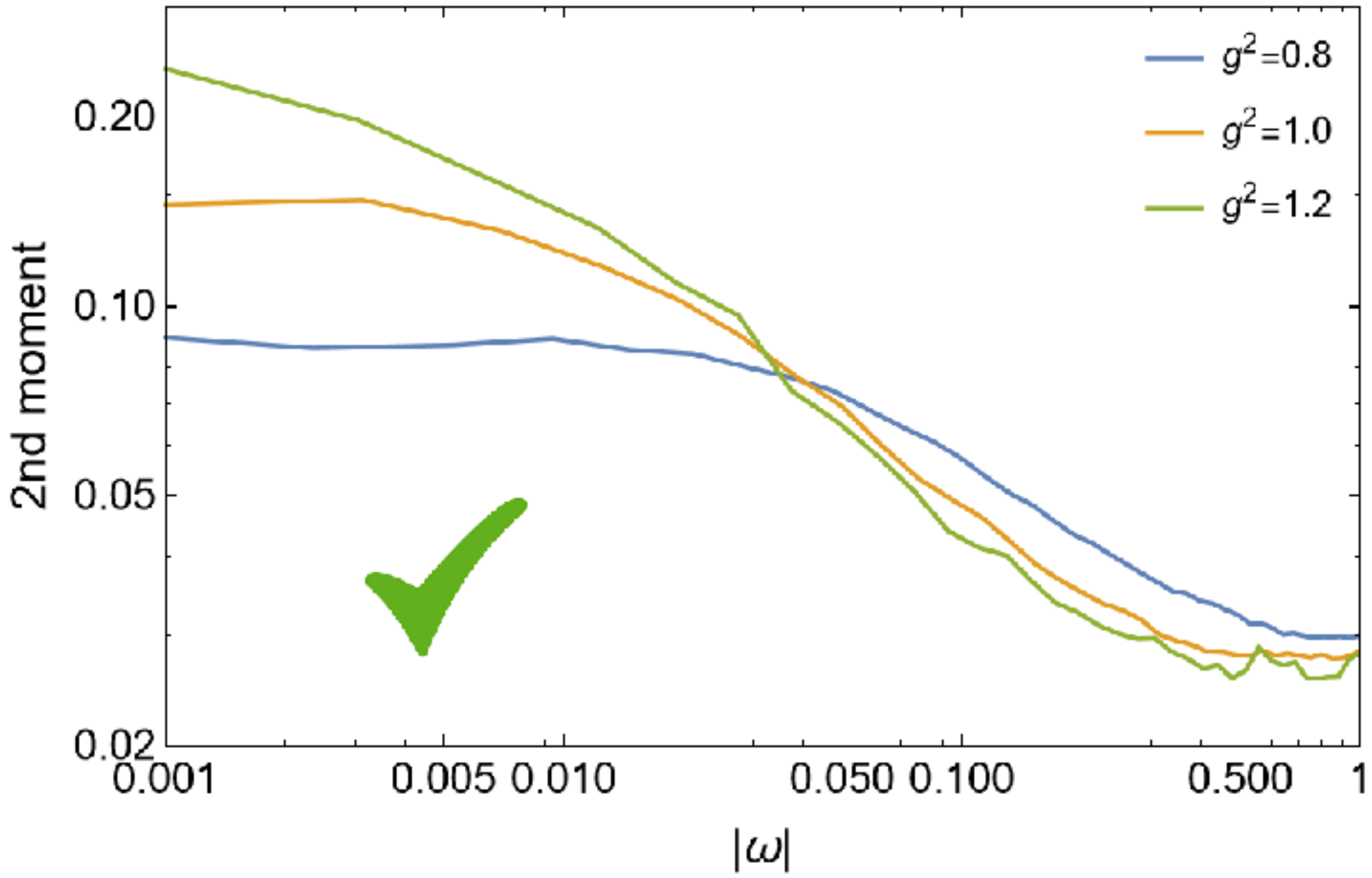


N = 3 Chain with $j_{\text{max}} = 7/2$: Off-Diagonal Part

Off-diagonal elements of H_{el} are Gaussian distributed



Spectral function at small |ω| exhibits a plateau



Plateau disappears when system is non-chaotic

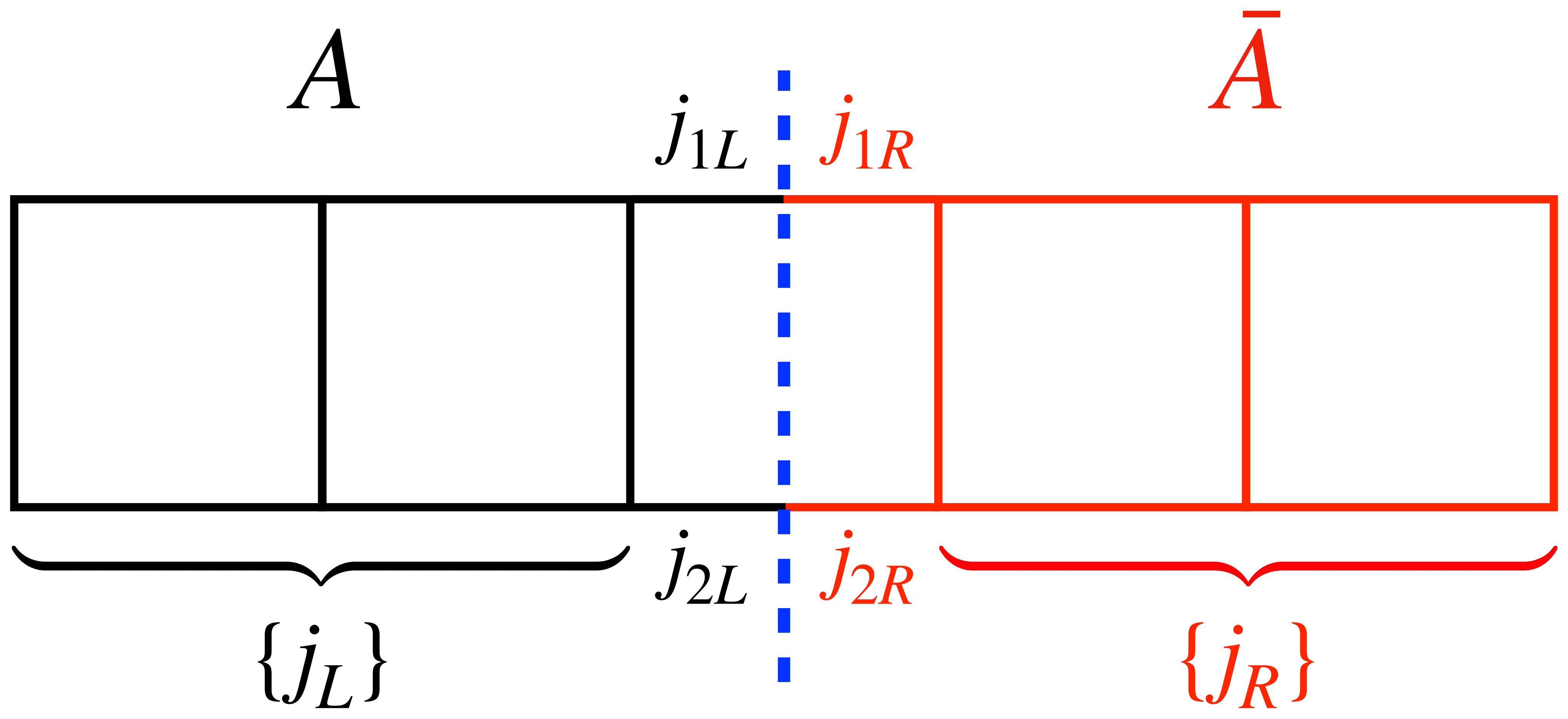
Entropy

Entanglement entropy

Page curve:

Entanglement entropy of subsystems first grows with size and then declines when the subsystem exceeds half the size of the full system [D.N. Page, PRL 71 (1993) 3743].

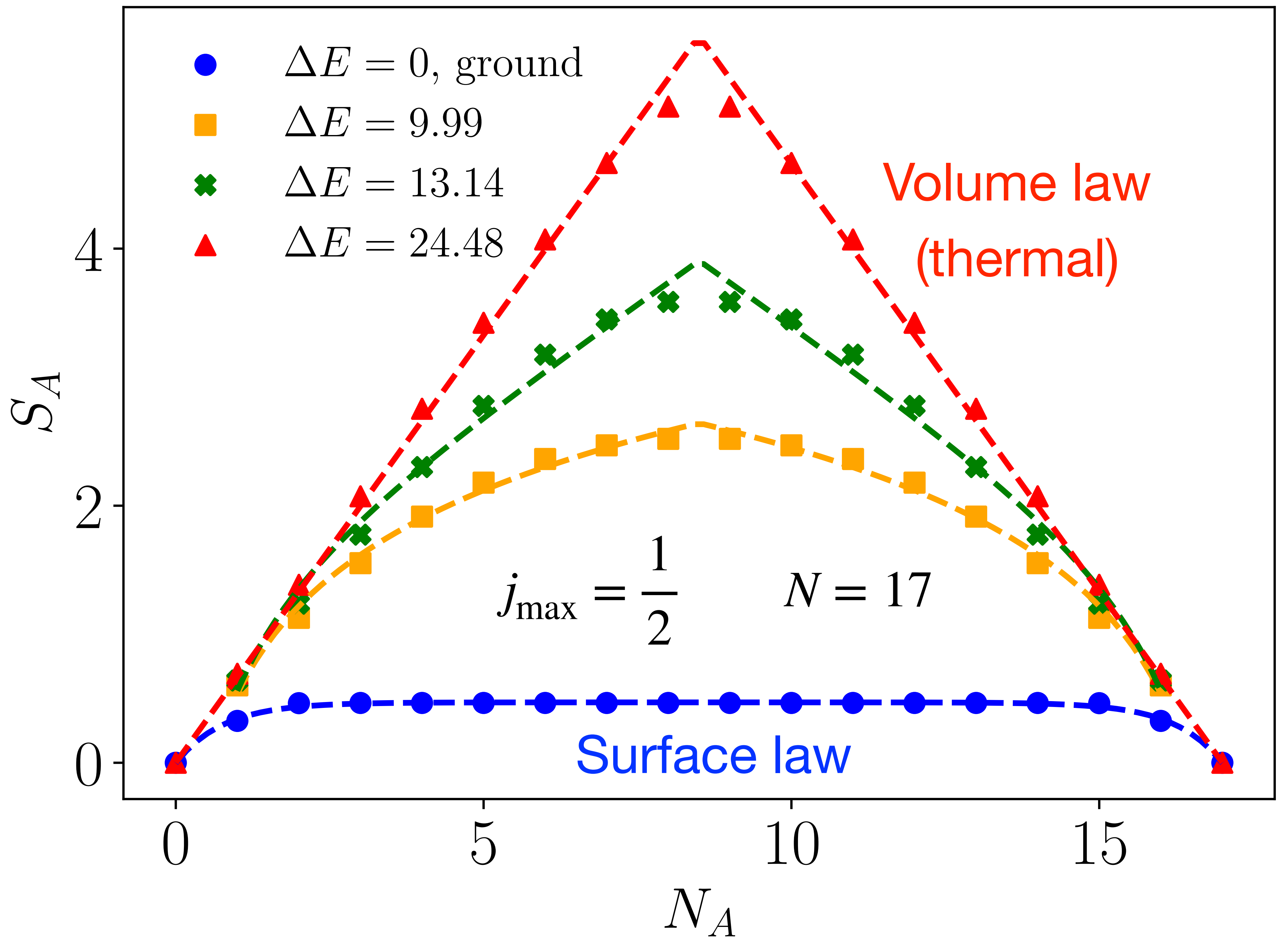
$S_A = \mathrm{Tr}_A(\rho_A \ln \rho_A)$ with $\rho_A = \mathrm{Tr}_{\bar{A}} |\psi\rangle \langle\psi|$

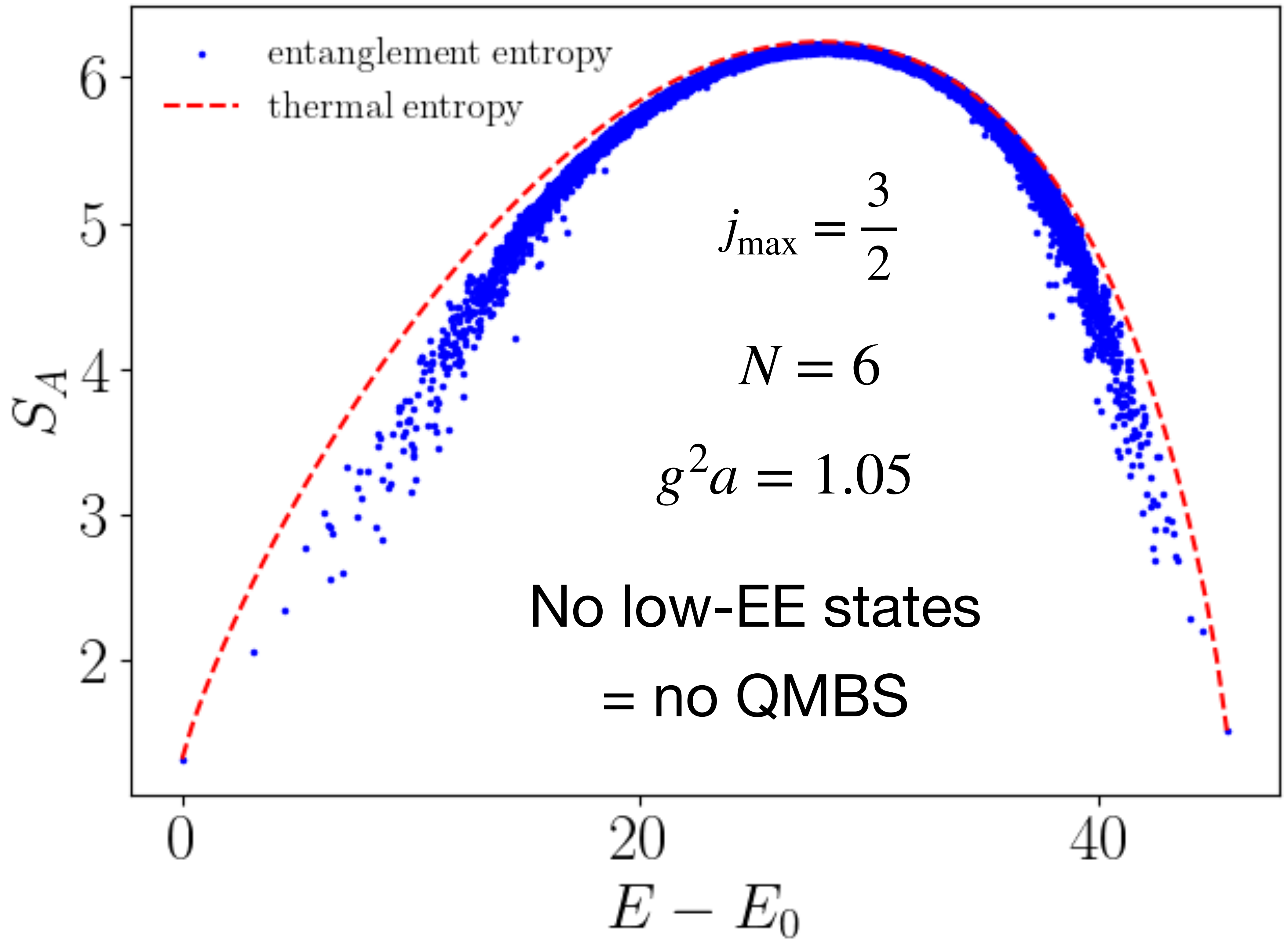


Ebner, BM, Schäfer, Seidl, Yao, 2401.15184

Page curve

$$S(N_A) = c_0 + \frac{c}{3}\ln(c_1) + \frac{c}{3}\left[\ln[\sinh(c_1^{-1}N_A)]\theta\left(\frac{N}{2} - N_A\right) + \ln\{\sinh[c_1^{-1}(N - N_A)]\}\theta\left(N_A - \frac{N}{2}\right)\right]$$



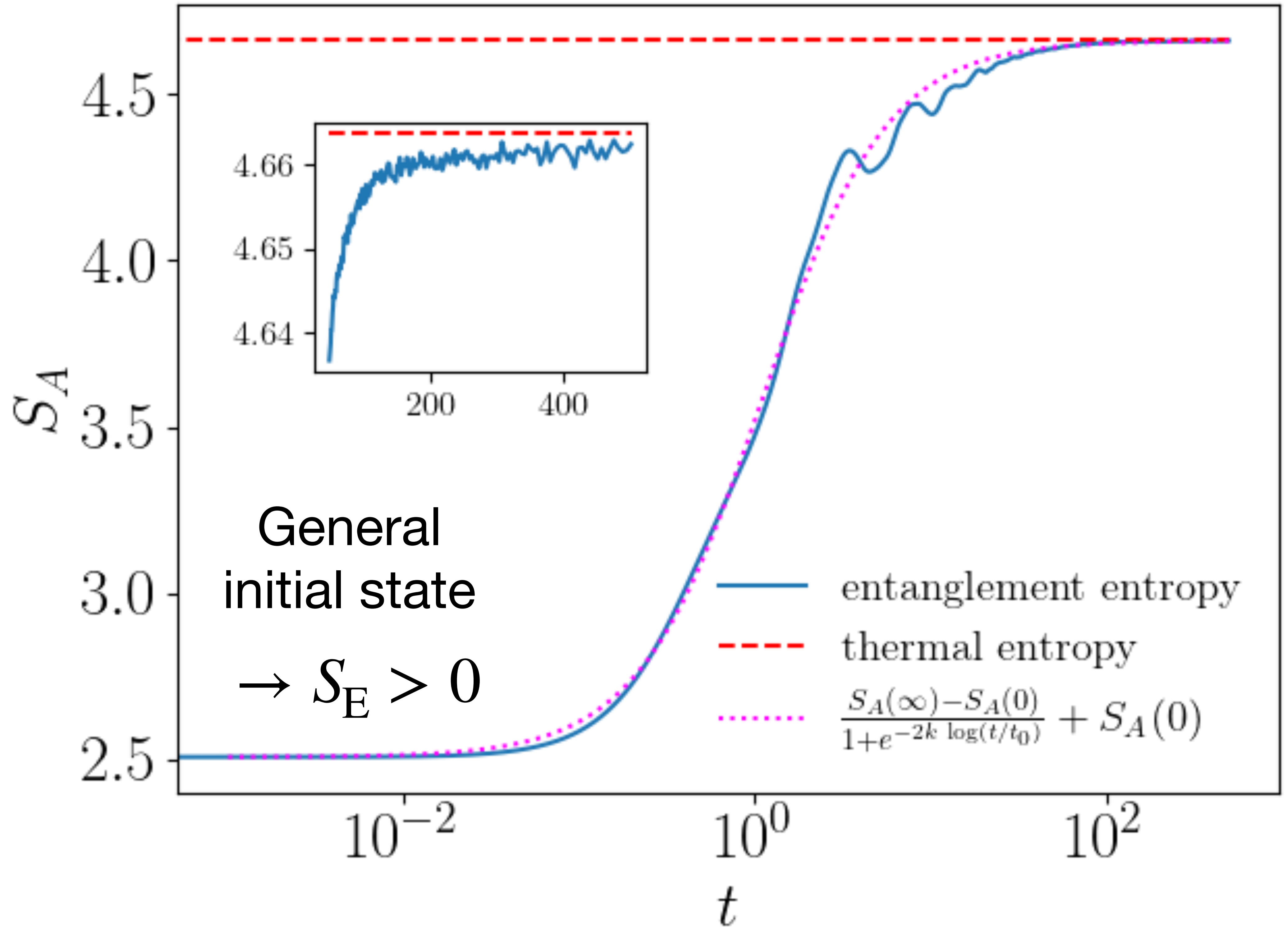


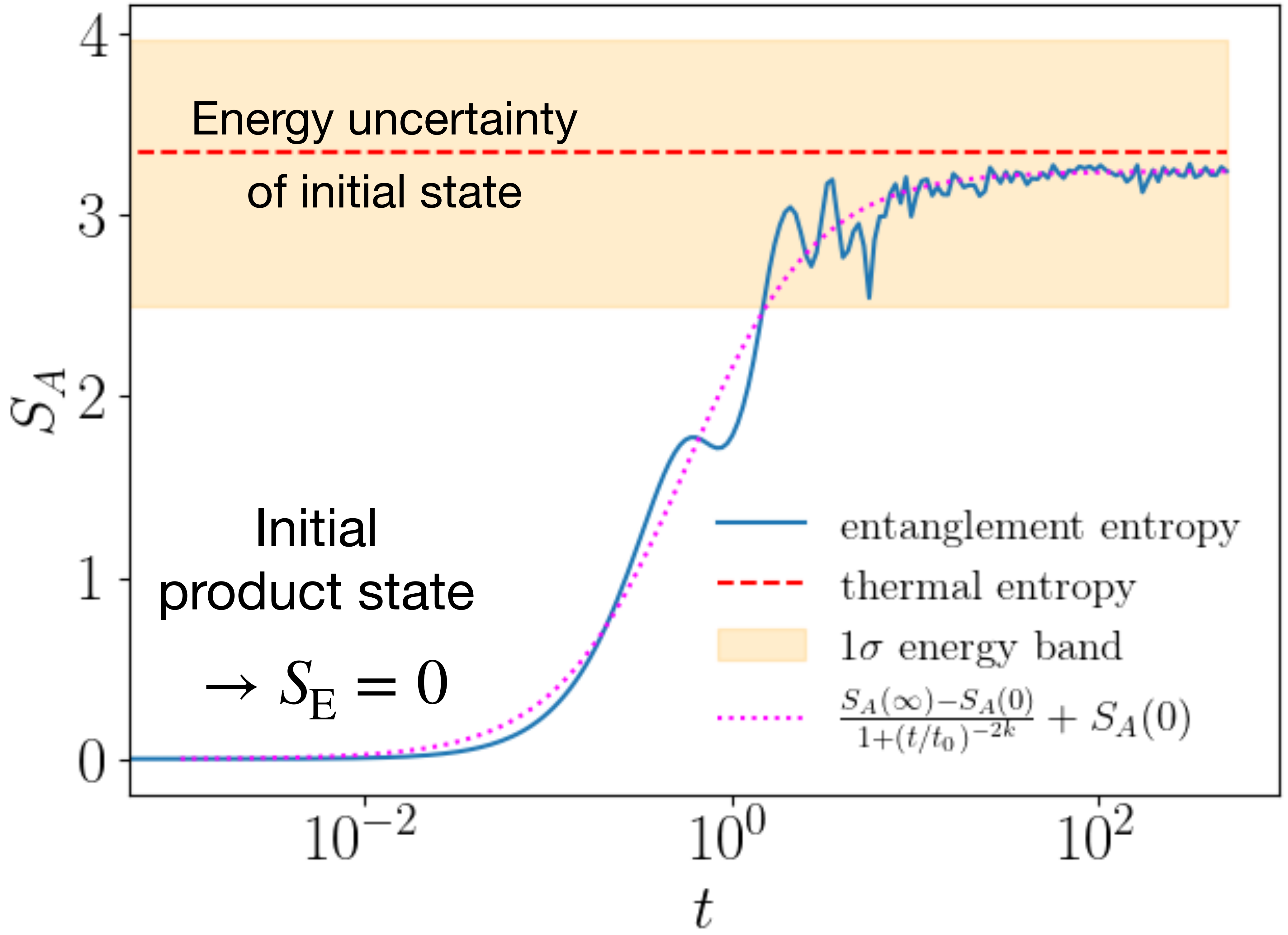
Ebner, BM, Schäfer, Seidl, Yao, *Phys.Rev.D* 110 (2024) 014505 [2401.15184]

Two-step thermalization

Step 1: Entanglement entropy introduced by local observation - depends on initial state

Step 2: Approach to micro-canonical (thermal) entropy - independent of initial state





L. Ebner, BM, L. Schmotzer, A. Schäfer, C. Seidl, X. Yao, *Commun.Phys.* 8 (2025) 368 [2411.04550]

“Magic”

What is “Magic”?

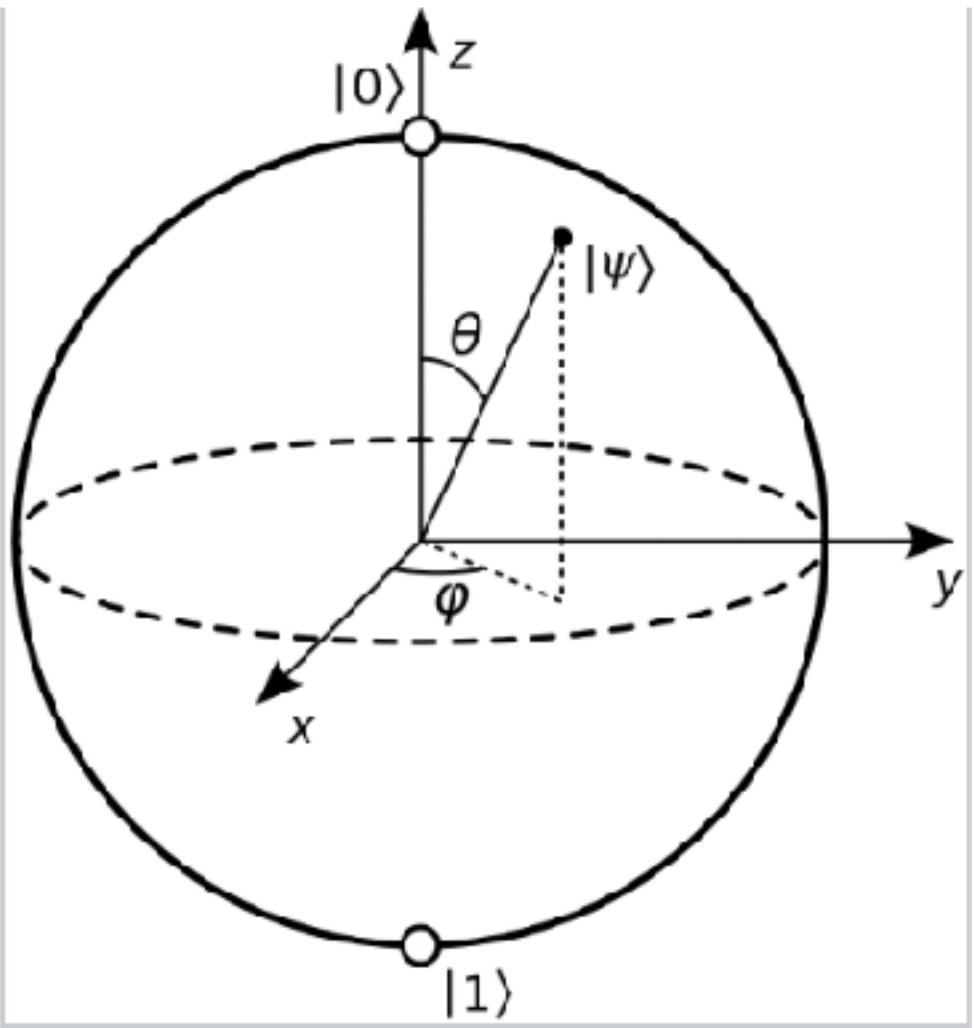
Unitary evolution of a quantum state can be expressed as a sequence of elementary quantum gates acting on qubits:

Operator	Gate(s)	Matrix
Pauli-X (X)	X, $\oplus$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
Pauli-Y (Y)	Y	$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$
Pauli-Z (Z)	Z	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
Hadamard (H)	H	$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$
Phase (S, P)	S	$\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$
$\pi/8$ (T)	T	$\begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}$
Controlled Not (CNOT, CX)	$\bullet$ / $\oplus$	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

Rotation operators

$R_x(\theta), R_y(\theta), R_z(\theta),$

$$P(\varphi) = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\varphi} \end{bmatrix}$$



Bloch sphere of a qubit

Clifford set: {CNOT, *H*, *S*} is not universal

Requires an additional gate, e.g. *T*-gate

Gottesman-Knill theorem (1998)

Quantum circuits consisting solely of operators from the Clifford group (Pauli matrices *I, X, Y, Z*) or equivalently the Clifford gate set {CNOT, *H, S*} can be perfectly simulated in polynomial time by a probabilistic digital computer.

Such quantum circuits are called "stabilizer" circuits. The amount of *non-stabilizer* quantum resources, such as *T*-gates, required to implement a unitary evolution is called *non-stablizerness* or "**magic**".

The quantitative determination of the "magic" resources required by a quantum evolution is difficult.

A lower limit is provided by the stabilizer Rényi entropies, i.e. Clifford averaged Rényi entropies:

$$M_n(\rho) = \frac{1}{1-n} \log \left[\sum_{P \in \mathcal{P}_N} 2^{-N} |\mathrm{Tr}(\rho P)|^{2n} \right]$$

Linearized: $M_2(\rho) = -\log[1 - M_{\mathrm{lin}}(\rho)]$

Anti-flatness of the entanglement spectrum of a subsystem *A*: $\mathcal{F}_A(\rho) = \mathrm{Tr}(\rho_A^3) - [\mathrm{Tr}(\rho_A^2)]^2$

$$\langle \mathcal{F}(\rho) \rangle_{\mathcal{P}} \propto M_{\mathrm{lin}}(\rho)$$

[E. Tirrito et al., PRA 109 (2024) L040401 [2304.01175]]

Anti-flatness

The *entanglement spectrum* of a density matrix is the spectrum of its *entanglement Hamiltonian*:

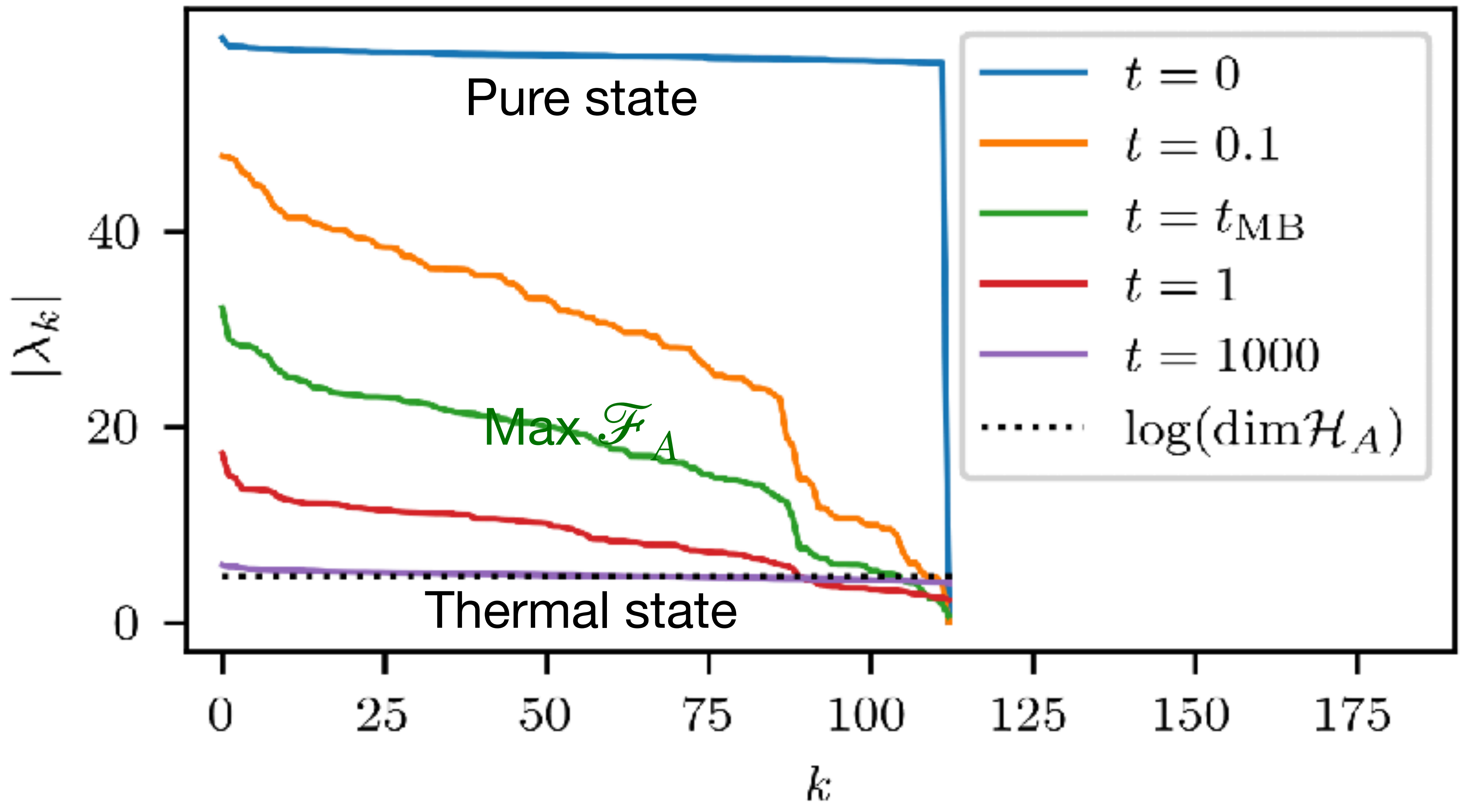
$$\rho = e^{-\tilde{H}} \quad \leftrightarrow \quad \tilde{H}|\nu\rangle = \omega_\nu |\nu\rangle$$

Examples of **flat** entanglement spectrum: Pure state: $\rho = |\psi\rangle\langle\psi| \quad \rightarrow \quad \omega_1 = 0, \omega_{\nu>1} = N$

Maximally mixed state: $\rho = \frac{1}{N} I_N \quad \rightarrow \quad \omega_\nu = \log N$

During (local) thermalization of a pure state, the reduced density matrix ρ_A first steepens and later flattens out again as the reduced density matrix becomes quasi-thermal and maximally mixed.

Maximal anti-flatness is reached during period of maximal entanglement entropy growth = time of most rapid thermalization.

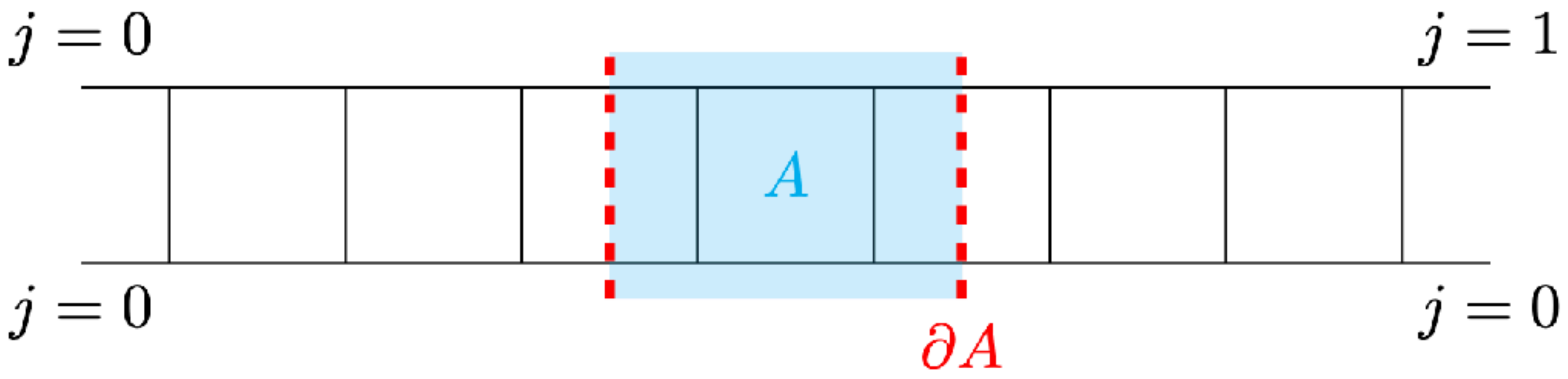


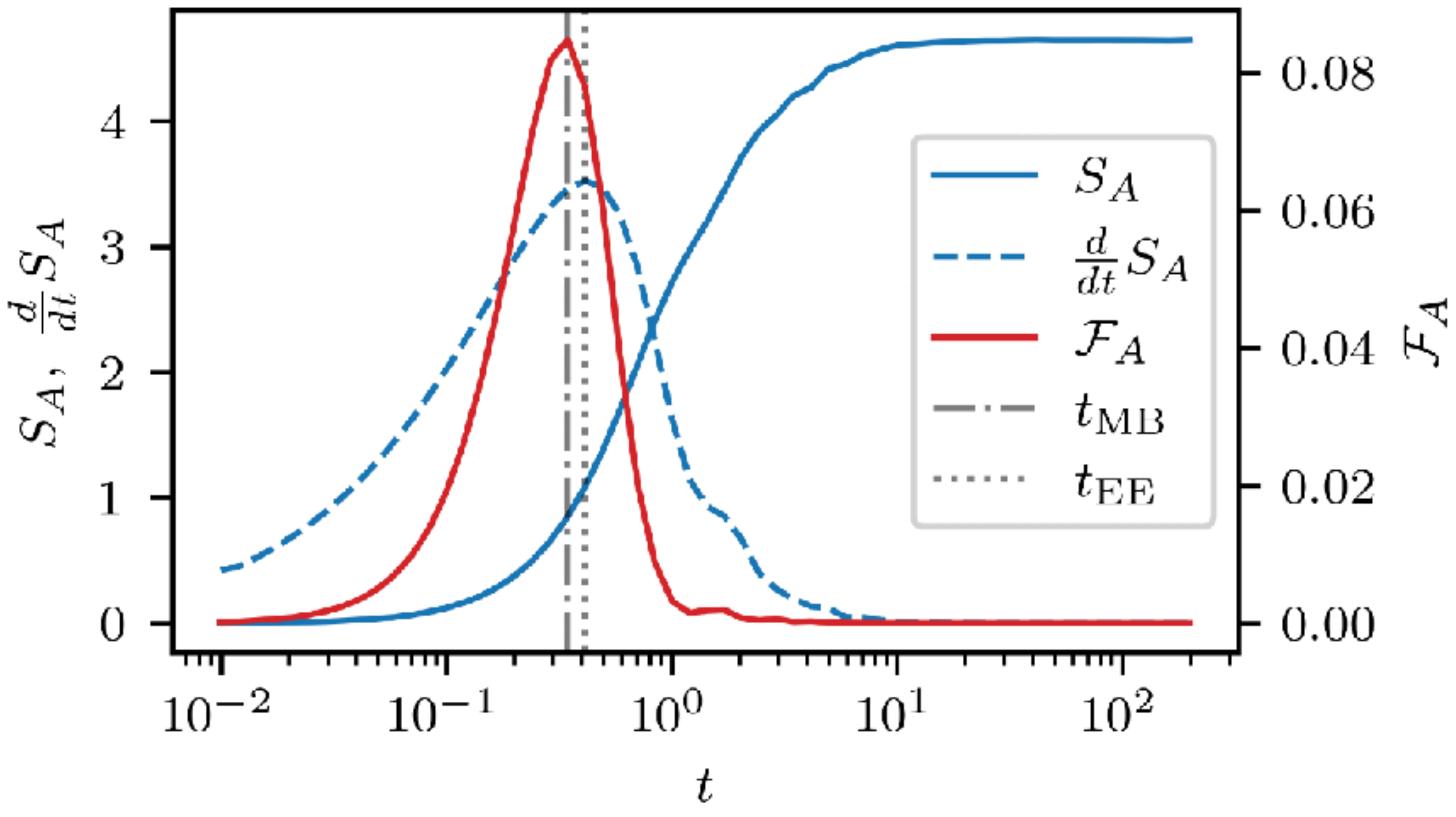
Results for SU(2) plaquette chain

[L. Ebner *et al.*, 2510.11681
PRL 136, 230403 (2026)]

External links of subsystem act like surface charges; they give incoherent contribution to S_E.

SU(2) on 7-plaquette chain

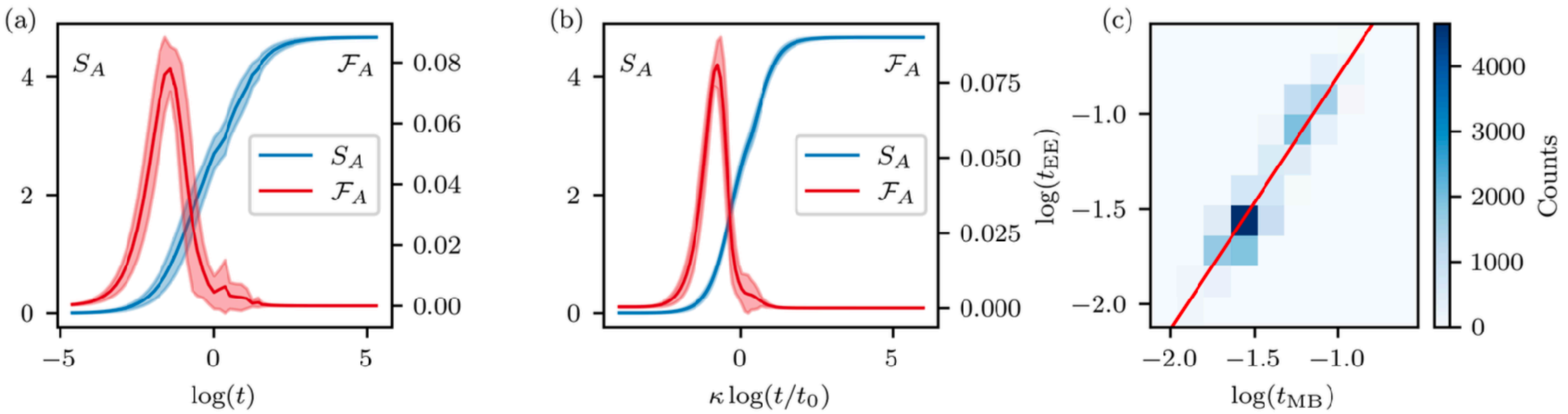




Initial state: Highly excited product state of electric energy on links obeying Gauss law. Can be thought of strong coupling eigenstate.

Anti-flatness peaks precisely where the growth rate of the entanglement entropy is maximal, i.e. when quantum correlations are rapidly rearranged.

Results II

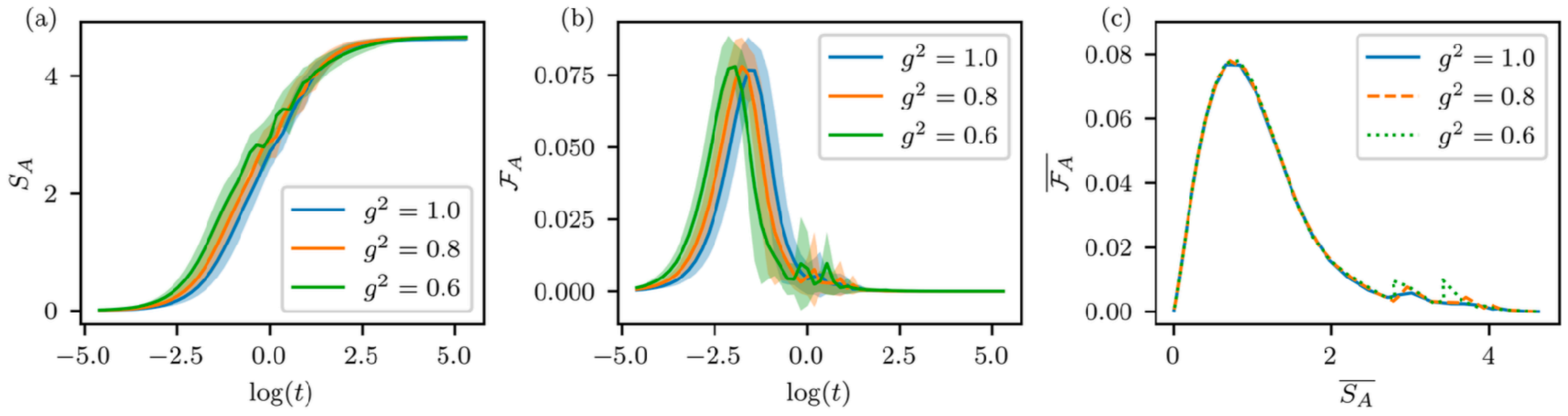


(b) Rescaled time evolution, where the time t is replaced by κ $\log(t/t_0)$ with state-dependent parameters κ and t_0, such that the thermalization of different states is synchronized.

(c) Joint distribution of magic barrier time t_{MB} and time of maximum entanglement entropy growth t_{EE} on a logarithmic scale for all 18389 electric basis states satisfying the Gauss law.

Results III

Coupling dependence of entanglement dynamics



(c) Parametric dependence of $\mathscr{F}_A$ on S_A for different ergodic couplings

The behavior seems to be universal for ergodic quantum systems; it is seen also in the transverse field Ising model.

SU(3) LGT

SU(3)

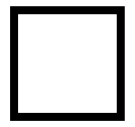
KS Hamiltonian: $$H = \frac{g^2\Sigma}{n_l}\sum_{\text{links}}(E_i^a)^2 + \frac{1}{g^2\Sigma}\sum_{\text{plaq}}(6 - U_P - U_P^\dagger)$$

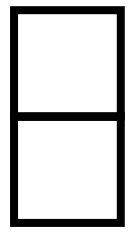
Square lattice $n_l = 2,\ \Sigma = 1$

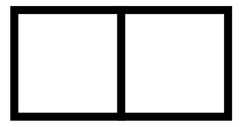
Hexagonal lattice $n_l = 3,\ \Sigma = \frac{3\sqrt{3}}{2}$

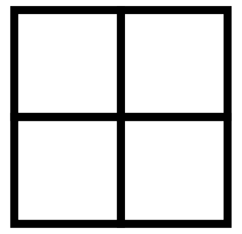
Representations:

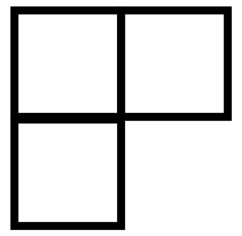
Singlet


3 = quark


$\bar{3}$ = quark


6 = di-quark


$\bar{6}$ = di-quark


8 = gluon

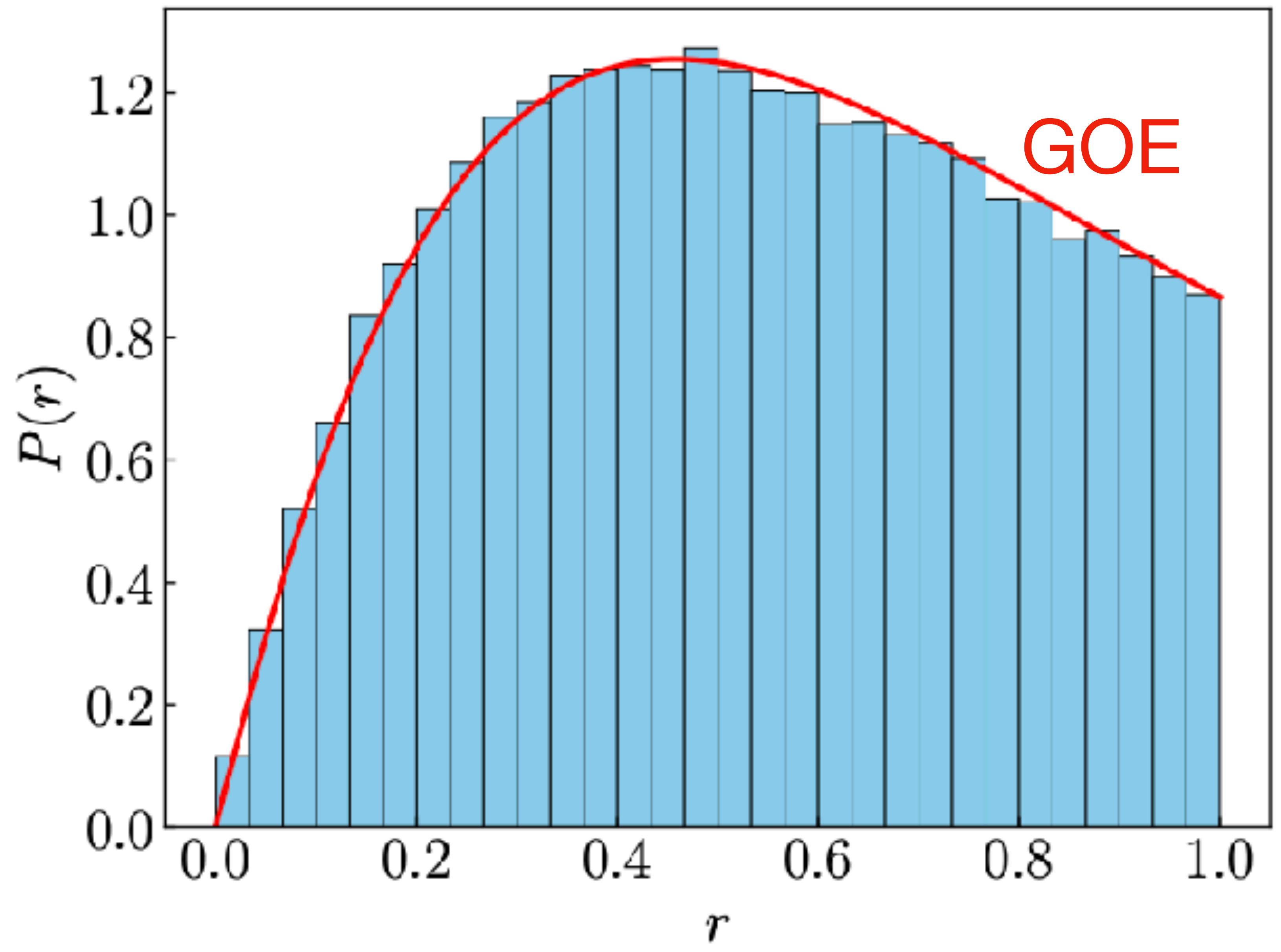
Minimal truncation of SU(3) gauge group

With this truncation, SU(3) on linear square plaquette chains or hexagonal (“honeycomb”) lattices can formulated as a quantum theory of qutrits (= a type of *N* = *3* Potts model).

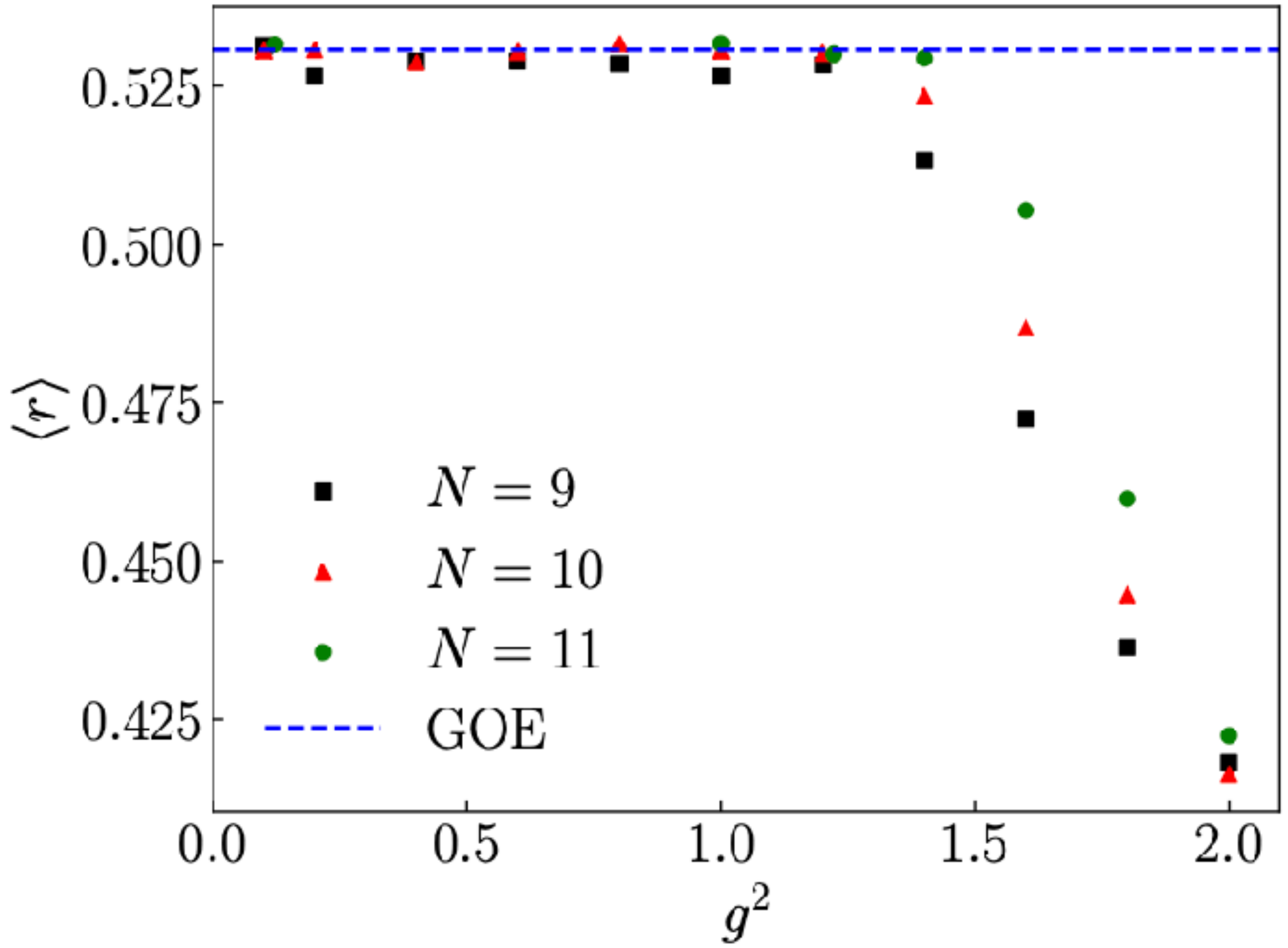
SU(3) LGT is chaotic

Energy gap ratio: $$r_\alpha = \frac{\min[\delta_\alpha, \delta_{\alpha-1}]}{\max[\delta_\alpha, \delta_{\alpha-1}]} \leq 1$$

GOE prediction: $$P_{\mathrm{GOE}}(r) = \frac{27}{4} \frac{r + r^2}{(1 + r + r^2)^{5/2}}$$

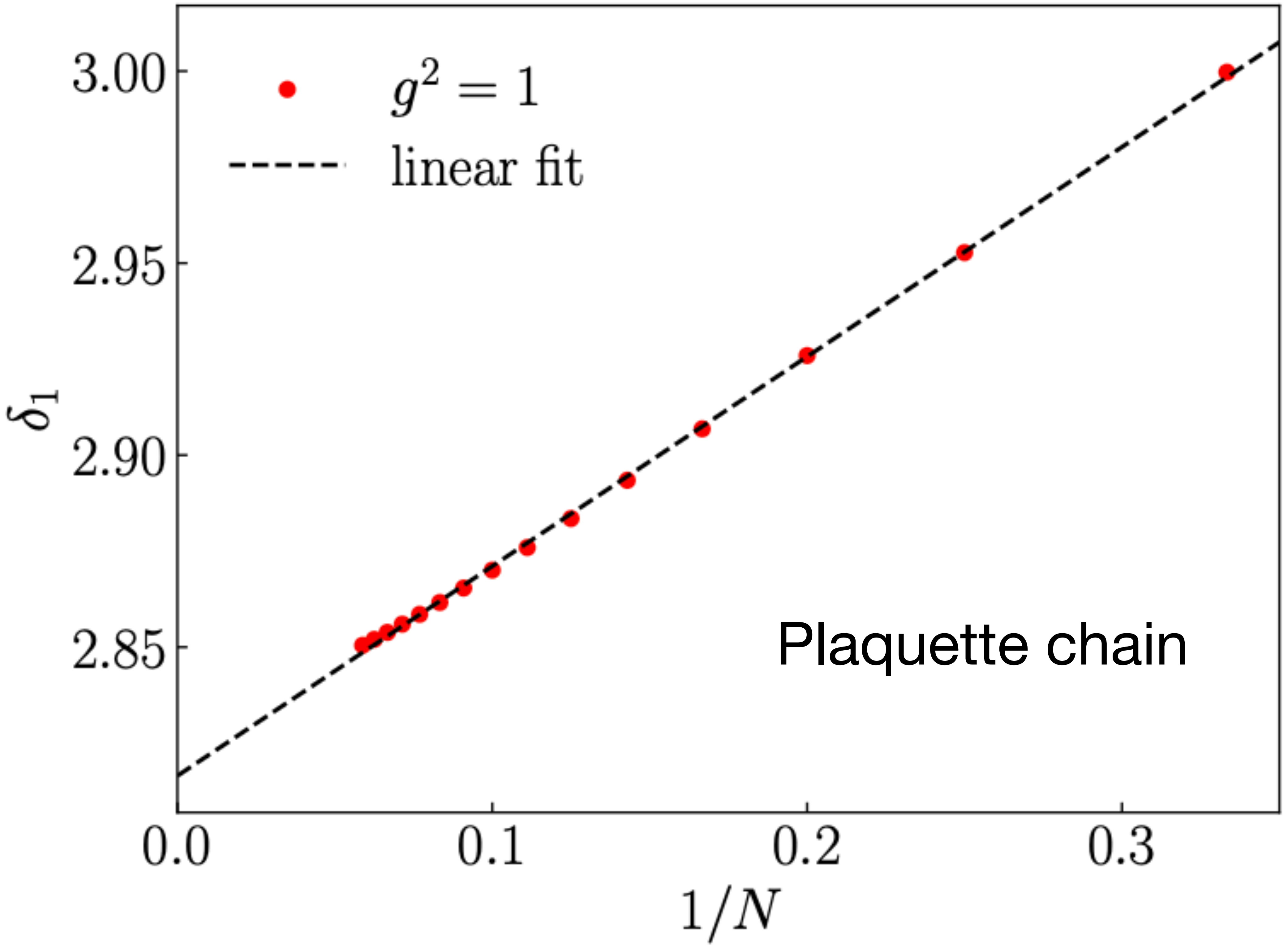


SU(3) is quantum chaotic in the physically relevant regime of weak coupling, even for for truncated theory on a short chain.

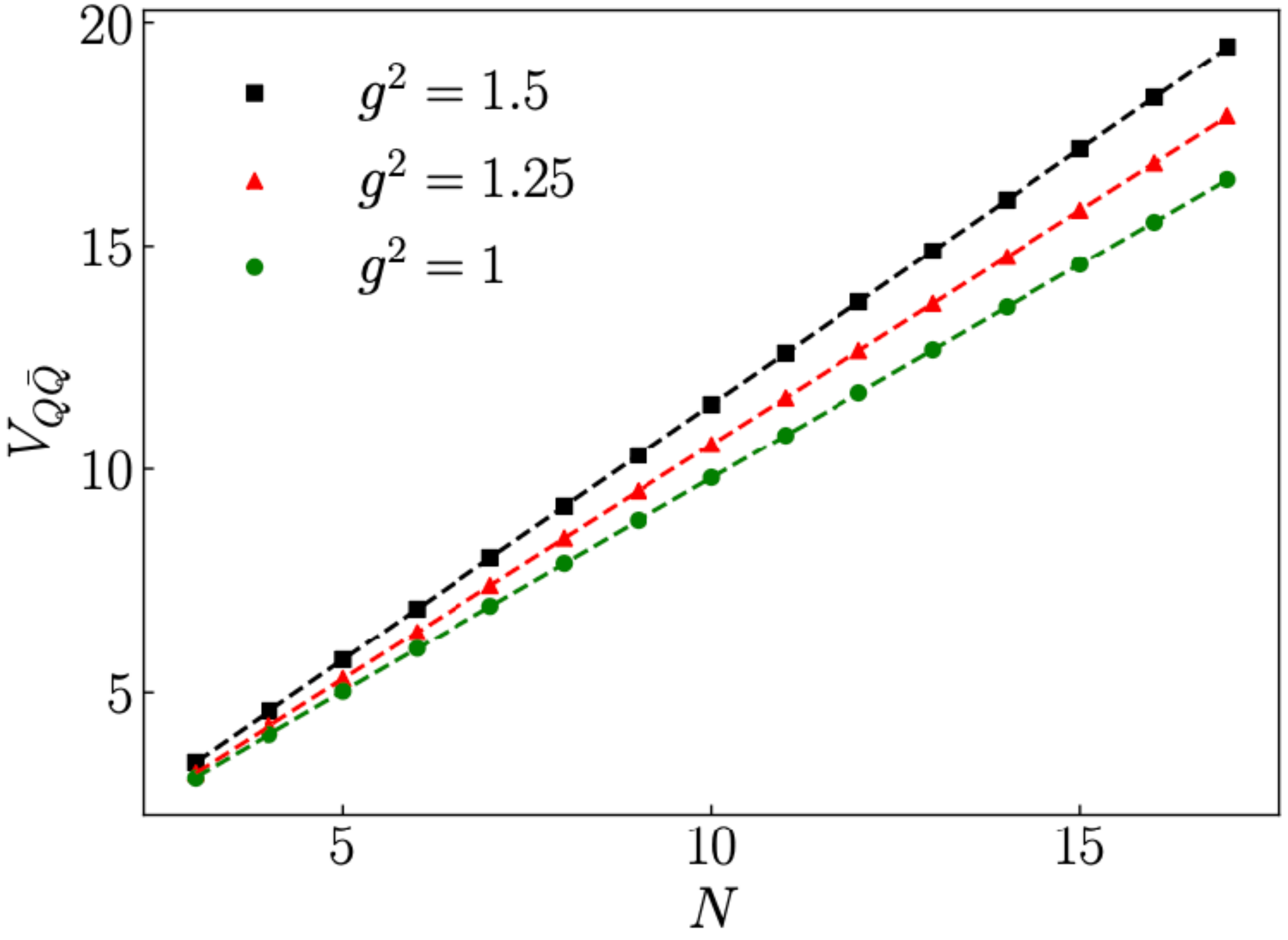


Ground state energy gap and $Q\bar{Q}$ potential

The ground state energy gap (“glue ball mass”) has a well defined large volume limit.



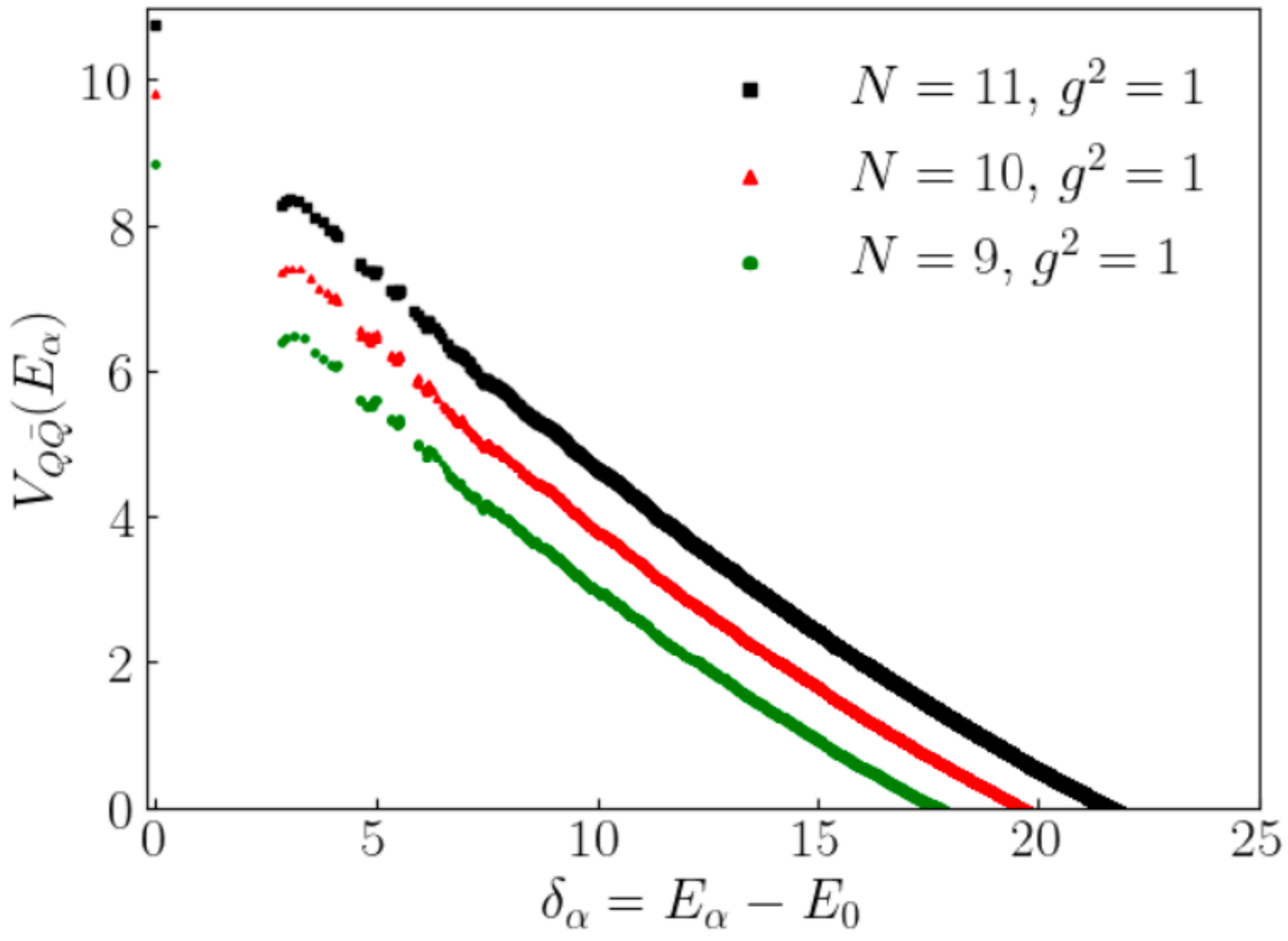
The quark-antiquark potential $V_{Q\overline{Q}}(r) = \sigma r$ is linear and has finite coupling corrections from quantum fluctuations that increase at weak coupling

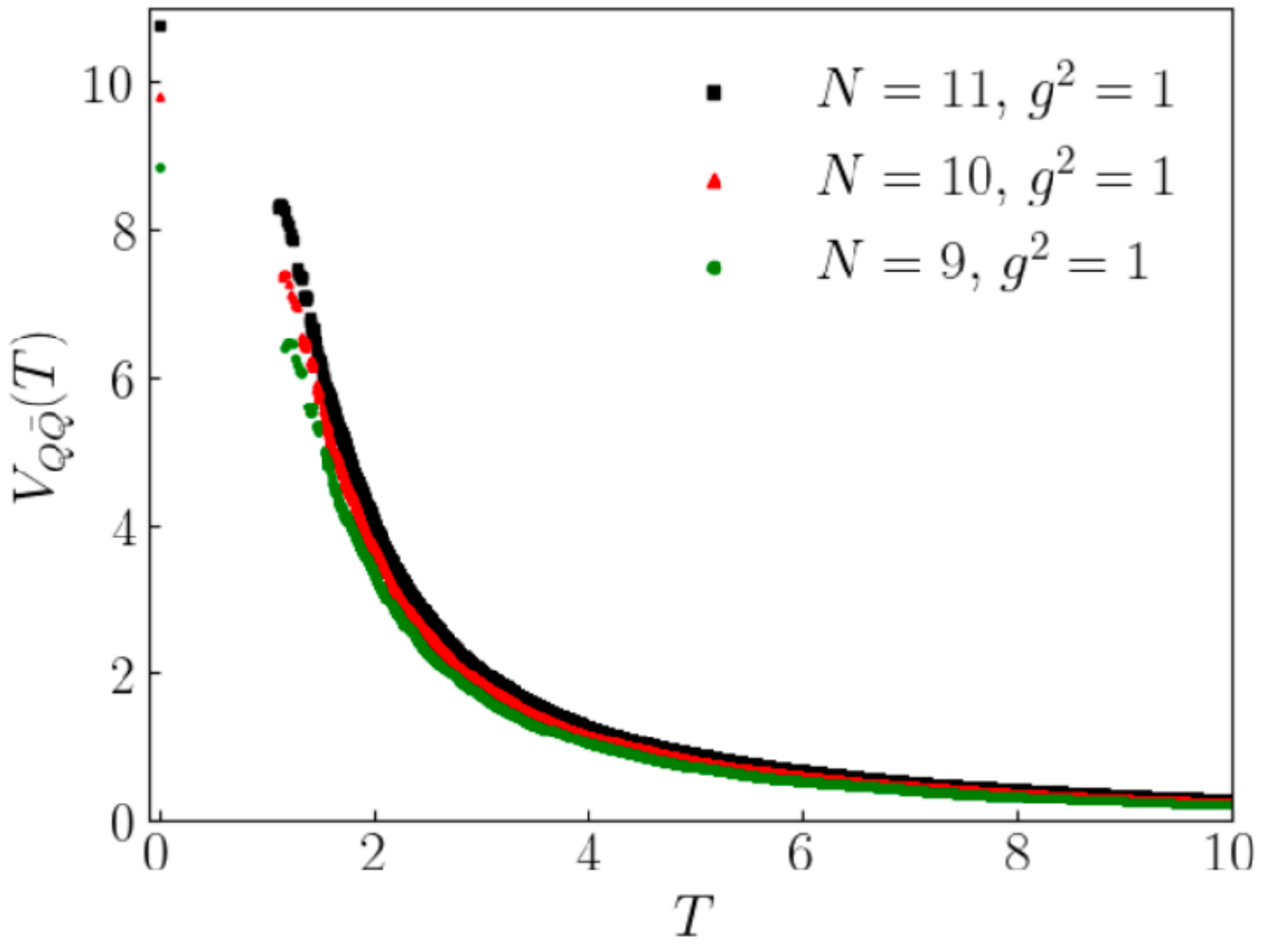


String “melting”

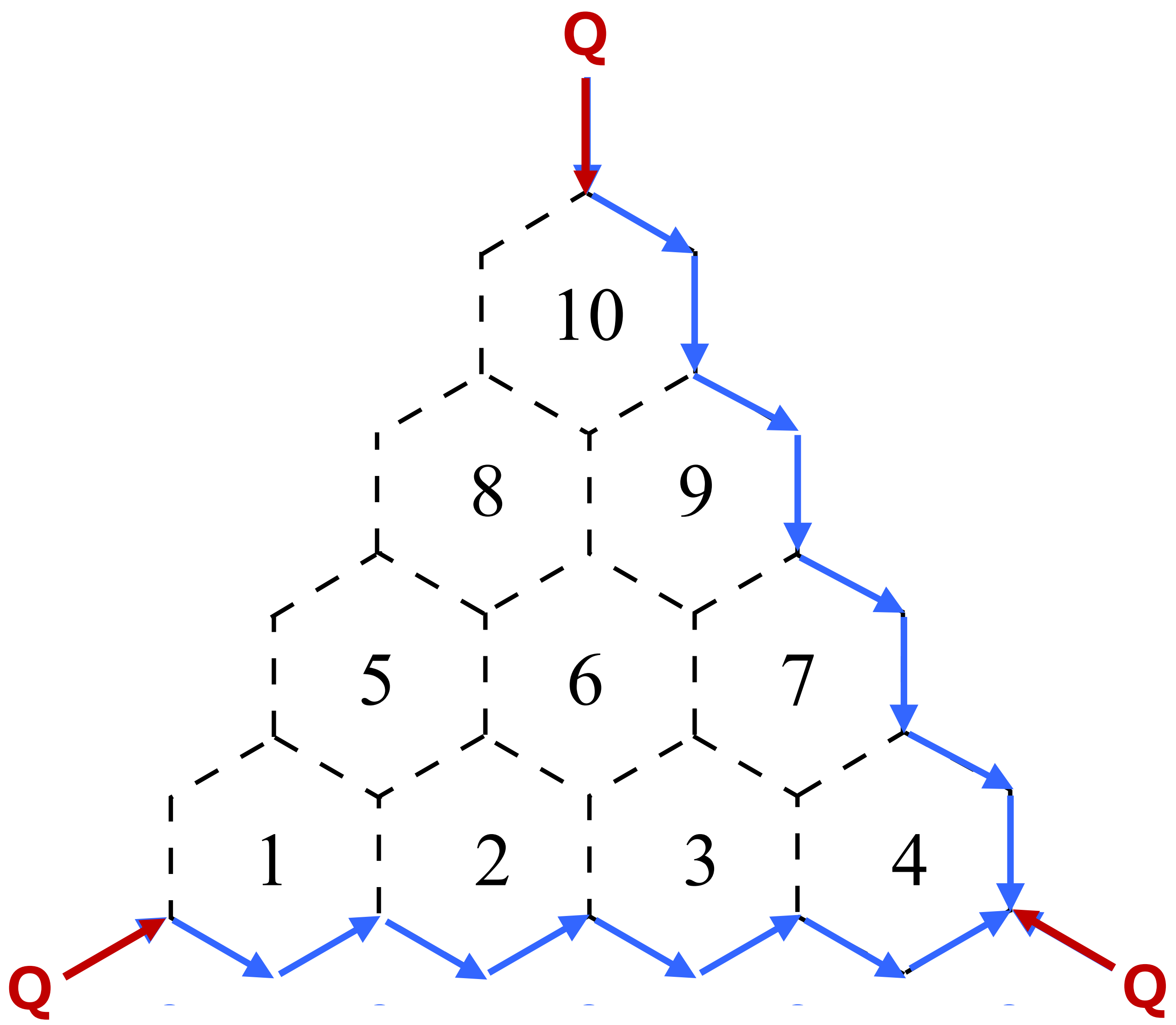
$V_{Q\overline{Q}}$ drops rapidly with excitation energy (or micro canonical temperature) for excited states.

For high temperatures the potential becomes independent of $Q\overline{Q}$ distance!

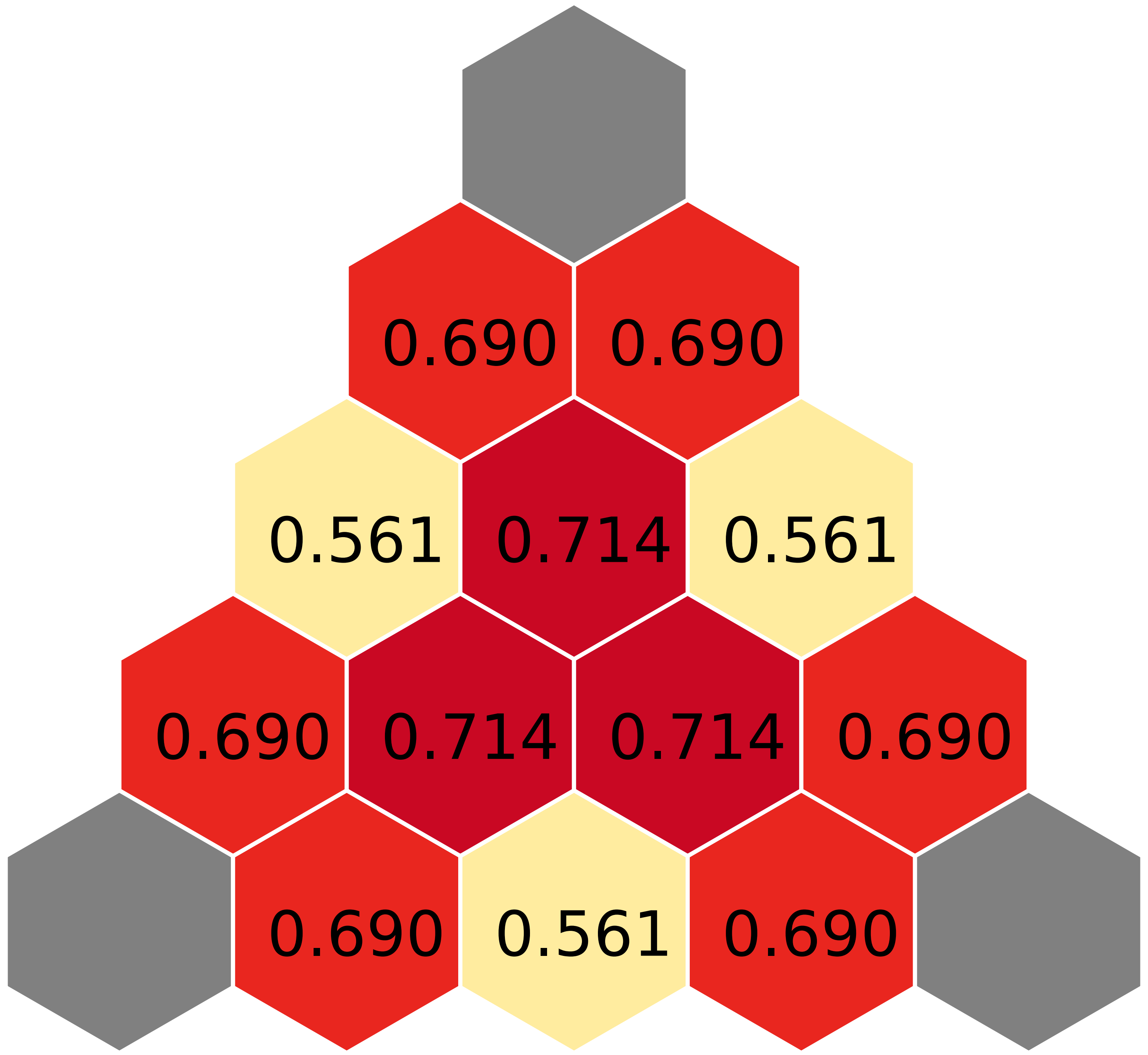




Proton toy model

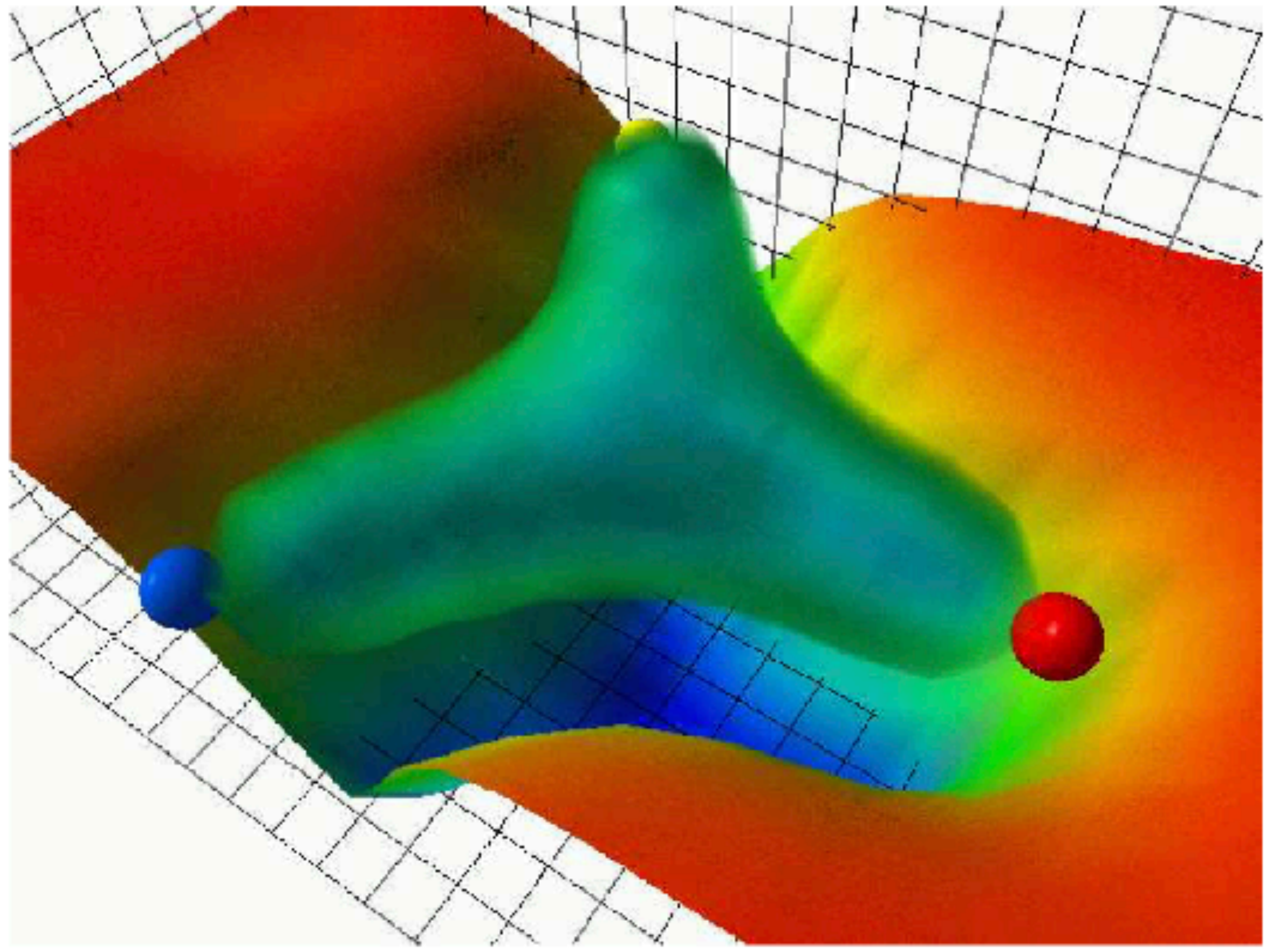


Three static quarks (*QQQ*) at the corners of a triangle lattice



Gauge field energy distribution

hep-lat/0501004



Gluon entanglement entropy

Gluons in proton are entangled with valence quarks: how to measure their entanglement entropy?

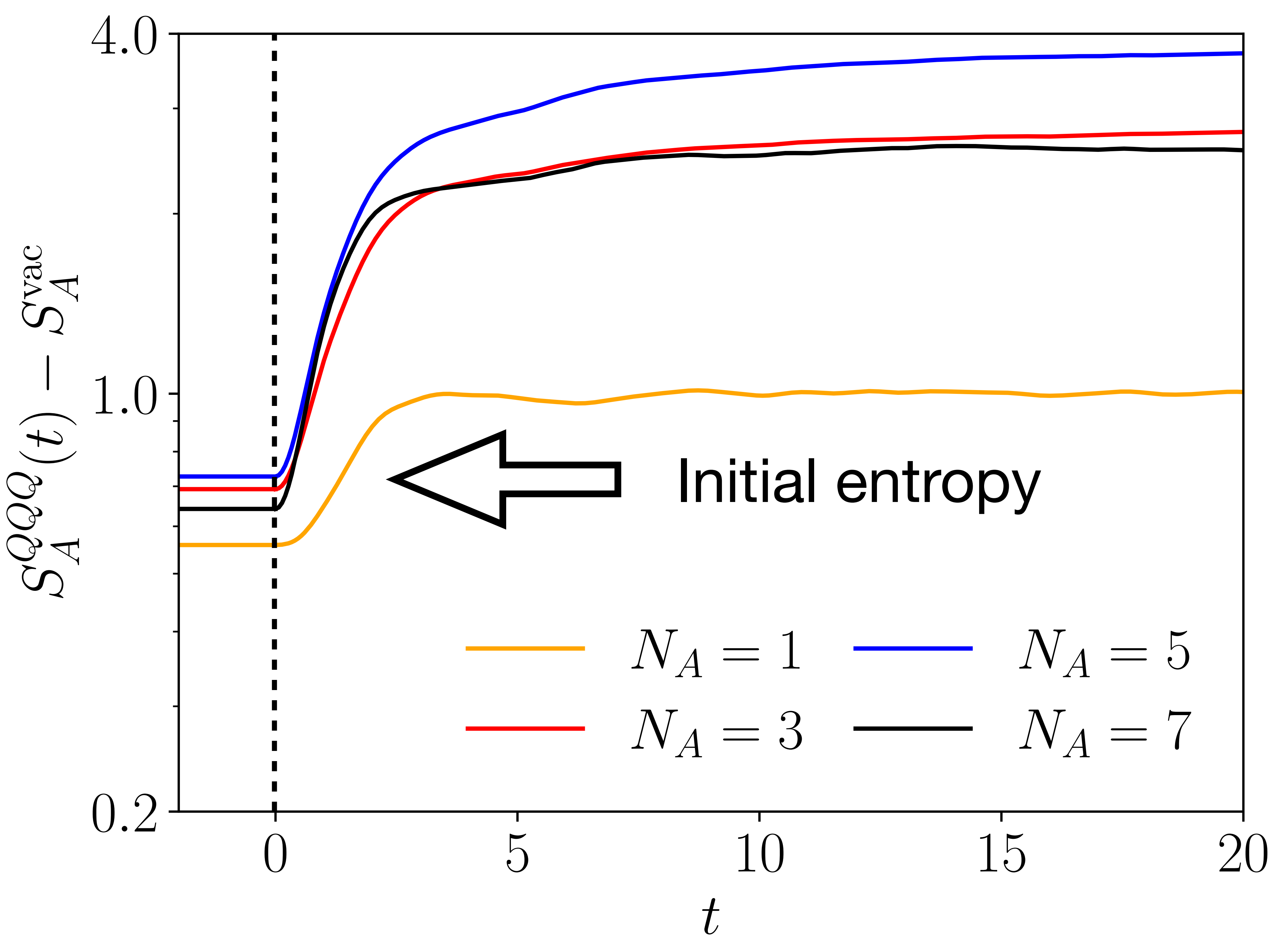
Kharzeev & Levin [1702.03489]:

$$S_{\text{glue}} = -\ln[xG(x)]$$

= density of partons resolved in deep inelastic scattering

Problem: Deep inelastic scattering destroys the proton state - to which extent does this reflect the proton or some highly excited state?

Model calculation: Suddenly remove all three quarks and calculate the entanglement entropy of regions of gauge field in the highly excited state [Horn, BM, Yao, 2605.11171



Summary

- First studies of thermalization behavior on small SU(2) lattices are encouraging
- Strong evidence for quantum chaos
- Evidence for two-step thermalization for local observables
- First results for SU(3) gauge theory on small lattices
- First studies of confining $Q\overline{Q}$ and QQQ potentials and string melting
- Proton toy model with static quarks

- Future plans: Higher SU(3) representations; tensor network calculations
- In process: Thermalization on IBM quantum computer; glasma formation
- 3-D lattices still look very hard; will require different QC architectures and better error correction at hardware level