QCD lecture 6b

November 19

Formulation of QCD in terms of functional integrals:

- **Path integrals in QM** <- this lecture
- Functional integrals, Grassmann variables
- Chiral anomaly by Fujikawa
- Atiyah-Singer theorem
- Topology of gauge fields, instantons in QM and in QCD
- Quantization of QED and QCD, ghosts

QM - reminder

Schrödinger eq. $$i\hbar \frac{\partial \Psi(x,t_b)}{\partial t_b} = H\Psi(x,t_b)$$

propagates solution from a =(x_a ,t_a) to b=(x_b ,t_b) $$\Psi(x,t_b) = e^{-\frac{i}{\hbar}H(t_b-t_a)}\Psi(x,t_a)$$

(remember H is an operator)

Define propagator: $$K(b,a) = <x_b|\, e^{-\frac{i}{\hbar}H(t_b-t_a)}\, |x_a>$$

recall Dirac notation $\Psi(x) = <x|\Psi>$ and plane wave solution $<x|p> = Ne^{\frac{i}{\hbar}px}$

complex conjugate $<p|x> = Ne^{-\frac{i}{\hbar}px}$

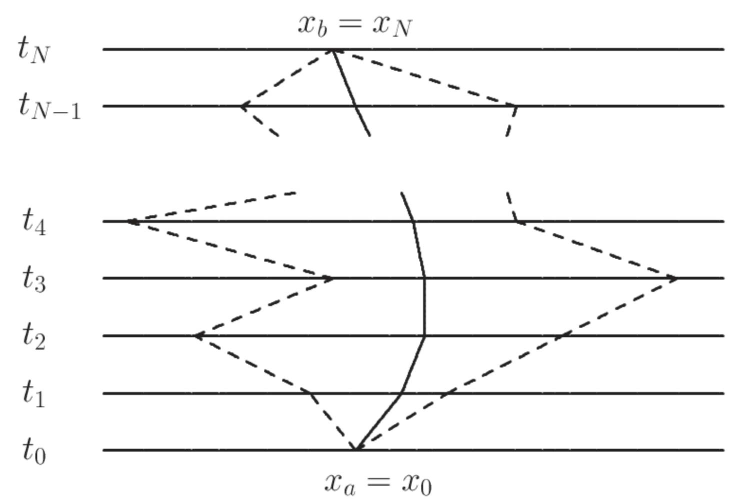
completness relation $$\sum_p |p><p| = \sum_x |x><x| = 1$$

We shall use the following normalization: $$<p|y> = \sqrt{\frac{1}{2\pi\hbar}}e^{-\frac{i}{\hbar}py}$$

Path integral for the propgator

$$K(b,a) = <x_b|\, e^{-\frac{i}{\hbar}H(t_b - t_a)}\, |x_a>$$

Discretize time



set $\hbar = m = 1$

“slice” evolution operator

$$e^{-i(t_b - t_b)H} = e^{-i\epsilon N H} = e^{-i\epsilon H}\, e^{-i\epsilon H} \ldots e^{-i\epsilon H}$$

insert inbetween unity $1 = \int dx_j\, |x_j><x_j|$

$$<x_b|e^{-i(t_b-t_a)H}|x_a> = \int <x_b|e^{-i\epsilon H}|x_{N-1}>\, dx_{N-1} <x_{N-1}|e^{-i\epsilon H}|x_{N-2}>$$
$$\ldots <x_2|e^{-i\epsilon H}|x_1>\, dx_1 <x_1|e^{-i\epsilon H}|x_a>$$

Path integral for the propgator

Decompose hamiltonian $H = \frac{p^2}{2m} + V(x) = K + V$

and use: $e^{-i\epsilon H} = e^{-i\epsilon(K+V)} = e^{-i\epsilon K} e^{-i\epsilon V} + O(\epsilon^2)$

which is true only for small ε

Baker-Cambell-Hausdorff: define C $e^A e^B = e^C$

then $C = A + B + \underbrace{\frac{1}{2}[A,B]}_{\sim \varepsilon^2} + \frac{1}{12}[A,[A,B]] + \frac{1}{12}[[A,B],B] + \ ...$

Therefore

$$K(b,a) = <x_b|e^{-i(t_b-t_a)H}|x_a>$$

$$= \int <x_b|e^{-i\epsilon K}|x_{N-1}> e^{-i\epsilon V(x_{N-1})}\, dx_{N-1} <x_{N-1}|e^{-i\epsilon K}|x_{N-2}>$$

$$\times\ e^{-i\epsilon V(x_{N-2})} dx_{N-2} \ \ ... \ \ dx_1 <x_1|e^{-i\epsilon K}|x_a> e^{-i\epsilon V(x_a)}$$

Path integral for the propgator

We need to calculate $$< x|e^{-\frac{i}{\hbar}\epsilon K}|y> \,=\, \int dp\, <x|e^{-\frac{i}{\hbar}\epsilon\frac{p^2}{2m}}|p><p|y>$$

(distinguish operators from eigenalues)

$$= \int dp\, <x|p>\, e^{\frac{-i\epsilon p^2}{\hbar 2m}}\, <p|y>$$

recall normalization $$<p|y>= \sqrt{\frac{1}{2\pi\hbar}}e^{-\frac{i}{\hbar}py}$$

$$<x|e^{-\frac{i}{\hbar}\epsilon K}|y>= \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} dp\, e^{\frac{-i\epsilon p^2}{2m\hbar}}\, e^{-\frac{i}{\hbar}(y-x)p} = \sqrt{\frac{m}{2i\pi\hbar\epsilon}}\, e^{im\frac{(y-x)^2}{2\epsilon\hbar}}$$

where we have used

$$\int_{-\infty}^{+\infty} dx\, e^{ax^2+bx} = \sqrt{\frac{\pi}{-a}}e^{-\frac{b^2}{4a}}\,, \qquad \mathrm{Re}\, a \leq 0$$

$$\frac{i}{2\hbar}m\left(\frac{y-x}{\epsilon}\right)^2\epsilon = \frac{i}{\hbar}\frac{mv^2}{2}\epsilon$$

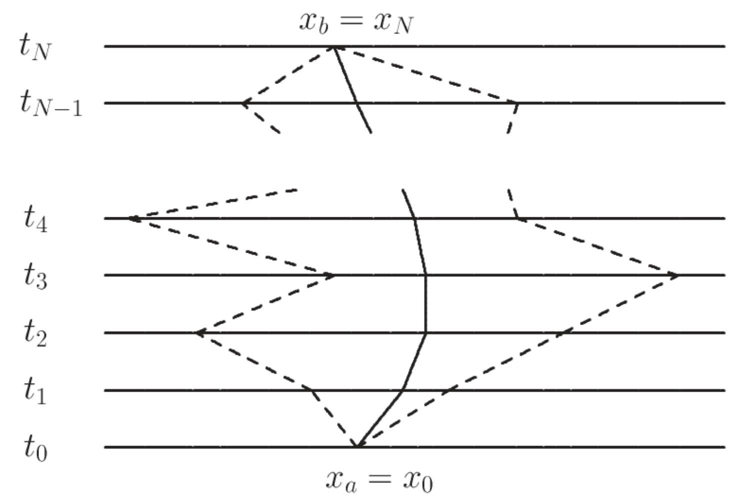
bur remember:

$$L_j = \tfrac{1}{2}m\left(\frac{x_{j+1}-x_j}{\epsilon}\right)^2 - V(x_j)$$

Path integral for the propgator

$$K(b,a) = \lim_{\epsilon\to 0} \sqrt{\frac{m}{2i\epsilon\hbar\pi}} \int \prod_{j=1}^{N-1} dx_j \sqrt{\frac{m}{2i\epsilon\hbar\pi}} \prod_{k=0}^{N-1} e^{\frac{i}{\hbar}\epsilon L_k}$$

$$\stackrel{\text{def}}{=} \int [\mathcal{D}x(t)]\, e^{\frac{i}{\hbar}\int_{t_a}^{t_b} dt L[x(t),\dot{x}(t)]}$$



Define functional integration measure integration over all traiectories from a to b

$$[\mathcal{D}x(t)] = dx_1 \ldots dx_{N-1} \left(\frac{m}{2i\epsilon\hbar\pi}\right)^{\frac{1}{2}N}$$

and use definition of action

$$\lim_{\epsilon\to 0} \sum_{j=0}^{N-1} \epsilon L_j = \int_{t_a}^{t_b} dt\, L(x(t),\dot{x}(t)) = S[x(t)]$$

to arrive at

$$K(b,a) = \int [\mathcal{D}x(t)]\, e^{\frac{i}{\hbar}S[x(t)]}$$

special role of the classical trajectory
i.e. stationary point of action

Euclidean path integral

Change $t \to -i\tau$

then $K(x_b, x_a, -i\tau) = <x_b|e^{-\frac{H}{\hbar}\tau}|x_a>$

$$= \sum_{n,n'} <x_b|E_n><E_n|\, e^{-\frac{H}{\hbar}\tau}|E_{n'}><E_{n'}|x_a>$$

$$= \sum_n e^{-\frac{E_n}{\hbar}\tau}\phi_n(x_b)\phi_n^*(x_a)\,.$$

for large τ only the ground state survives

Feynman-Kac formula $E_0 = -\lim_{\tau\to\infty}\left\{\frac{\hbar}{\tau}\ln\left(K(x_b, x_a, -i\tau)\right)\right\}$

In Euclidean one can perform computer simulations

$$K_n(x, x, -i\tau) = \int dx_1 dx_2 \ldots dx_n \left(\frac{1}{2\pi\epsilon}\right)^{\frac{1}{2}(n+1)} e^{-\epsilon\sum_{j=0}^{n}\left\{\frac{1}{2}\left(\frac{x_{j+1}-x_j}{\epsilon}\right)^2 + V(x_j)\right\}}$$