Higher-Order QED Calculations in KKMC

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Motivation: why higher-order QED in KKMC?

- This talk outlines the work we plan to pursue and complete within a one- to two-year timeframe.
- KKMC simulates

$$e^+e^- \to f\bar{f} + n\gamma, \qquad f = \mu, \tau, q, \nu$$

  with exclusive multi-photon radiative corrections.
- Precision observables at LEP, Belle II, FCC-ee, and radiative-return studies require control of:
  - soft and collinear photon logarithms
    (even better: amplitudes parts/terms leading to them),
  - ISR, FSR and ISR–FSR interference,
  - spin correlations, especially for $\tau^+\tau^-$,
  - realistic event selections: exact multi-particle phase space required.
- KKMC achieves this through YFS exclusive exponentiation, implemented at amplitude level in CEEX.

Precision targets: LEP1, LEP2, KKMC, FCC-ee

- LEP1/2-era fermion-pair predictions required roughly

$$0.2\% - 1\%$$

  precision, depending on event selection.

- KKMC with $\mathcal{O}(\alpha^2)$ CEEX was aimed at about

$$0.1\% \text{ technical precision,}$$

  and estimated physical precision

$$\delta\sigma/\sigma \simeq 0.2\%, \qquad \delta A_{\rm FB} \simeq 0.1\%.$$

- FCC-ee pushes QED effects to a much harder level. For some QED-sensitive observables, the relevant perturbative precision scale is

$$0.001\%.$$

What is needed for FCC-ee

Second-order CEEX is good for LEP2-level precision. FCC-ee needs third-order QED and partly fourth-order logarithmic information, eikonal parts even more.

Which perturbative orders are needed?

Define the large logarithm

$$L_f = \ln \frac{s}{m_f^2}.$$

- For a relative precision of about

  $$6 \times 10^{-3},$$

  one needs

  $$\mathcal{O}(\alpha L_f), \quad \mathcal{O}(\alpha), \quad \mathcal{O}(\alpha^2 L_f^2).$$

- For the next precision level,

  $$3 \times 10^{-4},$$

  one must add

  $$\mathcal{O}(\alpha^3 L_f^3), \qquad \mathcal{O}(\alpha^2 L_f).$$

- For FCC-ee-like precision,

  $$10^{-5},$$

  one also needs

  $$\mathcal{O}(\alpha^2), \qquad \mathcal{O}(\alpha^3 L_f^2), \qquad \mathcal{O}(\alpha^4 L_f^4).$$

Production × decay: starting point (cross section)

We start from the full production+decay cross section (including ISR/FSR photons):

$$\sigma \;=\; \frac{1}{F_{\rm ini}} \int d\Phi \, |\mathcal{M}|^2 \;=\; \frac{1}{F_{\rm ini}} \int d\Phi \, \mathcal{M} \, \mathcal{M}^\dagger. \tag{1}$$

- $F_{\rm ini}$: initial-state flux factor.
- $d\Phi$: Lorentz-invariant phase space for the full final state (decay products of $\tau^\pm$ and possible ISR/FSR photons, jets, ...).
- $\mathcal{M}$: complete matrix element for production and both decays.

The amplitude factorizes into production × propagators × decays, with spin (helicity) indices kept explicit.

- Extra complications related to: pair emissions and virtual counterparts.

"Virtual" for emissions: first order

Separate the one-loop amplitude into a universal IR part and an IR-finite part:

$$\mathcal{M}_0^1 = \alpha B \, \mathcal{M}_0^0 + M_0^1.$$

- $\mathcal{M}_0^1$: full one-loop amplitude.
- $\alpha B \, \mathcal{M}_0^0$: universal soft-virtual IR part.
- M_0^1: IR-finite hard virtual remainder.

Soft limit

When the virtual photon becomes soft, its contribution factorizes from the hard process:

$$\mathcal{M}^1_{\mathrm{soft}} \sim \alpha B \, \mathcal{M}_{\mathrm{Born}}.$$

Virtual soft factor B

For charged external lines i, j, the YFS virtual function is

$$B_{ij} = -\frac{i}{8\pi^3} Z_i Z_j \theta_i \theta_j \int \frac{d^4 k}{k^2 - \lambda^2} \left(\frac{2p_i\theta_i - k}{k^2 - 2k \cdot p_i\, \theta_i} + \frac{2p_j\theta_j + k}{k^2 + 2k \cdot p_j\, \theta_j} \right)^2 .$$

$$B = \sum_{i<j} B_{ij}.$$

- Z_i: charge of particle i.
- $\theta_i = +1$ for outgoing particles, $\theta_i = -1$ for incoming particles.
- λ: photon-mass IR regulator.

Repeated virtual soft insertions

The same soft factor appears at every order:

$$\mathcal{M}_0^0 = M_0^0,$$

$$\mathcal{M}_0^1 = M_0^1 + \alpha B\, M_0^0,$$

$$\mathcal{M}_0^2 = M_0^2 + \alpha B\, M_0^1 + \frac{(\alpha B)^2}{2!}\, M_0^0.$$

In general,

$$\mathcal{M}_0^n = \sum_{r=0}^{n} M_0^{n-r} \frac{(\alpha B)^r}{r!}.$$

Reason

Soft virtual photons are universal and factorize from the hard amplitude.

Virtual exponentiation

Summing over any number of virtual soft photons gives

$$\sum_{r=0}^{\infty} \frac{(\alpha B)^r}{r!} = e^{\alpha B}.$$

Therefore the virtual-soft-dressed amplitude is

$$\mathcal{M}_{\mathrm{virt}} = e^{\alpha B} \left[M_0^0 + M_0^1 + M_0^2 + \cdots \right]_{\mathrm{finite}}.$$

At cross-section level,

$$\left| e^{\alpha B} \right|^2 = e^{\alpha B + \alpha B^*} = e^{2\alpha \, \mathrm{Re}\, B}.$$

Virtual contribution to YFS

$$\boxed{\text{virtual soft factor} = e^{2\alpha \, \mathrm{Re}\, B}}$$

Including real unresolved photons

Unresolved real soft photons also exponentiate:

$$\sum_{n=0}^{\infty} \frac{1}{n!} \left[2\alpha \widetilde{B}(\Omega)\right]^n = e^{2\alpha \widetilde{B}(\Omega)}.$$

Combining real and virtual soft parts:

$$e^{2\alpha \operatorname{Re} B} e^{2\alpha \widetilde{B}(\Omega)} = e^{Y(\Omega)}.$$

Hence

$$Y(\Omega) = 2\alpha \operatorname{Re} B + 2\alpha \widetilde{B}(\Omega).$$

Important

$e^{Y(\Omega)}$ contains only the universal soft real+virtual contribution. Hard-photon effects remain in the finite residuals $\widetilde{\beta}_i$ or $\hat{\beta}_i$.

What are the logarithmic terms?

In QED corrections, large logarithms appear because the photon can be soft and/or collinear to a charged fermion.

$$L_f = \ln \frac{s}{m_f^2}$$

For ISR from the electron line,

$$L_e = \ln \frac{s}{m_e^2}.$$

Since m_e is very small, L_e can be large.

Soft logarithm

A soft photon gives an energy integral of the form

$$\int_{m_\gamma}^{E_{\max}} \frac{dk^0}{k^0} = \ln \frac{E_{\max}}{m_\gamma}.$$

- m_γ: photon-mass IR regulator.
- $E_{\max}$: soft-photon energy boundary.
- This logarithm is infrared divergent as $m_\gamma \to 0$.

$$\boxed{\text{soft log} \sim \ln m_\gamma}$$

YFS cancellation

The soft IR logarithm cancels between virtual B and real $\widetilde{B}$.

Collinear logarithm

A photon emitted nearly parallel to a charged fermion gives an angular integral:

$$\int_{m_f^2/s}^{1} \frac{d\theta^2}{\theta^2} \simeq \ln \frac{s}{m_f^2} = L_f\,.$$

- The fermion mass m_f regulates the collinear divergence.
- For electrons, m_e is very small, so L_e is large.

$$\boxed{\text{collinear log} = L_f = \ln \frac{s}{m_f^2}}$$

Soft-collinear logarithm

When the photon is both soft and collinear, the phase-space integral contains both enhancements:

$$\int \frac{dk^0}{k^0} \int \frac{d\theta^2}{\theta^2}.$$

This gives a double-logarithmic structure:

$$\ln \frac{E_{\max}}{E_{\min}} \ln \frac{s}{m_f^2}.$$

In perturbation theory

One photon can produce

$$\mathcal{O}(\alpha L).$$

Two photons can produce

$$\mathcal{O}(\alpha^2 L^2).$$

Three photons can produce

$$\mathcal{O}(\alpha^3 L^3).$$

Leading and subleading logarithms

At order α^n, the highest power of L is called the leading logarithm:

$$\mathcal{O}(\alpha^n L^n).$$

Examples:

$$\mathcal{O}(\alpha L) \quad \text{leading log at first order,}$$

$$\mathcal{O}(\alpha^2 L^2) \quad \text{leading log at second order,}$$

$$\mathcal{O}(\alpha^3 L^3) \quad \text{leading log at third order.}$$

Subleading examples:

$$\mathcal{O}(\alpha^2 L), \qquad \mathcal{O}(\alpha^3 L^2), \qquad \mathcal{O}(\alpha^2 L^0).$$

YFS idea: reorganize the QED series

In ordinary perturbation theory, real and virtual soft photons are separately infrared divergent.

Yennie–Frautschi–Suura reorganize the series as

$$d\sigma = e^{Y(\Omega)} \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{i=1}^{n} \left[d\Phi_{\gamma_i} \, \widetilde{S}(k_i) \right] \, d\Phi_f \, \left[\widetilde{\beta}_0 + \sum_i \frac{\widetilde{\beta}_1(k_i)}{\widetilde{S}(k_i)} + \sum_{i<j} \frac{\widetilde{\beta}_2(k_i, k_j)}{\widetilde{S}(k_i)\widetilde{S}(k_j)} + \cdots \right]$$

- e^Y contains universal virtual and unresolved-real soft effects.
- $\widetilde{S}(k)$ is the real soft eikonal factor.
- $\widetilde{\beta}_n$ are infrared-finite residuals.

Real soft factor

For emission of a soft photon with momentum k,

$$\widetilde{S}(k) = \sum_{i,j} \frac{\alpha}{4\pi^2} Z_i Z_j \theta_i \theta_j \left(\frac{p_i}{p_i \cdot k} - \frac{p_j}{p_j \cdot k} \right)^2 .$$

- Z_i is the charge of external line i.
- $\theta_i = +1$ for outgoing charged particles and -1 for incoming charged particles.
- The singular soft behavior is universal.

Matrix Element Factorization

The complicated matrix element is separated into a universal singular part and a finite residual part.

EEX: exclusive exponentiation at cross-section level

EEX works with spin-summed differential distributions:

$$d\sigma_{\mathrm{EEX}} = \sum_{n=0}^{\infty} \int d\Phi_{n+2}\, e^{Y}\, \bar{\rho}_n(k_1, \ldots, k_n),$$

where

$$\bar{\rho}_n = \prod_{i=1}^{n} \widetilde{S}(k_i)\, \bar{\beta}_0 + \sum_j \prod_{i \neq j} \widetilde{S}(k_i)\, \bar{\beta}_1(k_j) + \cdots .$$

- Exponentiation is done after squaring and spin summing.
- Very successful for KORALZ/BHLUMI.
- Less natural for ISR–FSR interference and spin-coherent effects.

CEEX: coherent exclusive exponentiation

CEEX performs exponentiation at the spin-amplitude level:

$$d\sigma_{\rm CEEX} = \sum_{n=0}^{\infty} \int d\Phi_{n+2} \sum_{\lambda,\sigma_1,\ldots,\sigma_n} \left| e^{\alpha B}\, \mathcal{M}^{\lambda}_{n,\sigma_1\ldots\sigma_n}(k_1,\ldots,k_n)\right|^2 .$$

For example,

$$\mathcal{M}^{\lambda}_{0} = \hat{\beta}^{\lambda}_{0},$$

$$\mathcal{M}^{\lambda}_{1,\sigma}(k) = \hat{\beta}^{\lambda}_{0} s_{\sigma}(k) + \hat{\beta}^{\lambda}_{1,\sigma}(k),$$

$$\mathcal{M}^{\lambda}_{2,\sigma_1\sigma_2} = \hat{\beta}^{\lambda}_{0} s_{\sigma_1}(k_1) s_{\sigma_2}(k_2) + \hat{\beta}^{\lambda}_{1,\sigma_1}(k_1) s_{\sigma_2}(k_2) + \hat{\beta}^{\lambda}_{1,\sigma_2}(k_2) s_{\sigma_1}(k_1) + \hat{\beta}^{\lambda}_{2,\sigma_1\sigma_2}.$$

Frame orientation: KKMCee (GPS/KS choice)

KKMCee::CEEX deals in calculating $\mathcal{M}^{prod}$. KKMCee adopts a frame convention optimized for Kleiss–Stirling (KS/GPS) spinors:

- Each particle spinor is constructed using a **universal auxiliary vector**, e.g. $\zeta = (1, 1, 0, 0)$; and use auxiliary vector η to fix the phase of the spinors.

Why auxiliary vectors?

- In Monte Carlo amplitude-level constructions (CEEX/YFS), **relative complex phases** must be consistent across diagrams and photon multiplicities.
- Cross sections are phase-invariant, but **interference terms and full spin density matrices** are not.

KS/GPS strategy

Fix *all* relative helicity phases by one global choice of auxiliary vectors (ζ, η):

$$\zeta^2 = 0, \qquad \eta^2 = -1, \qquad \zeta \cdot \eta = 0.$$

Second order: what is already achieved?

At $\mathcal{O}(\alpha^2)$ CEEX, KKMC includes:

- soft-photon exponentiation to all orders,
- exact two-hard-photon matrix element contributions,
- ISR and FSR radiation,
- ISR–FSR interference,
- spin-amplitude-level treatment,
- spin correlations in τ production and decay.

Typical LEP2-level precision estimate:

$$\delta\sigma/\sigma \sim 0.2\%, \qquad \delta A_{\rm FB} \sim 0.1\%.$$

Conclusion

Second order is already a precision framework, but not enough for FCC-ee targets.

Third order: what improvement is expected?

The leading third-order contribution is

$$\mathcal{O}(\alpha^3 L^3).$$

- Together with $\mathcal{O}(\alpha^2 L)$, it is needed for the

  $$3 \times 10^{-4}$$

  precision level.
- It reduces the dominant missing leading-logarithmic ISR uncertainty.
- It is already present in auxiliary EEX form in KKMC, but the desired future direction is CEEX-level implementation.

For FCC-ee

Third-order leading logs alone are not enough. One also needs subleading third-order terms, fourth-order leading logs, and improved non-logarithmic second-order pieces.

Current development directions

- Revive and extend physics tests of KKMCee.
- Understand and document Kleiss–Stirling/GPS spin-amplitude implementation.
- Improve electroweak form-factor treatment and weak corrections. Challenge: matching in amplitudes: two loop electroweak, third order QED.
- Improve the code documentation.
- Work toward third-order real-emission bremsstrahlung amplitudes.
- Develop the semi-analytic calculation for third order.
- Cross-check with other generators, for example Phokhara and Sherpa-YFS developments.

Long-term goal

A flexible precision framework for, Belle II and FCC-ee applications.

Summary

- YFS reorganizes QED perturbation theory so that soft real and virtual singularities cancel to all orders.
- EEX implements this at the cross-section level.
- CEEX implements it at the amplitude level and is therefore better for interference, spin and narrow-resonance effects.
- KKMC implements $\mathcal{O}(\alpha^2)$ CEEX and auxiliary $\mathcal{O}(\alpha^3 L^3)$ EEX.
- Second order gives LEP2-level precision, roughly 0.2% for cross sections.
- FCC-ee pushes the target toward 10^{-5} for the most sensitive QED components.
- Third order is necessary, but FCC-ee also requires subleading and non-logarithmic higher-order improvements.
- A semi-analytical tool needs to be developed to match the required perturbative order.

References

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- S. Jadach, B. F. L. Ward and Z. Was, *Coherent Exclusive Exponentiation for Precision Monte Carlo Calculations*, Phys. Rev. D **63** (2001) 113009, arXiv:hep-ph/0006359.
- S. Jadach and M. Skrzypek, *QED challenges at FCC-ee precision measurements?*, arXiv:1903.09895.

Conferences and Talks

- **Z. Was**: *Simulation of tau decays, ambiguities and anomalous couplings effects.*
  Tau 2025 — Marseille [proceedings] [slides]
- **A. Korchin**: *Spin effects in the tau-lepton pair induced by anomalous magnetic and electric dipole moments*
  Matter To The Deepest 2025 — Katowice [proceedings] [slides]
- **Z. Was**: *FCC precision requests: challenges for Monte Carlos and phenomenology tools*
  RADCOR 2025 — Puri [proceedings] [slides]
- **A. Tapadar**: *Phase space for tests and applications of photos Monte Carlo*
  RADCOR 2025 — Puri [proceedings] [slides]
- **J. M. John.**: *KKMC reweighing method*
  2nd IFJ PAN - IJCLab Heavy Flavour workshop 2025 — Paris [proceedings] [slides]
- **A. Tapadar.**: *Investigating β Angle Distributions in 3-Pion τ Decays*
  2nd IFJ PAN - IJCLab Heavy Flavour workshop 2025 — Paris [proceedings] [slides]
- **J. M. John. and A. Tapadar**: *β angle asymmetry in 3 pion τ decays*
  Online Paris — Online, 2025
- **Z. W., A. T., J. M. J., S. Banerjee. (University Louisvile USA), Hernandez. V. Michel (BNL USA)**: *Online learning meetings*
  Internal meetings — 2025-2026

Publications

- *Simulation of tau decays, ambiguities and anomalous couplings, Z. Was, A. Tapadar, J. M. John, S. Banerjee* (arXiv:2512.23475)
- *On τ spin use with KKMCee, J. M. John, A. Tapadar, Z. Was* (arXiv:2509.04400)
- Investigating β Angle Distributions in 3-Pion τ Decays (Paper with Orsay people) (author list pending)