QCD
Calculation of the winding number

Winding number of the SU(2) gauge transformation U is defined as

$$N_{\mathrm{w}} = \frac{1}{24\pi^2}\varepsilon^{ijk}\int d^3r \; \mathrm{Tr}\left[\left(U^\dagger\partial_i U\right)\left(U^\dagger\partial_j U\right)\left(U^\dagger\partial_k U\right)\right]. \tag{1}$$

Calculate (1) for $U = \exp\left(i\,\vec{n}\cdot\vec{\tau}\,P(r)\right)$ where $\vec{n} = \vec{r}/r$. What are the boundary conditions for $P(r)$ that ensure that N_{w} is an integer?

SOLUTION:

First decompose

$$U^\dagger\partial_i U = \frac{i}{2}\sum_{a=1}^{3}\xi_i^a\tau_a$$

We have

$$\begin{aligned} T &= \varepsilon^{ijk}\,\mathrm{Tr}\left[\left(U^\dagger\partial_i U\right)\left(U^\dagger\partial_j U\right)\left(U^\dagger\partial_k U\right)\right] \\ &= -\frac{i}{8}\varepsilon^{ijk}\underbrace{\mathrm{Tr}\left[\tau^a\tau^b\tau^c\right]}_{2i\varepsilon_{abc}}\xi_i^a\xi_j^b\xi_k^c \\ &= \frac{1}{4}\varepsilon^{ijk}\varepsilon_{abc}\,\xi_i^a\xi_j^b\xi_k^c. \end{aligned}$$

Due to the symmetry of U, elements of matrix ξ can be decomposed in the following way:

$$\xi_i^a = A\delta_{ia} + Bn_in_a + C\varepsilon_{iam}n_m.$$

Expression for T has therefore 27 terms

$$\begin{aligned}
T &= \frac{1}{4}\varepsilon^{ijk}\varepsilon_{abc}\left\{A^3\delta_{ia}\delta_{jb}\delta_{kc}\right.\\
&+ A^2B\left(\delta_{ia}\delta_{jb}n_kn_c + 2\text{ permutations}\right)\\
&+ AB^2\left(\delta_{ia}n_jn_bn_kn_c + 2\text{ permutations}\right) \leftarrow \text{ zero from asymm.}\\
&+ B^3\left(n_in_an_jn_bn_kn_c\right) \leftarrow \text{ zero from asymm.}\\
&+ A^2C\left(\delta_{ia}\delta_{jb}\varepsilon_{kcm}n_m + 2\text{ permutations}\right) \leftarrow \text{ zero from } \delta_{kc}\\
&+ AC^2\left(\delta_{ia}\varepsilon_{jbl}n_l\varepsilon_{kcm}n_m + 2\text{ permutations}\right)\\
&+ BC^2\left(n_in_a\varepsilon_{jbl}n_l\varepsilon_{kcm}n_m + 2\text{ permutations}\right)\\
&+ B^2C\left(n_in_an_jn_b\varepsilon_{kcm}n_m + 2\text{ permutations}\right) \leftarrow \text{ zero from asymm.}\\
&+ ABC\left(\delta_{ia}n_jn_b\varepsilon_{kcm}n_m + 5\text{ permutations}\right) \leftarrow \text{ zero } \sim \varepsilon_{nlm}n_nn_ln_m\\
&+ \left.C^3\varepsilon_{ian}n_n\varepsilon_{jbl}n_l\varepsilon_{kcm}n_m\right\} \leftarrow \text{ zero } \sim \varepsilon_{nlm}n_nn_ln_m
\end{aligned}$$

but only 4 (up to permutations) are not zero. We shall use the following identities:

$$\begin{aligned}
\varepsilon^{ijk}\varepsilon_{abc}\delta_{ia} &= \delta_{jb}\delta_{kc} - \delta_{jc}\delta_{kb},\\
\varepsilon^{ijk}\varepsilon_{abc}\delta_{ia}\delta_{jb} &= 2\delta_{kc},\\
\varepsilon^{ijk}\varepsilon_{abc}\delta_{ia}\delta_{jb}\delta_{kc} &= 6.
\end{aligned}$$

Check zero:

$$\begin{aligned}
ABC \quad &: \quad \varepsilon^{ijk}\varepsilon_{ibc}\varepsilon_{kcm}n_jn_bn_m = \left(\delta_{jb}\delta_{kc} - \delta_{jc}\delta_{kb}\right)\varepsilon_{kcm}n_jn_bn_m\\
&= \quad -\varepsilon_{bjm}n_jn_bn_m = 0
\end{aligned}$$

We get

$$\begin{aligned}
T &= \frac{1}{4}\left\{6A^3 + 6A^2B\right.\\
&+ 3AC^2\varepsilon^{ijk}\varepsilon_{abc}\delta_{ia}\varepsilon_{jbl}\varepsilon_{kcm}n_ln_m\\
&+ \left.3\,BC^2\varepsilon^{ijk}\varepsilon_{abc}\varepsilon_{jbl}\varepsilon_{kcm}n_in_ln_an_m\right\}.
\end{aligned}$$

Two expressions are more difficult:

$$AC^2 : (\delta_{jb}\delta_{kc} - \delta_{jc}\delta_{kb})\,\varepsilon_{jbl}\varepsilon_{kcm}n_l n_m = 2n^2 = 2.$$

Next (flipping indices):

$$\begin{aligned} BC^2 &: \quad \varepsilon^{jik}\varepsilon_{jbl}\,\varepsilon_{cab}\varepsilon_{ckm}n_i n_l n_a n_m \\ &= \quad (\delta_{ib}\delta_{kl} - \delta_{il}\delta_{kb})\,(\delta_{ak}\delta_{bm} - \delta_{am}\delta_{bk})\,n_i n_l n_a n_m \\ &= \quad (\delta_{im}\delta_{al} - \delta_{am}\delta_{il} - \delta_{il}\delta_{am} + 3\delta_{am}\delta_{il})\,n_i n_l n_a n_m \\ &= \quad 2. \end{aligned}$$

Finally we get

$$\begin{aligned} T &= \frac{3}{2}\left\{A^3 + A^2B + AC^2 + BC^2\right\} \\ &= \frac{3}{2}\left\{A^2(A+B) + (A+B)C^2\right\} \\ &= \frac{3}{2}(A^2 + C^2)(A+B). \end{aligned}$$

Next we have to calculate A,B and C. To this we expand

$$U = \cos P + i(\boldsymbol{n}\cdot\boldsymbol{\tau})\sin P$$

and use

$$\begin{aligned} \partial_i r &= n_i, \\ \partial_i n_k &= \frac{1}{r}(\delta_{ik} - n_i n_k). \end{aligned}$$

We have

$$\begin{aligned} \partial_i U &= \partial_i \cos P + i(\boldsymbol{n}\cdot\boldsymbol{\tau})\partial_i \sin P + i(\partial_i\boldsymbol{n}\cdot\boldsymbol{\tau})\sin P \\ &= n_i P'\Big(- \sin P + i(\boldsymbol{n}\cdot\boldsymbol{\tau})\cos P\Big) + i\frac{1}{r}\Big(\delta_{ik} - n_i n_k\Big)\tau_k \sin P \\ &= n_i P'\Big(\cos P + i(\boldsymbol{n}\cdot\boldsymbol{\tau})\sin P\Big)\, i(\boldsymbol{n}\cdot\boldsymbol{\tau}) + i\frac{1}{r}\Big(\tau_i - n_i(\boldsymbol{n}\cdot\boldsymbol{\tau})\Big)\sin P \\ &= n_i P' U i(\boldsymbol{n}\cdot\boldsymbol{\tau}) + i\frac{1}{r}\Big(\tau_i - n_i(\boldsymbol{n}\cdot\boldsymbol{\tau})\Big)\sin P. \end{aligned}$$

Next

$$\begin{aligned} U^\dagger\partial_i U &= in_i(\boldsymbol{n}\cdot\boldsymbol{\tau})P^{'} + i\frac{1}{r}\Big(\cos P - i(\boldsymbol{n}\cdot\boldsymbol{\tau})\sin P\Big)\Big(\tau_i - n_i(\boldsymbol{n}\cdot\boldsymbol{\tau})\Big)\sin P \\ &= in_i(\boldsymbol{n}\cdot\boldsymbol{\tau})P^{'} \\ &\quad + i\frac{\sin P}{r}\Big(\tau_i\cos P - n_i(\boldsymbol{n}\cdot\boldsymbol{\tau})\cos P - i(\boldsymbol{n}\cdot\boldsymbol{\tau})\tau_i\sin P + in_i\sin P\Big). \end{aligned}$$

Note that

$$\mathrm{Tr}\left(U^\dagger\partial_i U\tau_a\right) = \frac{i}{2}\xi_i^b\,\mathrm{Tr}\left(\tau_b\tau_a\right) = i\xi_i^a.$$

We use

$$\begin{aligned} \mathrm{Tr}\left(\tau_k\tau_a\right) &= 2\delta_{ka}, \\ \mathrm{Tr}\left(\tau_m\tau_i\tau_a\right) &= 2i\varepsilon_{mia} = 2i\varepsilon_{iam}. \end{aligned}$$

Hence

$$\xi_i^a = 2P'n_in_a + 2\frac{\sin P\cos P}{r}\delta_{ia} - 2\frac{\sin P\cos P}{r}n_in_a + 2\frac{\sin^2 P}{r}\varepsilon_{iam}n_m$$

which gives

$$A = 2\frac{\sin P\cos P}{r},\ B = 2\left(P' - \frac{\sin P\cos P}{r}\right),\ C = 2\frac{\sin^2 P}{r}.$$

Substituting this to T we get

$$\begin{aligned} T &= \frac{3}{2}(A+B)(A^2+C^2) \\ &= \frac{3}{2}2P'\,4\frac{\sin^2 P}{r^2}\left(\cos^2 P + \sin^2 P\right) \\ &= 12\,P'\frac{\sin^2 P}{r^2}. \end{aligned}$$

Finally the winding number is equal to:

$$\begin{aligned} N_{\mathrm{w}} &= \frac{1}{2\pi^2}4\pi\int\limits_0^\infty drr^2P'\frac{\sin^2 P}{r^2} \\ &= \frac{2}{\pi}\int\limits_{P(0)}^{P(\infty)} dP\sin^2 P \\ &= \frac{1}{\pi}\left(P - \frac{1}{2}\sin(2P)\right)\Bigg|_{P(0)}^{P(\infty)}. \end{aligned}$$

For boundary conditions $P(0) = n_0\pi$ and $P(\infty) = n_\infty\pi$ we have.

$$N_W = n_\infty - n_0.$$

Note that in order to nullify $\sin(2P)$ entering N_w above, we could in principle choose $n_{0,\infty}$ half-integer. However, in this case $\sin P \neq 0$, and the U matrix would have an angle dependent (due to $i(\boldsymbol{n}\cdot\boldsymbol{\tau})\sin P$) limit, that is not a unique limit at the boundaries.

Typical boundary condition is $n_\infty = 0$ and $n_0 = -N_\mathrm{w}$.