QCD lecture 9

December 10

Anomaly of the axial current

Remember that the free lagrangian changes due to the anomaly in the following way

$$\mathcal{L}(x) \to \mathcal{L}(x) + \alpha(x)\mathcal{A}(x)$$

But also the lagrangian changes

.

$$\mathcal{L}_0 = i\,\bar{\psi}\gamma^\mu\partial_\mu\psi \to i\,\left(\psi^\dagger e^{-i\alpha\gamma_5}\right)\gamma^0\,\gamma^\mu\partial_\mu\left(e^{i\alpha\gamma_5}\psi\right)$$

$$= i\,\bar{\psi}e^{i\alpha\gamma_5}\gamma^\mu e^{i\alpha\gamma_5}\left(i\gamma_5\left(\partial_\mu\alpha\right) + \partial_\mu\right)\psi$$

$$= i\,\bar{\psi}\gamma^\mu\partial_\mu\psi - \underbrace{\bar{\psi}\gamma^\mu\gamma_5\psi}_{=J_5^\mu}\,\left(\partial_\mu\alpha\right)$$

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$$\mathcal{L}(x) \to \mathcal{L}(x) + \alpha(x)\mathcal{A}(x) - J_5^\mu(x)\,\partial_\mu\alpha(x)$$

We need to integrate this to get the action, integrate last term by parts and require that the total change of action is zero:

$$\langle \partial_\mu J_5^\mu(x) \rangle_A = \frac{g^2}{16\pi^2}\epsilon_{\mu\nu\rho\sigma}\,F_a^{\mu\nu}(x)\,F_b^{\rho\sigma}(x)\,\mathrm{tr}\,(t^a t^b t)$$

where $\langle\cdot\rangle_A$ is an average over the fermion fields, in a fixed gauge field configuration.

Atiyah-Singer theorem

Dirac matrices: $\gamma^0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \; \gamma^i = \begin{bmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{bmatrix}$

hermitean antihermitean

Dirac operator is neither hermitean not antihermitean. Let's go to Euclidean space

$$x^0 = ix^4 \rightarrow \partial_0 = \frac{\partial}{\partial x^0} = -i\frac{\partial}{\partial x^4} = -i\partial_4 \qquad A^0 = iA^4 \qquad \gamma^0 = i\gamma^4$$

Then:
$$\begin{aligned} \not{D}_x &= \gamma^0\partial_0 + \gamma^k\partial_k - ig\left(A_a^0\gamma^0 - A_a^k\gamma^k\right)t^a \\ &= \gamma^4\partial_4 + \gamma^k\partial_k + ig\left(A_a^4\gamma^4 + A_a^k\gamma^k\right)t^a \\ &= \sum_{j=1}^{4}\left(\partial_j + igA_a^j t^a\right)\gamma^j \end{aligned}$$

is hermitean becuse all gamma matrices are antihermitean.

Atiyah-Singer theorem

Dirac operator in Euclidean space can be therefore diagonalized in an orthonormal basis of eigenfunctions ϕ_k

$$\not{D}_x \phi_k(x) = \lambda_k \phi_k(x) \,,$$
$$\int d^4x_E \; \phi_k^\dagger(x)\phi_{k'}(x) = \delta_{kk'}$$

$$\sum_k \phi_k(x)\phi_k^\dagger(y) = \delta(x-y)$$

Anomaly function for $t = 1$ can be rewritten as:

$$\mathcal{A}(x) = -2 \lim_{y\to x, M\to+\infty} \mathrm{tr}\left\{\gamma^5\, t\, \mathcal{F}\left(-\frac{\not{D}_x^2}{M^2}\right)\right\}\delta(x-y)$$

$$\begin{aligned}
\mathcal{A}(x) &= -2 \lim_{y\to x, M\to+\infty} \mathrm{tr}\left\{\gamma^5\, \mathcal{F}\left(-\frac{\not{D}_x^2}{M^2}\right)\sum_k \phi_k(x)\phi_k^\dagger(y)\right\} \\
&= -2 \lim_{y\to x, M\to+\infty} \sum_k \mathrm{tr}\left\{\phi_k^\dagger(y)\,\gamma^5\, \mathcal{F}\left(-\frac{\not{D}_x^2}{M^2}\right)\phi_k(x)\right\} \\
&= -2 \lim_{M\to+\infty} \sum_k \mathcal{F}\left(-\frac{\lambda_k^2}{M^2}\right)\phi_k^\dagger(x)\gamma^5\phi_k(x) \,.
\end{aligned}$$

Atiyah-Singer theorem

We can connect this result with the previous one, rewritten in Euclidean metric

$$\frac{g^2}{32\pi^2}\int d^4x_E\ \epsilon_{ijkl}\,F^a_{ij}(x)\,F^b_{kl}(x)\,\mathrm{tr}(t^a t^b)$$
$$= -\frac{1}{2}\int d^4x_E\ \mathcal{A}(x) = \lim_{M\to+\infty}\sum_k \mathcal{F}\left(-\frac{\lambda_k^2}{M^2}\right)\int d^4x_E\ \phi_k^\dagger(x)\gamma^5\phi_k(x)$$

We can relate eigenvalues of $\phi_k(x)$ to eigenvalues of $\gamma^5\phi_k(x)$
since $\{\gamma^5, \not{D}\} = 0$

So have $\not{D}_x(\gamma^5\phi_k(x)) = -\lambda_k(\gamma^5\phi_k(x))$ This means that for $\lambda_k \neq 0$
functions $\phi_{k'} \equiv \gamma^5\phi_k$ and ϕ_k are different eigenfunctions of $\not{D}_x$ hence

$$\int d^4x_E\ \phi_k^\dagger(x)\gamma^5\phi_k(x) = \int d^4x_E\,\phi_k^\dagger(x)\phi_{k'}(x) = 0$$

Therefore only eigenfunctions with $\lambda_k = 0$ so called zero modes contribute
to the anomaly.

Atiyah-Singer theorem

Anomaly expressed in terms of the zero modes

$$\frac{g^2}{32\pi^2}\int d^4x_E\ \epsilon_{ijkl}\,F^a_{ij}(x)\,F^b_{kl}(x)\,\mathrm{tr}(t^a t^b) = \sum_{k|\lambda_k=0}\int d^4x_E\ \phi^\dagger_k(x)\gamma^5\phi_k(x)$$

Since $\{\gamma^5, \not{D}_x\} = 0$ zero modes can be chosen to be also eigenstates of γ^5
so called left and right zero modes

$$\not{D}_x\phi_R(x) = 0\,,\quad \gamma^5\phi_R(x) = +\phi_R(x)$$
$$\not{D}_x\phi_L(x) = 0\,,\quad \gamma^5\phi_L(x) = -\phi_L(x)$$

Because zero modes are normalized

$$\frac{g^2}{32\pi^2}\int d^4x_E\ \epsilon_{ijkl}\,F^a_{ij}(x)\,F^b_{kl}(x)\,\mathrm{tr}(t^a t^b) = n_R - n_L$$

where n_R and n_L are numbers of right and left zero modes. The difference is an integer. This formula is called Attiyah-Singer index theorem.
There exist nonperturbative, nontrivial configurations of the gauge field with the above property – instantons.

θ term and strong CP problem

Recall QCD Lagrangian: $$\mathcal{L} = -\frac{1}{2}\mathrm{Tr}\left[\boldsymbol{F}_{\mu\nu}\boldsymbol{F}^{\mu\nu}\right] + \sum_{f=1}^{6}\left[\overline{q}_f\, i\gamma^\mu D_\mu q_f - m_f \overline{q}_f\, q_f\right]$$

In principle we could add a new term that has the same dimension as $\boldsymbol{F}_{\mu\nu}\boldsymbol{F}^{\mu\nu}$

$$\mathcal{L}_\theta \equiv \frac{g^2\theta}{32\pi^2}\,\epsilon_{\mu\nu\rho\sigma}\ \mathrm{tr}\left(F^{\mu\nu}\,F^{\rho\sigma}\right) = \frac{g^2\theta}{32\pi^2}\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F^a_{\mu\nu}F^a_{\rho\sigma}$$

called the θ-term. This is precisely the anomaly multiplied by a dimesionless coupling constant θ. This term, however, can be expressed as a total derivative and therefore does not contribute to the equations of motion:

$$\partial_\mu K^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\,F^a_{\mu\nu}\,F^a_{\rho\sigma}$$

where (exercise)

$$K^\mu \equiv \epsilon^{\mu\nu\rho\sigma}\left[A^a_\nu F^a_{\rho\sigma} - \frac{g}{3}\,f^{abc}\,A^a_\nu A^b_\rho A^c_\sigma\right]$$

or

$$K^\mu \equiv 2\epsilon^{\mu\nu\rho\sigma}\ \mathrm{tr}\left[A_\nu F_{\rho\sigma} + \frac{2ig}{3}A_\nu A_\rho A_\sigma\right]$$

and $\mathcal{L}_\theta = \frac{g^2\theta}{32\pi^2}\,\partial_\mu K^\mu$ This term is Lorentz and gauge invariant but violates CP.

θ term and strong CP problem

θ term is related to the neutron electric dipole moment: $|\theta| \lesssim 10^{-10}$

Why is it so small? One would naturally expect θ ~ 1. This is called a strong CP problem.

Relation to the quark masses

$$\psi_f \longrightarrow e^{i\gamma_5 \alpha_f} \psi_f$$

This transformation is anomalous

$$[D\psi D\overline{\psi}] \longrightarrow \exp\left(-\frac{i}{32\pi^2}\int d^4x\, \epsilon^{\mu\nu\rho\sigma} F^a_{\mu\nu} F^a_{\rho\sigma} \sum_f \alpha_f\right) [D\psi D\overline{\psi}]$$

The same effect would be caused by a change of coupling θ
(if we add theta term to the lagrangian)

$$\theta \to \theta - 2\sum_f \alpha_f$$

θ term and strong CP problem

Let's allow for the complex fermion masses: this would violate P and CP

$$\sum_f M_f \, \overline{\psi}_f \frac{1+\gamma_5}{2} \psi_f + \sum_f M_f^* \, \overline{\psi}_f \frac{1-\gamma_5}{2} \psi_f$$

Transformation $\psi_f \longrightarrow e^{i\gamma_5 \alpha_f} \psi_f$ results in (exercise)

$$\sum_f e^{2i\alpha_f} M_f \, \overline{\psi}_f \frac{1+\gamma_5}{2} \psi_f + \sum_f e^{-2i\alpha_f} M_f^* \, \overline{\psi}_f \frac{1-\gamma_5}{2} \psi_f$$

which is equivalent to $M_f \to e^{2i\alpha_f} M_f$

Since any change of θ can be undone by a chiral transformation of quarks,
physical quantities cannot depend separately on θ and M_f but on the combination:

$$e^{i\theta} \prod_f M_f$$

which is invariant. So θ term would have no efect if at least one quark mass were zero.

Possible solution to the CP problem – axion (not discussed here)

Topology of gauge fields

Since $\mathcal{L}_\theta$ is a full derivative, we can apply Stokes' theorem to calculate the action

$$\int d^4x_E \, \mathcal{L}_\theta = \frac{g^2\theta}{32\pi^2}\int d^4x_E \, \partial_\mu K^\mu = \frac{g^2\theta}{32\pi^2}\lim_{R\to\infty}\int_{S_{3,R}} dS_\mu \, K^\mu$$

3-dim sphere of radius R

Recall non-Abelian gauge transformation (now we use Ω rather than U):

$$A_\mu(x) \quad \to \quad A_\mu^\Omega(x) \equiv \Omega^{-1}(x)\, A_\mu(x)\, \Omega(x) + \frac{i}{g}\,\Omega^{-1}(x)\big(\partial_\mu\Omega(x)\big)$$

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pure gauge

If all color sources are placed in a finite region of space time, we can assume that the gauge fields on the 3-sphere are pure gauge plus a small correction:

$$A_\mu(x) = a_\mu(x) + \frac{i}{g}\,\Omega^\dagger(\widehat{x})\,\partial_\mu\,\Omega(\widehat{x})$$

and matrix $\Omega(\widehat{x})$ depends only on the direction of x^μ Since $\frac{\partial}{\partial x^\mu} = \frac{1}{|x|}\frac{\partial}{\partial \hat{x}^\mu}$

$$A_\mu \to \frac{1}{|x|} \text{ for } |x| \to \infty$$

Topology of gauge fields

If $$A_\mu \to \frac{1}{|x|} \text{ for } |x| \to \infty$$

then $$K^\mu \equiv 2\epsilon^{\mu\nu\rho\sigma}\,\mathrm{tr}\left[A_\nu F_{\rho\sigma} + \frac{2ig}{3} A_\nu A_\rho A_\sigma\right] \underset{|x|\to+\infty}{\longrightarrow} \frac{4ig}{3}\epsilon^{\mu\nu\rho\sigma}\,\mathrm{tr}\left(A_\nu A_\rho A_\sigma\right) \sim |x|^{-3}$$

One can show that $F_{\rho\sigma}(x)$ for pure gauge goes to 0 faster than $|x|^{-2}$

Therefore $$\int d^4x_E\,\mathcal{L}_\theta = \frac{\theta}{24\pi^2}\lim_{R\to\infty}\int_{\mathcal{S}_{3,R}} dS\,\widehat{x}_\mu\,\epsilon^{\mu\nu\rho\sigma} \times \mathrm{tr}\left(\Omega^\dagger(\partial_\nu\Omega)\Omega^\dagger(\partial_\rho\Omega)\Omega^\dagger(\partial_\sigma\Omega)\right)$$

Since $dS \sim R^3$ the integral is finite and we can drop $\lim$ Therefore the integral depends only on $\Omega(\widehat{x})$ - unitary matrix that maps a 3-dim sphere in Euclidean space-time onto the gauge group

$$\Omega : \quad \mathcal{S}_3 \longmapsto \mathcal{G}$$

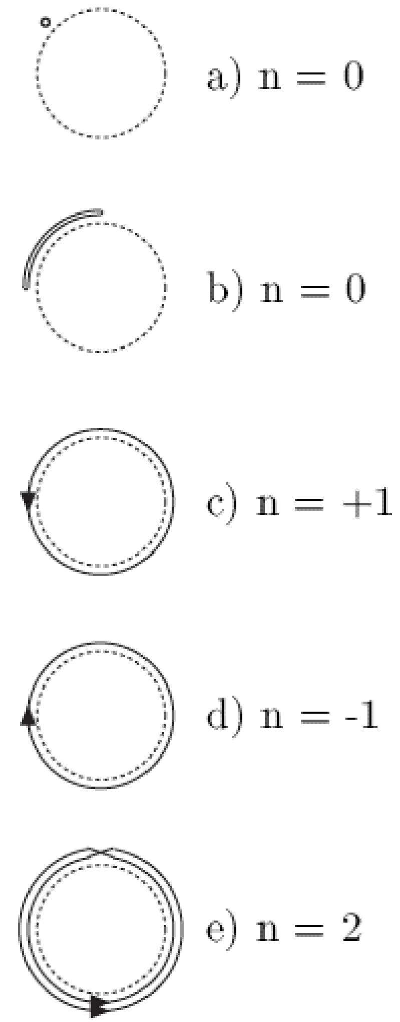
Topology of mappings

Consider baby-model: mapping of
1 dim sphere (circle) onto U(1) group,
which is also a circle.

One can characterize these mappings by
a winding number.

Mappings from one class cannot be deformed
into a mapping of another class. They are called
homotopy classes:

$$\pi_1(\mathrm{U}(1)) = \mathbb{Z}$$



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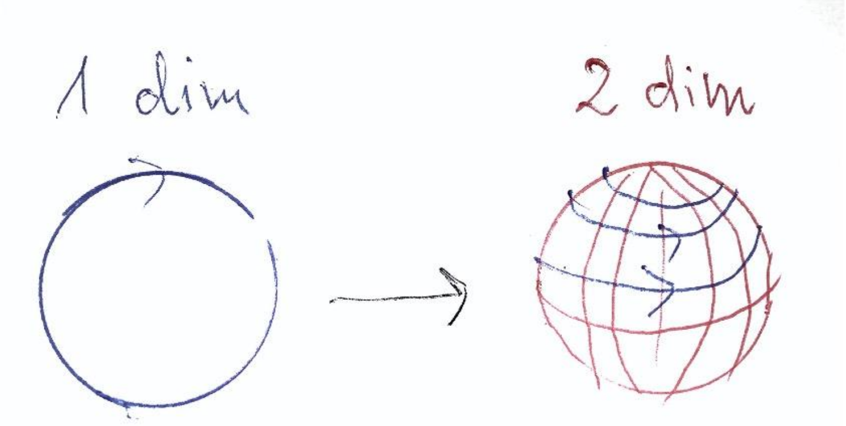
Topology of mappings

Some mappings can be always shrunk
to a point.



Homotopy classes

We have a mapping $S^3_{space} \to SU(3)_{color}$

To discuss topology it is convenent to restrict discussion to an $\mathrm{SU}(2)$ subgroup of $\mathrm{SU}(N)$

For $U \in SU(2)$ we have the following parametrization $U = u_0 + i u_a \tau_a$
with u_α real satisfying $u_0^2 + u_a u_a = 1$ But this is equation of a 3-sphere!

So in practice we have the following mapping

$$S^3_{space} \to S^3_{group}$$

It is known (generalization of our 1 dim example)

$$\pi_d\left(S^d\right) = Z$$

Homotopy classes

Mapping of a 3 dimensional sphere is characterized by homotopy class $\pi_3(\mathcal{G})$

So for SU(*N*) where *N* > 1

$$\pi_3(\mathrm{SU}(N)) = \mathbb{Z}$$

We see now that anomaly, that is an integer

$$\frac{g^2}{32\pi^2}\int d^4x_E\ \epsilon_{ijkl}\, F^a_{ij}(x)\, F^b_{kl}(x)\, \mathrm{tr}(t^a t^b) = n_R - n_L$$

is related to the topology of gauge fields.

Now we can understand notation $\langle \partial_\mu J^\mu_5(x) \rangle_A = \frac{g^2}{16\pi^2}\epsilon_{\mu\nu\rho\sigma}\, F^{\mu\nu}_a(x)\, F^{\rho\sigma}_b(x)\, \mathrm{tr}(t^a t^b t)$;

Instantons - preliminaries

Consider QCD in temporal gauge $A_0^a = 0$ There are still residual time-independent gauge transformations possible that preserve this condition denoted by U
We shall assume that they approach a constant at spacial infinty, chosen to be unity (vacuum):

$$U(\vec{x}) \to 1 \qquad \text{for} \qquad |\vec{x}| \to \infty$$

This means that all points at spacial infity correspond to the same value of U, so we can identify them (squeeze to a point), which means that $R^3{}_{space}$ compactified to a sphere, so that we have a mapping

$$S^3_{space} \to SU(3)_{color}$$

These mappings fall into a distinct tolological classes characterized by an integer n

$$U^{(n)}(x)$$

For pure gauge field

$$A_\mu^{(n)} = -\frac{i}{g} U^{(n)} \partial_\mu U^{(n)\dagger}$$

(field tensor is zero! – exercise) in a given class n we cannot penetrate to another class m within a pure gauge configuration

Instantons - preliminaries

In order to continously deform $A_\mu^{(n)} \to A_\mu^{(m)}$ we have to consider field configurations with nonminimal action $S_E > 0$

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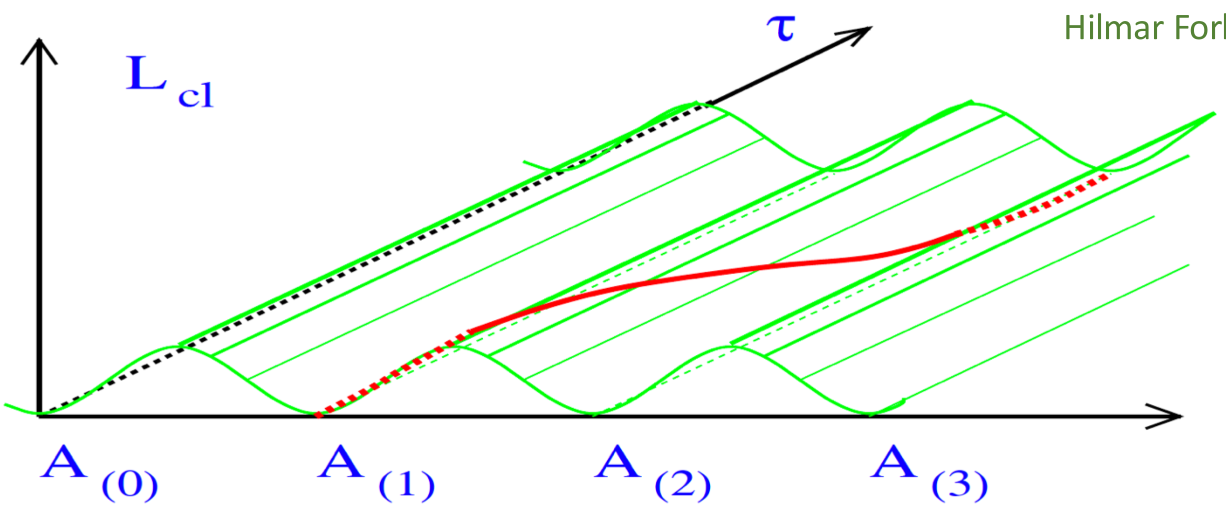

$$n = \frac{1}{24\pi^2}\int d^3\mathbf{x}\, \epsilon^{ijk}\left[(U^\dagger\partial_i U)(U^\dagger\partial_j U)(U^\dagger\partial_k U)\right]$$

Example (hedgehog) $U = \exp[i(r\cdot\tau)/r\,P(r)]$ $P(0) = n\pi,\ \ P(\infty) = 0$

Exercise: calculate n