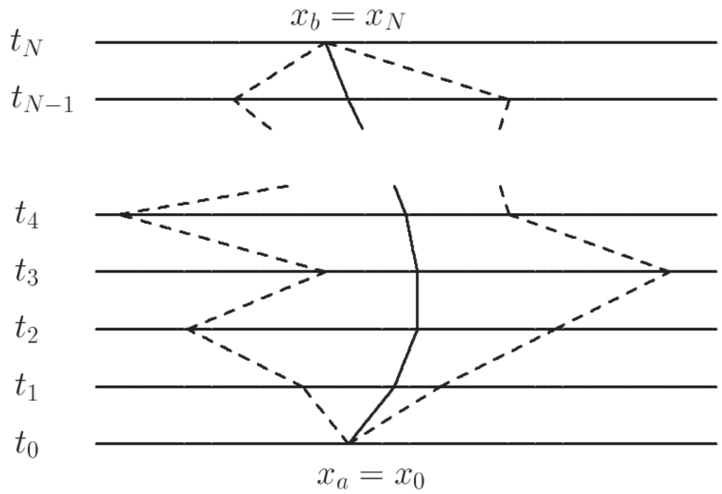
QCD lecture 7

November 26, 2025

Path integral for the propgator

$$K(b,a) = \lim_{\epsilon\to 0} \sqrt{\frac{m}{2i\epsilon\hbar\pi}} \int \prod_{j=1}^{N-1} dx_j \sqrt{\frac{m}{2i\epsilon\hbar\pi}} \prod_{k=0}^{N-1} e^{\frac{i}{\hbar}\epsilon L_k}$$

$$\stackrel{\text{def}}{=} \int [\mathcal{D}x(t)]\, e^{\frac{i}{\hbar}\int_{t_a}^{t_b} dt L[x(t),\dot{x}(t)]}$$



Define functional integration measure integration over all traiectories from a to b

$$[\mathcal{D}x(t)] = dx_1 \dots dx_{N-1} \left(\frac{m}{2i\epsilon\hbar\pi}\right)^{\frac{1}{2}N}$$

and use definition of action

$$\lim_{\epsilon\to 0} \sum_{j=0}^{N-1} \epsilon L_j = \int_{t_a}^{t_b} dt\, L(x(t),\dot{x}(t)) = S[x(t)]$$

to arrive at

$$K(b,a) = \int [\mathcal{D}x(t)]\, e^{\frac{i}{\hbar}S[x(t)]}$$

special role of the classical trajectory
i.e. stationary point of action

Gaussian functional integrals

Assume that path integral is the way we formulate QM (and QFT). All properties and equations are derived from the path integral. In practice we deal with Gaussian functional integrals:

$$L(\dot{x}, x, t) = a(t)\,\dot{x}^2(t) + b(t)\,\dot{x}x + c(t)\,x^2 + d(t)\dot{x} + e(t)\,x + f(t)$$

Propagator:

$$K(x_b, x_a, t_b - t_a) = \int [\mathcal{D}x(t)]\, e^{\frac{i}{\hbar}\int_{t_a}^{t_b} dt L(\dot{x},x,t)}$$

To evaluate K decompose the quantal trajectory into the classicel one $\bar{x}(t)$

$$\delta S[x(t)] = 0 \qquad \text{gives} \qquad \bar{x}(t)$$

and a fluctuation $y(t)$:

$$x(t) = \bar{x}(t) + y(t)\,, \qquad y(t_b) = y(t_a) = 0$$

Since terms linear in y vanish

$$S[\bar{x}(t) + y(t)] = S[\bar{x}(t)] + \tfrac{1}{2}\delta^2 S[y(t)]$$

$$= S[\bar{x}] + \frac{1}{2!}\int_0^T \left[\frac{\partial^2 L}{\partial \dot{x}^2}\dot{y}^2 + 2\frac{\partial^2 L}{\partial x \partial \dot{x}}\dot{y}y + \frac{\partial^2 L}{\partial x^2}y^2\right] dt$$

for convenience $T = t_b - t_a$

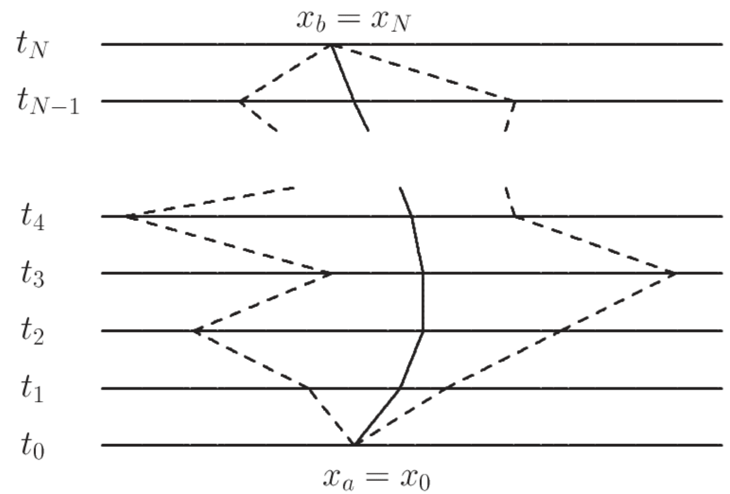
Gaussian functional integrals

Since $\bar{x}(t)$ is fixed we have $\mathcal{D}x(t) = \mathcal{D}y(t)$
and

$$K(x_b, x_a, t_b - t_a) = F(t_b - t_a)\, e^{\frac{i}{\hbar} S[\bar{x}(t)]}$$

where

$$F(t_b - t_a) = \int [\mathcal{D}y(t)]\, e^{\frac{i}{\hbar}\frac{1}{2}\delta^2 S[y(t)]}$$



Recall:

$$\delta^2 S = \int_0^T \left[\dot{y}\frac{\partial^2 L}{\partial \dot{x}^2}\dot{y} + 2y\frac{\partial^2 L}{\partial x \partial \dot{x}}\dot{y} + y\frac{\partial^2 L}{\partial x^2} y \right] dt$$

identities:
(integration by parts)

$$\dot{y}\frac{\partial^2 L}{\partial \dot{x}^2}\dot{y} = \underline{\frac{d}{dt}\left(y\frac{\partial^2 L}{\partial \dot{x}^2}\dot{y}\right)} - y\frac{d}{dt}\left(\frac{\partial^2 L}{\partial \dot{x}^2}\dot{y}\right)$$

$$y(0) = y(T) = 0$$

$$2y\frac{\partial^2 L}{\partial x \partial \dot{x}}\dot{y} = \underline{\frac{d}{dt}\left(\frac{\partial^2 L}{\partial x \partial \dot{x}} y^2\right)} - y\frac{d}{dt}\left(\frac{\partial^2 L}{\partial x \partial \dot{x}}\right) y$$

we get
definition of D

$$\delta^2 S = -\int_0^T dt\, y \left[\frac{d}{dt}\left(\frac{\partial^2 L}{\partial \dot{x}^2}\frac{d}{dt}\right) + \left(\frac{d}{dt}\frac{\partial^2 L}{\partial x \partial \dot{x}}\right) - \frac{\partial^2 L}{\partial^2 x}\right] y = \int_0^T dt\, y D(t) y$$

Gaussian functional integrals

$$\delta^2 S = -\int_0^T dt\, y \left[\frac{d}{dt}\left(\frac{\partial^2 L}{\partial \dot{x}^2}\frac{d}{dt}\right) + \left(\frac{d}{dt}\frac{\partial^2 L}{\partial x \partial \dot{x}}\right) - \frac{\partial^2 L}{\partial^2 x}\right] y = \int_0^T dt\, y D(t) y$$

D is a Sturm-Liouville operator $\quad D(t) y_n(t) = \lambda_n y_n(t)\,, \quad n = 1, 2, 3, \ldots\,, \quad \lambda_1 < \lambda_2 < \ldots$

Example: $\quad L = \frac{1}{2} m \dot{x}^2 - V(x)$

$$D(t) = -m\frac{\partial^2}{\partial t^2} - \frac{\partial^2 V}{\partial x^2}\bigg|_{x=\bar{x}(t)}$$

Use y_n basis to expand $\quad y(t) = \sum_{n=1}^{\infty} a_n\, y_n(t) \quad$ then $\quad \delta^2 S[y] = \sum_{n=1}^{\infty} \lambda_n a_n^2$

and $\quad [\mathcal{D}y(t)] \sim \prod_{n=1}^{\infty} da_n$

$$F(T) \sim \prod_{n=1}^{\infty} da_n \, \exp\left(\frac{i}{2\hbar}\lambda_n a_n^2\right) \sim \sqrt{\frac{1}{\prod_n \lambda_n}} = \sqrt{\frac{1}{\det D(t)}}$$

Path integral revisited

We have performed *dp* integral using a specific form of the hamiltonian

$$H = \frac{p^2}{2m} + V(x)$$

however we do need to use this information. We only have to remember

$$\mathcal{L} = p\dot{q} - H(p,q)$$

Let's recalculate

$$\begin{aligned} \langle x, t+\varepsilon | \, y, t \rangle &= \langle x | \, e^{-iH\varepsilon} \, | y \rangle \\ &= \int \frac{dp}{2\pi} e^{ip(x-y)} e^{-iH\varepsilon} \\ &= \int \frac{dp}{2\pi} \exp i \left[p \frac{(x-y)}{\varepsilon} - H(p,x) \right] \varepsilon \\ &= \int \frac{dp}{2\pi} \exp i \left[p\dot{x} - H(p,x) \right] \varepsilon \end{aligned}$$

Hence:

$$K(b,a) \sim \int \mathcal{D}[x(t)] \int \mathcal{D}[p(t)] \exp\left(\frac{i}{\hbar} \int dt \, \left[p\dot{x} - H(p,x) \right] \right)$$

Transition amplitudes

Consider matrix element of a position operator Q measuring expectation value of the position at time t_1

$$\langle q_f | e^{-i\mathcal{H}(t_f - t_1)} \, Q \, e^{-i\mathcal{H}(t_1 - t_i)} | q_i \rangle$$

We have

$$Q \quad \rightarrow \quad \int dq dq' \, |q\rangle \underbrace{\langle q|Q|q'\rangle}_{q\,\delta(q-q')} \langle q'| = \int dq \; q \; |q\rangle\langle q|$$

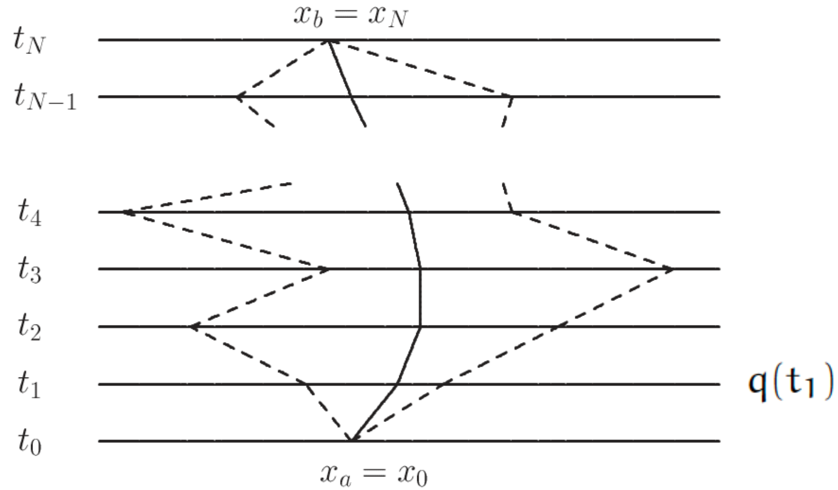
which lead to

$$\langle q_f | e^{-i\mathcal{H}(t_f - t_1)} \, Q \, e^{-i\mathcal{H}(t_1 - t_i)} | q_i \rangle = \int_{\substack{q(t_i)=q_i \\ q(t_f)=q_f}} [Dq(t)] \; q(t_1) \; e^{i\mathcal{S}[q(t)]}$$

Similarly for $t_2 > t_1$

$$\langle q_f | e^{-i\mathcal{H}(t_f - t_2)} \, Q \, e^{-i\mathcal{H}(t_2 - t_1)} \, Q \, e^{-i\mathcal{H}(t_1 - t_i)} | q_i \rangle =$$

$$= \int_{\substack{q(t_i)=q_i \\ q(t_f)=q_f}} [Dq(t)] \; q(t_1) \, q(t_2) \; e^{i\mathcal{S}[q(t)]}$$



Transition amplitudes

F. Gelis: A Stroll Through Field Theory

Define time dependent operator $Q(t) \equiv e^{i\mathcal{H}t}\, Q\, e^{-i\mathcal{H}t}$ and $|q, t\rangle \equiv e^{i\mathcal{H}t}\, |q\rangle$

then

$$\langle q_f, t_f | Q(t_2) Q(t_1) | q_i, t_i \rangle \underset{t_2 > t_1}{=} \int\limits_{\substack{q(t_i)=q_i \\ q(t_f)=q_f}} [Dq(t)]\; q(t_1)\, q(t_2)\; e^{i\mathcal{S}[q(t)]}$$

operators functions

Note that l.h.s is very different when $t_1 > t_2$, whereas r.h.s. is the same because classical trajectories commute. Introduce time ordering T

$$\mathrm{T}(Q(t_1)Q(t_2)) = \theta(t_1 - t_2)Q(t_1)Q(t_2) + \theta(t_2 - t_1)Q(t_2)Q(t_1)$$

then

$$\langle q_f, t_f | \mathrm{T}\left(Q(t_1)Q(t_2)\right) | q_i, t_i \rangle = \int\limits_{\substack{q(t_i)=q_i \\ q(t_f)=q_f}} [Dq(t)]\; q(t_1)\, q(t_2)\; e^{i\mathcal{S}[q(t)]}$$

generally

$$\langle q_f, t_f | \mathrm{T}\left(Q(t_1)\cdots Q(t_n)\right) | q_i, t_i \rangle = \int\limits_{\substack{q(t_i)=q_i \\ q(t_f)=q_f}} [Dq(t)]\; q(t_1)\cdots q(t_n)\; e^{i\mathcal{S}[q(t)]}$$

Functional sources an derivatives

One can derive transtion amplitudes with the help of generating functional

$$Z_{fi}[j(t)] \equiv \langle q_f, t_f | T \exp i \int_{t_i}^{t_f} dt\, j(t)\, Q(t) | q_i, t_i \rangle$$

where $j(t)$ is some arbitrary function of time and $Q(t)$ is and operator
Amplidudes are given as functional derivatives

$$\langle q_f, t_f | T \left(Q(t_1) \cdots Q(t_n) \right) | q_i, t_i \rangle = \left. \frac{\delta^n Z_{fi}[j]}{i^n \delta j(t_1) \cdots \delta j(t_n)} \right|_{j \equiv 0}$$

Functional derivatives act essentially as regular differenciation with one additional property

$$\frac{\delta j(t)}{\delta j(t')} = \delta(t - t')$$

values of function $j(t)$ at different times are independent variables

Generating functional has path integral representation (Lagrange)

$$Z_{fi}[j(t)] = \int_{\substack{q(t_i)=q_i \\ q(t_f)=q_f}} [Dq(t)]\; e^{iS[q(t)] + i \int_{t_i}^{t_f} dt\, j(t) q(t)}$$

Functional sources an derivatives

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values of function $j(t)$ at different times are independent variables

Generating functional has path integral representation (Hamilton)

$$Z_{fi}[j(t)] = \int_{\substack{q(t_i)=q_i \\ q(t_f)=q_f}} [Dp(t) Dq(t)] \times \exp\left\{ i \int_{t_i}^{t_f} dt \left(p(t)\dot{q}(t) - \mathcal{H}(p(t), q(t)) + j(t) q(t) \right) \right\}$$

Ground state projection

Initial and final states do not have to be position eigenstates. Consider some operator O and some state ψ

$$\psi(q) \equiv \langle q|\psi\rangle$$

Then

$$\langle\psi_f, t_f|\mathcal{O}|\psi_i, t_i\rangle = \int dq_i dq_f\, \psi_f^*(q_f)\,\psi_i(q_i)\,\langle q_f, t_f|\mathcal{O}|q_i, t_i\rangle$$

In practice we often need matrix element when ininitial and final states are the ground states:

$$\begin{aligned} |q_i, t_i\rangle &= e^{i\mathcal{H}t_i}|q_i\rangle \\ &= \sum_{n=0}^{\infty} e^{i\mathcal{H}t_i}|n\rangle\langle n|q_i\rangle \\ &= \sum_{n=0}^{\infty} \psi_n^*(q_i)\, e^{iE_n t_i}|n\rangle \end{aligned}$$

Assume that $E_0 = 0$ (shifting energy) and multiply the hamiltonian by $1 - i0^+$

Then all factors $\exp(i(1-i0^+)E_n t_i)$ go to 0 for $t_i \to -\infty$ except for the ground state

Ground state projection

With $1 - i0^+$ prescripton

$$\lim_{t_i \to -\infty} |q_i, t_i\rangle = \psi_0^*(q_i)\,|0\rangle \qquad \lim_{t_f \to +\infty} \langle q_f, t_f| = \psi_0(q_f)\,\langle 0|$$

The generating functional is then vacuum expectation value and reads (Hamilton)

$$Z[j(t)] = \int [Dp(t)Dq(t)] \times \exp\left\{i \int dt\,\left(p(t)\dot{q}(t) - (1 - i0^+)\mathcal{H}(p(t), q(t)) + j(t)q(t)\right)\right\}$$

or (Lagrange)

$$Z[j(t)] = \int [Dq(t)] \times \exp\left\{i \int dt\,\left((1 + i0^+)\frac{m\dot{q}^2(t)}{2} - (1 - i0^+)V(q(t)) + j(t)q(t)\right)\right\}$$

Normalization $Z[0] = 1$

Functional integral for scalar field

One can easily translate the QM functional formalism to QFT with the help of the following correspondence

$$\begin{array}{ccc} q(t) & \longleftrightarrow & \phi(x) \\ p(t) & \longleftrightarrow & \Pi(x) \\ j(t) & \longleftrightarrow & j(x) \end{array}$$

and the analogue of the generating functional reads

$$Z[j] = \int [D\Pi(x) D\phi(x)] \times \exp\left\{ i \int d^4x \, \left(\Pi(x)\dot{\phi}(x) - (1 - i0^+)\mathcal{H}(\Pi, \phi) + j(x)\phi(x)\right)\right\}$$

The hamiltonian reads

$$\mathcal{H} = \frac{1}{2}\Pi^2 + \frac{1}{2}(\nabla\phi)\cdot(\nabla\phi) + \frac{1}{2}m^2\phi^2 + V(\phi)$$

and can be obtained from the Lagrangian

$$\mathcal{L} = \int d^3\mathbf{x} \; \left\{\tfrac{1}{2}(\partial_\mu\phi(x))(\partial^\mu\phi(x)) - \tfrac{1}{2}m^2\phi^2(x)\right\}$$

Functional integral for scalar field

Since the hamiltonian is quadratic in Π we can perform Gaussian integral

$$Z[j] = \int [D\phi(x)] \ \exp\left\{ i \int d^4x \ \left(\mathcal{L}(\phi) + j(x)\phi(x) \right) \right\}$$

where

$$\mathcal{L}(\phi) \equiv \frac{1}{2}(1+i0^+)\dot{\phi}^2 - \frac{1}{2}(1-i0^+)\left((\nabla\phi)\cdot(\nabla\phi) + m^2\phi^2\right) - (1-i0^+)V(\phi)$$

Note that $1 - i0^+$ in front of V plays no role if interaction vanishes for large times. Then

$$Z[j] = \exp\left\{ -i \int d^4x \ V\left(\frac{\delta}{i\delta j(x)} \right) \right\} Z_0[j]$$

where

$$Z_0[j] \equiv \int [D\phi(x)] \ \exp\left\{ i \int d^4x \ \left(\mathcal{L}_0(\phi) + j(x)\phi(x) \right) \right\}$$

and

$$\mathcal{L}_0(\phi) = \frac{1}{2}(1+i0^+)\dot{\phi}^2 - \frac{1}{2}(1-i0^+)\left((\nabla\phi)\cdot(\nabla\phi) + m^2\phi^2\right)$$

Scalar propagator

The free functional integral can be easily performed, because it is Gaussian in ϕ

$$\mathcal{L}_0(\phi) \;=\; \frac{1}{2}(1+i0^+)\dot{\phi}^2 - \frac{1}{2}(1-i0^+)\big((\nabla\phi)\cdot(\nabla\phi)+m^2\phi^2\big)$$

$$Z_0[j] \;\equiv\; \int [D\phi(x)]\;\exp\left\{i\int d^4x\,\big(\mathcal{L}_0(\phi)+j(x)\phi(x)\big)\right\}$$

Recall

$$\int_{-\infty}^{+\infty} dx\, e^{ax^2+bx} = \sqrt{\frac{\pi}{-a}}\,e^{-\frac{b^2}{4a}}\,, \qquad \mathrm{Re}\,a \le 0$$

and we get

$$Z_0[j] = \exp\left\{-\frac{1}{2}\int d^4x d^4y\; j(x)j(y)\; G_F^0(x,y)\right\}$$

where $G_F^0(x,y)$ is an inverse of $i\Big[(1+i0^+)\partial_0^2-(1-i0^+)(\nabla^2-m^2)\Big]$ which is obtained by integration by parts:

$$(\dot{\phi})^2 \to -\phi\,\partial_0^2\phi, \qquad (\nabla\phi)\cdot(\nabla\phi) \to -\phi\,\nabla^2\phi$$

Scalar propagator

Inverse of $i\Big[(1+i0^+)\partial_0^2 - (1-i0^+)(\nabla^2 - m^2)\Big]$ can be evaluated in momenum space

$$i\partial_0 \to k_0, \qquad -i\boldsymbol{\nabla} \to \boldsymbol{k}$$

yielding $$\frac{i}{(1+i0^+)k_0^2 - (1-i0^+)(\mathbf{k}^2+m^2)}$$

This is of course the same result as the one obtained in the canonical approach

$$\widetilde{G}_F^0(k) = \frac{i}{k^2 - m^2 + i0^+}$$

Exercise: show that the pole structure of the two expressions is the same

Fermions
and Grassmann variables

Hermann Günther Grassmann (1809 Szczecin – 1877 Szczecin)

Fermion fields anticommute. How to take this into account in functional integral?
Introduce Grassmann variables:

$$\psi_i \; (i = 1 \cdots N)$$

$$\{\psi_i, \psi_j\} = 0$$

Linear space spanned by ψ_i's is called Grassmann algebra

Consider first $N = 1$ $\quad \psi^2 = 0$

any function has a form $f(\psi) = a + \psi b$ where a is a number and $\{b, b\} = \{b, \psi\} = 0$

so $f(\psi) = a + \psi b = a - b\psi$

We have to define left and right derivatives $\overrightarrow{\partial}_\psi \, f(\psi) = b$, $\quad f(\psi) \, \overleftarrow{\partial}_\psi = -b$

Berezin integral: $\int d\psi \, \alpha \, f(\psi) = \alpha \int d\psi \, f(\psi)$ and $\int d\psi \, \partial_\psi f(\psi) = 0$

The only solution consistent with these requirements $\int d\psi \, f(\psi) = b$

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Berezin integral: $\displaystyle\int d\psi \; \alpha \, f(\psi) = \alpha \int d\psi \; f(\psi)$ and $\displaystyle\int d\psi \; \partial_\psi f(\psi) = 0$

$$\int d\psi \; 1 = 0$$

$$\int d\psi \; \psi = 1$$

The only solution consistent with these requirements $\displaystyle\int d\psi \; f(\psi) = b$

Functions of Grassmann variables

Consider now N Grassmann variables $\psi \equiv (\psi_1, \cdots, \psi_N)$ $\{\psi_i, \psi_j\} = 0$

The most general function: $f(\psi) = \sum_{p=0}^{N} \frac{1}{p!} \psi_{i_1} \psi_{i_2} \cdots \psi_{i_p} C_{i_1 i_2 \cdots i_p}$

Only linear terms in each variable are possible. Note that it must be $C_{i_1 \cdots i_N} \equiv \gamma\, \epsilon_{i_1 \cdots i_N}$

alternatively $\frac{1}{N!} \psi_{i_1} \cdots \psi_{i_N}\, \gamma\, \epsilon_{i_1 \cdots i_N} = \psi_1 \cdots \psi_N\, \gamma$

For consitency with previous definition $\int d^N\psi\, f(\psi) = \gamma$

Terms where at least one variable is missing do not contribute to the integral because

Integration measure $d^N\psi \equiv d\psi_N\, d\psi_{N-1} \cdots d\psi_1$ assures that $\int d\psi\, 1 = 0$

$$\int d^N\psi\, \psi_1 \cdots \psi_N = \int d\psi_N \cdots \underbrace{\Big(\int d\psi_2 \underbrace{\Big(\int d\psi_1\, \psi_1 \Big)}_{1} \psi_2 \Big)}_{1} \cdots \psi_N = 1$$

Change of variables

Consider $\psi_i \equiv J_{ij}\,\theta_j$ where $\theta_1 \cdots \theta_N$ are also Grassmann variables

Last term of the function $f(\psi)$

$$\begin{aligned}\psi_{i_1} \cdots \psi_{i_N}\ \epsilon_{i_1 \cdots i_N}\ \gamma &= (J_{i_1 j_1}\theta_{j_1}) \cdots (J_{i_N j_N}\theta_{j_N})\ \epsilon_{i_1 \cdots i_N}\ \gamma \\ &= \det(J)\ \theta_{j_1} \cdots \theta_{j_N}\ \epsilon_{j_1 \cdots j_N}\ \gamma\,.\end{aligned}$$

From this we conclude

$$\underbrace{\int d^N\psi\ f(\psi)}_{\gamma} = [\det(J)]^{-1}\ \underbrace{\int d^N\theta\ f(\psi(\theta))}_{\det(J)\ \gamma}$$

(same as for scalar integral)

Gaussian integral

Consider $\psi \equiv (\psi_1, \cdots, \psi_N) \quad \{\psi_i, \psi_j\} = 0$

$$I(\mathbf{M}) \equiv \int d^N\psi \; \exp\left(\tfrac{1}{2}\psi_i M_{ij}\psi_j\right)$$

where M is $N \times N$ antisymmetric numeric matrix (real or complex). Such integral is non-zero only if N is even. For $N = 2$

$$\mathbf{M} = \begin{pmatrix} 0 & \mu \\ -\mu & 0 \end{pmatrix}$$

and $\exp\left(\tfrac{1}{2}\psi_i M_{ij}\psi_j\right) = 1 + \mu\,\psi_1\psi_2$

Hence: $I(\mathbf{M}) = \mu = \left[\det(\mathbf{M})\right]^{1/2}$

For general even N we can always "diagonalize" $\mathbf{M}$ by special orthogonal matrix

$$\mathbf{M} = \mathbf{Q} \underbrace{\begin{pmatrix} 0 & \mu_1 & & & \\ -\mu_1 & 0 & & & \\ & & 0 & \mu_2 & \\ & & -\mu_2 & 0 & \\ & & & & \ddots \end{pmatrix}}_{\mathbf{D}} \mathbf{Q}^T$$

Define $\mathbf{Q}^T\psi \equiv \theta,$

Gaussian integral

After change of variables we get

$$\mathrm{I}(\mathbf{M}) = \left[\det(\mathbf{Q})\right]^{-1} \underbrace{\int \mathrm{d}^N\theta \; \exp\left(\tfrac{1}{2}\,\theta^\top \mathbf{D}\theta\right)}_{\mu_1\mu_2\cdots=[\det(\mathbf{D})]^{1/2}}$$

But $\det(\mathbf{Q}) = +1$ and we have

$$\mathrm{I}(\mathbf{M}) = \left[\det(\mathbf{D})\right]^{1/2} = \left[\det(\mathbf{M})\right]^{1/2}$$

This is inverse with respect to the Gaussian integral for ordinary variables

We will also need integrals with Grassmann sources η_i

$$\mathrm{I}(\mathbf{M},\eta) \equiv \int \mathrm{d}^N\psi \; \exp\left(\tfrac{1}{2}\,\psi_i M_{ij}\psi_j + \eta_i\psi_i\right)$$

Changing variables

$$\psi_i' \equiv \psi_i - M_{ij}^{-1}\eta_j$$

we obtain

$$\mathrm{I}(\mathbf{M},\eta) = \left[\det(\mathbf{M})\right]^{1/2} \exp\left(-\tfrac{1}{2}\eta^\top \mathbf{M}^{-1}\eta\right)$$

Gaussian integral for 2N variables

Consider $J(\mathbf{M}) \equiv \int d^N\xi d^N\psi \; \exp(\psi_i M_{ij} \xi_j)$ where ψ and ξ are independent

Then (exercise) $J(\mathbf{M}) = \det(\mathbf{M})$

Complex Grassmann variables

Define $\chi \equiv \frac{\psi + i\xi}{\sqrt{2}}$, $\overline{\chi} \equiv \frac{\psi - i\xi}{\sqrt{2}}$ and inverse $\psi = \frac{\overline{\chi} + \chi}{\sqrt{2}}$, $\xi = \frac{i(\overline{\chi} - \chi)}{\sqrt{2}}$

Integrations

$$d\xi d\psi = i \, d\chi d\overline{\chi} \,,$$

$$\psi\xi = -i\overline{\chi}\chi \,,$$

$$\int d\chi d\overline{\chi} \; \overline{\chi}\chi = \int d\xi d\psi \; \psi\xi = 1$$

which leads to

$$\int d\chi d\overline{\chi} \; \exp(\mu \overline{\chi}\chi) = \mu$$

or generally

$$\int d\chi_N d\overline{\chi}_N \cdots d\chi_1 d\overline{\chi}_1 \; \exp(\overline{\chi}^T \mathbf{M} \chi) = \det(\mathbf{M})$$

with sources

$$\int d\chi_N d\overline{\chi}_N \cdots d\chi_1 d\overline{\chi}_1 \; \exp(\overline{\chi}^T \mathbf{M}\chi + \overline{\eta}^T \chi + \overline{\chi}^T \eta) = \det(\mathbf{M}) \; \exp(-\overline{\eta}^T \mathbf{M}^{-1} \eta)$$