QCD lecture 6a

November 19

Literature: T-P Cheng and L-F Li,
Gauge theory of elementary particle physics
Claderon Press, Oxford, 1984
chapter 6.2

Axial anomaly

In massless QCD there are two conserved currents

$$\bar{u}(p')\gamma^\mu u(p) \qquad \bar{u}(p')\gamma^\mu\gamma_5 u(p)$$

We will show that when loop corrections are included only one current remains conserved. We now from gauge invariancde that the vector current must be conserved, so it is the axial-vector current that is not conserved.
This phenomenon is called axial anomaly

symmetry of lagrangian broken by quantum corrections

Conserved currents

$$\begin{aligned} q_\mu j^\mu(q) &= q_\mu \overline{u}(p')\gamma^\mu u(p) \\ &= \overline{u}(p')(\not{p}' - \not{p})u(p) \\ &= 0 \end{aligned}$$

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \qquad \{\gamma^\mu, \gamma_5\} = 0$$

$$\begin{aligned} q_\mu j_5^\mu(q) &= q_\mu \overline{u}(p')\gamma^\mu\gamma_5 u(p) \\ &= \overline{u}(p')(\not{p}' - \not{p})\gamma_5 u(p) \\ &= \overline{u}(p')(\not{p}'\gamma_5 + \gamma_5\not{p})u(p) \\ &= 2m\,\overline{u}(p')\gamma_5 u(p) \end{aligned}$$

Chiral symmetry

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$$

$$\{\gamma^\mu, \gamma_5\} = 0$$

Dirac equation in **chrial representation** for gamma matrices

$$\gamma^0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \gamma^i = \begin{bmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{bmatrix}, \quad \gamma_5 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

splits into two equations

$$(i\partial_t - i\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})\,\psi_L - m\psi_R = 0, \qquad (i\partial_t + i\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})\,\psi_R - m\psi_L = 0,$$

where $\psi = \begin{bmatrix} \psi_L \\ \psi_R \end{bmatrix}$. Note that for massless fermions these eqs. are *independent*.

Projection operators: $P_L = \frac{1}{2}(1-\gamma_5),\ P_R = \frac{1}{2}(1+\gamma_5)$ project solutions of

given chirality (eigen value of γ_5)

$$\psi_- = \begin{bmatrix} \psi_L \\ 0 \end{bmatrix}, \quad \psi_+ = \begin{bmatrix} 0 \\ \psi_R \end{bmatrix}$$

Chiral symmetry and the Lorentz group

Lorentz group SO(1,3) has six generators (plus translations)

$$\boldsymbol{J} = (J^{23}, J^{31}, J^{12}), \qquad \boldsymbol{K} = (J^{01}, J^{02}, J^{03})$$

$$[J^i, J^j] = i\,\varepsilon^{ijk} J^k, \qquad [J^i, K^j] = i\,\varepsilon^{ijk} K^k, \qquad [K^i, K^j] = -i\varepsilon^{ijk} J^k$$

Define: $J^i_\pm = \frac{1}{2}(J^i \pm iK^i)$

Commutation relations read then

$$[J^i_\pm, J^j_\pm] = i\varepsilon_{ijk} J^k_\pm,$$
$$[J^i_\pm, J^j_\mp] = 0.$$

and imply isomorphism:

$$SO(1,3) \simeq SU(2) \otimes SU(2)$$

Axial anomaly

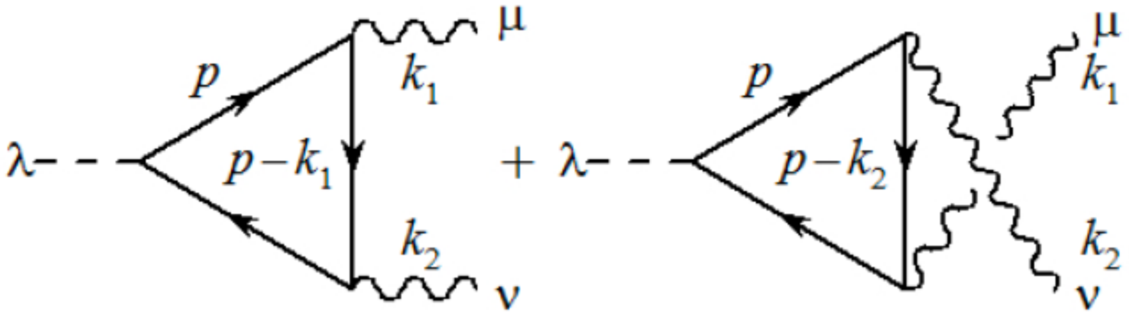
pseudoscalar density

Gauge invariance of QED (and QCD): $q_\mu j^\mu(q) = q_\mu \overline{u}(p')\gamma^\mu u(p) = 0$

divergence of axial-vector current: $q_\mu j_5^\mu(q) = q_\mu \overline{u}(p')\gamma^\mu \gamma_5 u(p) = 2m\overline{u}(p')\gamma_5 u(p)$

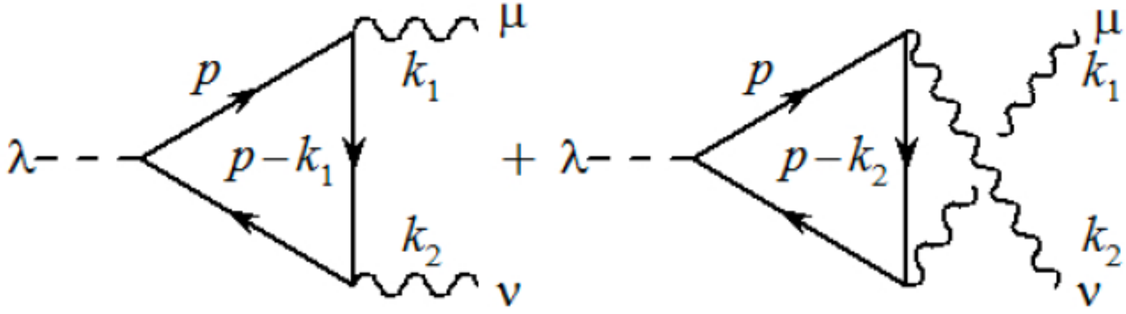
Axial current is conserved for massless fermions: chiral symmetry

It is not possible to maintain both symmetries when loop corrections are included. This is called: AXIAL ANOMALY



photons are bosons and they are not distinguishable hence amplitude has to be symmetrized

Naïve current conservation



$$q = k_1 + k_2$$

Skipping coupling constants (charges) the amplitude reads:

$$\begin{aligned} T_{\mu\nu\lambda} \;\; = \;\; & -i\int \frac{d^4p}{(2\pi)^4}\,\mathrm{Tr}\left[\frac{i}{\not p - m}\gamma_\lambda\gamma_5\frac{i}{(\not p - \not q) - m}\gamma_\nu\frac{i}{(\not p - \not k_1) - m}\gamma_\mu\right] \\ & -i\int \frac{d^4p}{(2\pi)^4}\,\mathrm{Tr}\left[\frac{i}{\not p - m}\gamma_\lambda\gamma_5\frac{i}{(\not p - \not q) - m}\gamma_\mu\frac{i}{(\not p - \not k_2) - m}\gamma_\nu\right] \end{aligned}$$

Naively we expect:

$$k_1^\mu T_{\mu\nu\lambda} = k_2^\nu T_{\mu\nu\lambda} = 0 \qquad q^\lambda T_{\mu\nu\lambda} = 2mT_{\mu\nu}$$

Naïve current conservation

Vector current, first diagram:

$$k_1^\mu T_{\mu\nu\lambda} \sim \mathrm{Tr}\left[\gamma_\lambda\gamma_5 \frac{i}{(\not{p}-\not{q})-m}\gamma_\nu \frac{i}{(\not{p}-\not{k}_1)-m}\not{k}_1 \frac{i}{\not{p}-m}\right]$$

use trick:

$$\not{k}_1 = (\not{p}-m) - ((\not{p}-\not{k}_1)-m)$$

we get:

$$= \ i\,\mathrm{Tr}\left[\gamma_\lambda\gamma_5 \frac{i}{(\not{p}-\not{q})-m}\gamma_\nu \frac{i}{(\not{p}-\not{k}_1)-m}\right] - i\,\mathrm{Tr}\left[\gamma_\lambda\gamma_5 \frac{i}{(\not{p}-\not{q})-m}\gamma_\nu \frac{i}{\not{p}-m}\right]$$

Naïve current conservation

Vector current, first diagram:

$$k_1^\mu T_{\mu\nu\lambda} \sim \mathrm{Tr}\left[\gamma_\lambda\gamma_5 \frac{i}{(\not{p}-\not{q})-m}\gamma_\nu \frac{i}{(\not{p}-\not{k}_1)-m}\not{k}_1 \frac{i}{\not{p}-m}\right]$$

use trick:

$$\not{k}_1 = (\not{p}-m) - ((\not{p}-\not{k}_1)-m)$$

we get:

$$= \; i\,\mathrm{Tr}\left[\gamma_\lambda\gamma_5 \frac{i}{(\not{p}-\not{q})-m}\gamma_\nu \frac{i}{(\not{p}-\not{k}_1)-m}\right] - i\,\mathrm{Tr}\left[\gamma_\lambda\gamma_5 \frac{i}{(\not{p}-\not{q})-m}\gamma_\nu \frac{i}{\not{p}-m}\right]$$

same trick with the second diagram gives

$$= \; i\,\mathrm{Tr}\left[\gamma_\lambda\gamma_5 \frac{i}{(\not{p}-\not{q})-m}\gamma_\nu \frac{i}{\not{p}-m}\right] - i\,\mathrm{Tr}\left[\gamma_\lambda\gamma_5 \frac{i}{(\not{p}-\not{k}_2)-m}\gamma_\nu \frac{i}{\not{p}-m}\right]$$

Naïve current conservation

$$k_1^\mu T_{\mu\nu\lambda} \sim \int \frac{d^4p}{(2\pi)^4}$$

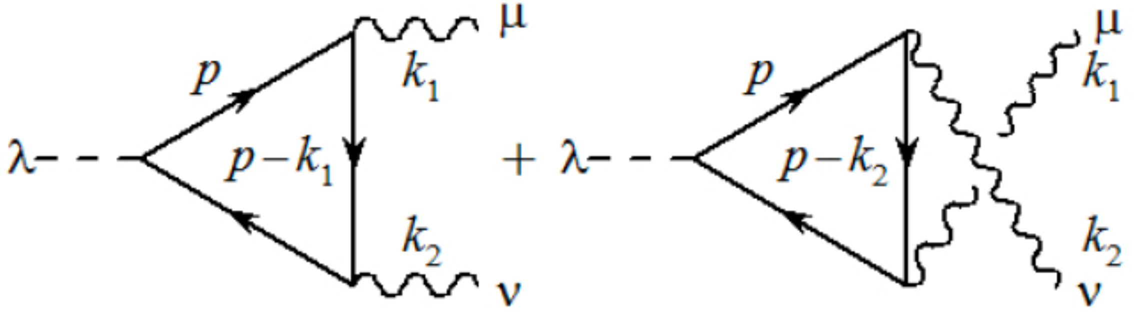
$$\left\{ \mathrm{Tr}\left[\gamma_\lambda\gamma_5 \frac{i}{(\not{p}-\not{q})-m}\gamma_\nu \frac{i}{(\not{p}-\not{k}_1)-m}\right] - \mathrm{Tr}\left[\gamma_\lambda\gamma_5 \frac{i}{(\not{p}-\not{k}_2)-m}\gamma_\nu \frac{i}{\not{p}-m}\right]\right\}$$

change variable in the first integral $p \to p + k_1$

$$q = k_1 + k_2$$

It seems we get zero

Naïve current conservation



$$q = k_1 + k_2$$

Skipping coupling constants (charges) the amplitude reads:

$$T_{\mu\nu\lambda} = -i\int \frac{d^4p}{(2\pi)^4}\,\mathrm{Tr}\left[\frac{i}{\not p - m}\gamma_\lambda\gamma_5\frac{i}{(\not p - \not q) - m}\gamma_\nu\frac{i}{(\not p - \not k_1) - m}\gamma_\mu\right]$$
$$-i\int \frac{d^4p}{(2\pi)^4}\,\mathrm{Tr}\left[\frac{i}{\not p - m}\gamma_\lambda\gamma_5\frac{i}{(\not p - \not q) - m}\gamma_\mu\frac{i}{(\not p - \not k_2) - m}\gamma_\nu\right]$$

Naively we expect:

$$q^\lambda T_{\mu\nu\lambda} = 2mT_{\mu\nu}$$

Axial current

To calculate $q^\lambda T_{\mu\nu\lambda}$

we use the following trick:

$$\begin{aligned} \not{q}\gamma_5 &= -\gamma_5 \not{q} \\ &= \gamma_5 \left[(\not{p} - \not{q}) - m \right] - \gamma_5 \left[\not{p} - m \right] \\ &= \gamma_5 \left[(\not{p} - \not{q}) - m \right] + \left[\not{p} - m \right] \gamma_5 + 2m\gamma_5 \end{aligned}$$

and for the first diagram we obtain

$$\begin{aligned} q^\lambda \left[\frac{i}{\not{p} - m} \gamma_\lambda \gamma_5 \frac{i}{(\not{p} - \not{q}) - m} \right] &= 2m \frac{i}{\not{p} - m} \gamma_5 \frac{i}{(\not{p} - \not{q}) - m} \\ &+ i \frac{i}{\not{p} - m} \gamma_5 + i\gamma_5 \frac{i}{(\not{p} - \not{q}) - m} \end{aligned}$$

Axial current

Sum from the two diagrams $$q^\lambda T_{\mu\nu\lambda} = 2mT_{\mu\nu} + \Delta^{(1)}_{\mu\nu} + \Delta^{(2)}_{\mu\nu}$$

$$\begin{aligned}
&\Delta^{(1)}_{\mu\nu} + \Delta^{(2)}_{\mu\nu} \\
= &\int \frac{d^4p}{(2\pi)^4} \mathrm{Tr}\left[\frac{i}{\not{p} - m}\gamma_5\gamma_\nu \frac{i}{(\not{p} - \not{k}_1) - m}\gamma_\mu + \gamma_5 \frac{i}{(\not{p} - \not{q}) - m}\gamma_\nu \frac{i}{(\not{p} - \not{k}_1) - m}\gamma_\mu\right] \\
+ &\int \frac{d^4p}{(2\pi)^4} \mathrm{Tr}\left[\frac{i}{\not{p} - m}\gamma_5\gamma_\mu \frac{i}{(\not{p} - \not{k}_2) - m}\gamma_\nu + \gamma_5 \frac{i}{(\not{p} - \not{q}) - m}\gamma_\mu \frac{i}{(\not{p} - \not{k}_2) - m}\gamma_\nu\right]
\end{aligned}$$

Axial current

$$\Delta^{(1)}_{\mu\nu} = \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr}\left[\frac{i}{\not{p} - m}\gamma_5\gamma_\nu \frac{i}{(\not{p} - \not{k}_1) - m}\gamma_\mu - \frac{i}{(\not{p} - \not{k}_2) - m}\gamma_5\gamma_\nu \frac{i}{(\not{p} - \not{q}) - m}\gamma_\mu\right]$$

$$\Delta^{(2)}_{\mu\nu} = \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr}\left[\frac{i}{\not{p} - m}\gamma_5\gamma_\mu \frac{i}{(\not{p} - \not{k}_2) - m}\gamma_\nu - \frac{i}{(\not{p} - \not{k}_1) - m}\gamma_5\gamma_\mu \frac{i}{(\not{p} - \not{q}) - m}\gamma_\nu\right]$$

The question is: are $\Delta^{(1,2)}_{\mu\nu}$ equal zero?

Changing variables $p \to p + k_2$

seems to nullify $\Delta^{(1,2)}_{\mu\nu}$. $p \to p + k_1$

However, $\Delta^{(1,2)}_{\mu\nu} \sim \int^{\infty} dp\,p^3 \frac{1}{p^2} \sim \int^{\infty} dp\,p$ are UV divergent

Due to the minus sign the divergence is only linear

The same applies to vector current conservation.

Axial and vector current conservation

$$p \to p + k_1$$

$$\Delta_{\nu\lambda} = \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr}\left[\gamma_\lambda\gamma_5 \frac{i}{(\not p - \not q) - m}\gamma_\nu \frac{i}{(\not p - \not k_1) - m} - \gamma_\lambda\gamma_5 \frac{i}{(\not p - \not k_2) - m}\gamma_\nu \frac{i}{\not p - m}\right]$$

$$\Delta^{(1)}_{\mu\nu} = \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr}\left[\frac{i}{\not p - m}\gamma_5\gamma_\nu \frac{i}{(\not p - \not k_1) - m}\gamma_\mu - \frac{i}{(\not p - \not k_2) - m}\gamma_5\gamma_\nu \frac{i}{(\not p - \not q) - m}\gamma_\mu\right]$$

$$\Delta^{(2)}_{\mu\nu} = \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr}\left[\frac{i}{\not p - m}\gamma_5\gamma_\mu \frac{i}{(\not p - \not k_2) - m}\gamma_\nu - \frac{i}{(\not p - \not k_1) - m}\gamma_5\gamma_\mu \frac{i}{(\not p - \not q) - m}\gamma_\nu\right]$$

$$p \to p + k_2$$

$$p \to p + k_1$$

It seems that by changing integration variables one can nullify all three Deltas. However, because all these integrals are linearly divergent, change of variables may result in a finite value for the difference of two infinities.

Mathematical diggression

Consider the integral that is naively zero:

$$\int_{-\infty}^{\infty} dx \, [f(x+a) - f(x)]$$

However, if

$$f(\pm\infty) \neq 0.$$

we can calculate this integral by Taylor expansion:

$$\int_{-\infty}^{\infty} dx \, [f(x+a) - f(x)] = a \, [f(\infty) - f(-\infty)] + \frac{a^2}{2} \, [f'(\infty) - f'(-\infty)] + \ldots$$

it may happen that $\neq 0$

Mathematical diggression

Consider Euclidean integral:
$$\Delta(\vec{a}) = \int d^n\vec{r}\,[f(\vec{r}+\vec{a}) - f(\vec{r})]$$

expand in a
$$= \int d^n\vec{r}\,\vec{a}\cdot\vec{\nabla} f(\vec{r}) + \ldots$$

apply Gauss theorem
$$= \vec{a}\cdot\vec{n}\,S_n(R)\,f(\vec{R})$$

where $\vec{n} = \frac{\vec{R}}{R}$ and $S_n(R)$ is a surface of the n sphere, R is regulator.

For even n
$$S_n(R) = \frac{2\pi^{n/2}}{(n/2-1)!}R^{n-1} = \begin{cases} 2\pi R & \text{for} \quad n=2 \\ 2\pi^2 R^3 & \text{for} \quad n=4 \end{cases}$$

In Minkowski space

$$\Delta(a) = 2i\pi^2 a^\mu \lim_{R\to\infty} R^2 R_\mu\, f(R)$$

Shift in full amplitude

$$
\begin{aligned}
T_{\mu\nu\lambda} &= -i \int \frac{d^4p}{(2\pi)^4} \operatorname{Tr}\left[\frac{i}{\not p - m}\gamma_\lambda\gamma_5 \frac{i}{(\not p - \not q) - m}\gamma_\nu \frac{i}{(\not p - \not k_1) - m}\gamma_\mu\right] \\
&\quad -i \int \frac{d^4p}{(2\pi)^4} \operatorname{Tr}\left[\frac{i}{\not p - m}\gamma_\lambda\gamma_5 \frac{i}{(\not p - \not q) - m}\gamma_\mu \frac{i}{(\not p - \not k_2) - m}\gamma_\nu\right]
\end{aligned}
$$

define shift vector
and amplitude difference:

$$a = \alpha k_1 + (\alpha - \beta) k_2$$

$$\Delta_{\mu\nu\lambda}(a) = T_{\mu\nu\lambda}(p \to p + a) - T_{\mu\nu\lambda}$$

Strategy:

$$
\begin{aligned}
q^\lambda T_{\mu\nu\lambda}(a) &= q^\lambda \left(T_{\mu\nu\lambda}(a) - T_{\mu\nu\lambda}(0)\right) + q^\lambda T_{\mu\nu\lambda}(0) \\
&= q^\lambda \Delta_{\mu\nu\lambda}(a) + 2mT_{\mu\nu} + \Delta^{(1)}_{\mu\nu} + \Delta^{(2)}_{\mu\nu}
\end{aligned}
$$

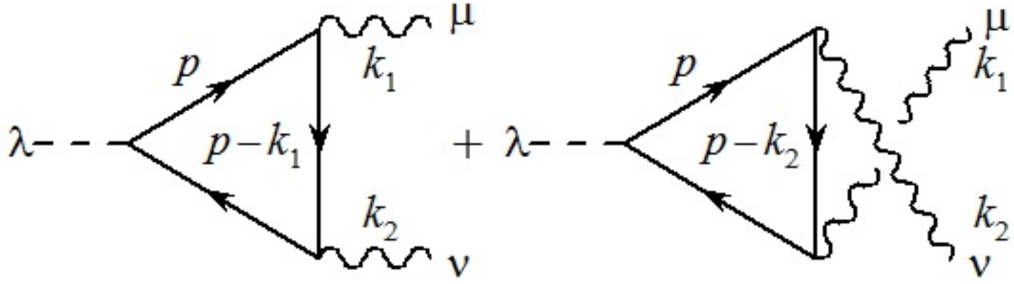
$$
\begin{aligned}
k_1^\mu T_{\mu\nu\lambda}(a) &= k_1^\mu \left(T_{\mu\nu\lambda}(a) - T_{\mu\nu\lambda}(0)\right) + k_1^\mu T_{\mu\nu\lambda}(0) \\
&= k_1^\mu \Delta_{\mu\nu\lambda}(a) + \Delta_{\nu\lambda}
\end{aligned}
$$

chose a in a way that vector current is conserved
and see what comes out for the axial current

Shift in full amplitude

Calculate
(all i's give -)

$$\Delta_{\mu\nu\lambda}(a) = -\int \frac{d^4p}{(2\pi)^4}\left\{\mathrm{Tr}\left[\frac{1}{\not p + \not a - m}\gamma_\lambda\gamma_5\frac{1}{(\not p + \not a - \not q) - m}\gamma_\nu\frac{1}{(\not p + \not a - \not k_1) - m}\gamma_\mu\right] - \mathrm{Tr}\left[\frac{1}{\not p - m}\gamma_\lambda\gamma_5\frac{1}{(\not p - \not q) - m}\gamma_\nu\frac{1}{(\not p - \not k_1) - m}\gamma_\mu\right]\right\} + (\mu \longleftrightarrow \nu, k_1 \leftrightarrow k_2)\,.$$



Shift in full amplitude

Calculate
(all i's give -)

$$\Delta_{\mu\nu\lambda}(a) = -\int \frac{d^4p}{(2\pi)^4}\left\{\mathrm{Tr}\left[\frac{1}{\not p + \not a - m}\gamma_\lambda\gamma_5\frac{1}{(\not p + \not a - \not q) - m}\gamma_\nu\frac{1}{(\not p + \not a - \not k_1) - m}\gamma_\mu\right] - \mathrm{Tr}\left[\frac{1}{\not p - m}\gamma_\lambda\gamma_5\frac{1}{(\not p - \not q) - m}\gamma_\nu\frac{1}{(\not p - \not k_1) - m}\gamma_\mu\right]\right\} + (\mu \longleftrightarrow \nu, k_1 \leftrightarrow k_2)\,.$$

Expand in a

$$\Delta_{\mu\nu\lambda}(a) = -\int \frac{d^4p}{(2\pi)^4} a^\sigma \frac{\partial}{\partial p^\sigma}\mathrm{Tr}\left[\frac{1}{\not p - m}\gamma_\lambda\gamma_5\frac{1}{(\not p - \not q) - m}\gamma_\nu\frac{1}{(\not p - \not k_1) - m}\gamma_\mu\right] + (\mu \longleftrightarrow \nu, k_1 \leftrightarrow k_2)\,.$$

Shift in full amplitude

Calculate
(all i's give -)

$$\Delta_{\mu\nu\lambda}(a) = -\int \frac{d^4p}{(2\pi)^4}\left\{\mathrm{Tr}\left[\frac{1}{\not p+\not a-m}\gamma_\lambda\gamma_5\frac{1}{(\not p+\not a-\not q)-m}\gamma_\nu\frac{1}{(\not p+\not a-\not k_1)-m}\gamma_\mu\right] - \mathrm{Tr}\left[\frac{1}{\not p-m}\gamma_\lambda\gamma_5\frac{1}{(\not p-\not q)-m}\gamma_\nu\frac{1}{(\not p-\not k_1)-m}\gamma_\mu\right]\right\} + (\mu\longleftrightarrow\nu, k_1\leftrightarrow k_2)\,.$$

Expand in a

$$\Delta_{\mu\nu\lambda}(a) = -\int \frac{d^4p}{(2\pi)^4}a^\sigma\frac{\partial}{\partial p^\sigma}\mathrm{Tr}\left[\frac{1}{\not p-m}\gamma_\lambda\gamma_5\frac{1}{(\not p-\not q)-m}\gamma_\nu\frac{1}{(\not p-\not k_1)-m}\gamma_\mu\right] + (\mu\longleftrightarrow\nu, k_1\leftrightarrow k_2)\,.$$

large p limit

$$\frac{1}{p^6}\mathrm{Tr}\left[\not p\gamma_\lambda\gamma_5\not p\gamma_\nu\not p\gamma_\mu\right]$$

Shift in full amplitude

Calculate
(all i's give -)

$$\Delta_{\mu\nu\lambda}(a) = -\int \frac{d^4p}{(2\pi)^4}\left\{\mathrm{Tr}\left[\frac{1}{\not p + \not a - m}\gamma_\lambda\gamma_5\frac{1}{(\not p + \not a - \not q) - m}\gamma_\nu\frac{1}{(\not p + \not a - \not k_1) - m}\gamma_\mu\right] - \mathrm{Tr}\left[\frac{1}{\not p - m}\gamma_\lambda\gamma_5\frac{1}{(\not p - \not q) - m}\gamma_\nu\frac{1}{(\not p - \not k_1) - m}\gamma_\mu\right]\right\} + (\mu \longleftrightarrow \nu, k_1 \leftrightarrow k_2)\,.$$

Expand in a

$$\Delta_{\mu\nu\lambda}(a) = -\int \frac{d^4p}{(2\pi)^4} a^\sigma \frac{\partial}{\partial p^\sigma}\mathrm{Tr}\left[\frac{1}{\not p - m}\gamma_\lambda\gamma_5\frac{1}{(\not p - \not q) - m}\gamma_\nu\frac{1}{(\not p - \not k_1) - m}\gamma_\mu\right] + (\mu \longleftrightarrow \nu, k_1 \leftrightarrow k_2)\,.$$

large p limit

$$\frac{1}{p^6}\mathrm{Tr}\left[\not p\gamma_\lambda\gamma_5\not p\gamma_\nu\not p\gamma_\mu\right]$$

apply expansion

$$\Delta_{\mu\nu\lambda}(a) = -\frac{i}{(2\pi)^4}2\pi^2 a^\sigma \lim_{P\to\infty} P^3\frac{P_\sigma}{P}\mathrm{Tr}\left[\not P\gamma_\lambda\gamma_5\not P\gamma_\nu\not P\gamma_\mu\right]\frac{1}{P^6}$$

$$+ (\mu \longleftrightarrow \nu, k_1 \leftrightarrow k_2)$$

$$\Delta(a) = 2i\pi^2 a^\mu \lim_{R\to\infty} R^2 R_\mu\, f(R)$$

Shift in full amplitude

Calculate
(all i's give -)

$$\Delta_{\mu\nu\lambda}(a) = -\int \frac{d^4p}{(2\pi)^4} \left\{ \mathrm{Tr}\left[\frac{1}{\not p + \not a - m}\gamma_\lambda\gamma_5 \frac{1}{(\not p + \not a - \not q) - m}\gamma_\nu \frac{1}{(\not p + \not a - \not k_1) - m}\gamma_\mu\right] - \mathrm{Tr}\left[\frac{1}{\not p - m}\gamma_\lambda\gamma_5 \frac{1}{(\not p - \not q) - m}\gamma_\nu \frac{1}{(\not p - \not k_1) - m}\gamma_\mu\right]\right\} + (\mu \longleftrightarrow \nu, k_1 \leftrightarrow k_2)\,.$$

Expand in a

$$\Delta_{\mu\nu\lambda}(a) = -\int \frac{d^4p}{(2\pi)^4} a^\sigma \frac{\partial}{\partial p^\sigma} \mathrm{Tr}\left[\frac{1}{\not p - m}\gamma_\lambda\gamma_5 \frac{1}{(\not p - \not q) - m}\gamma_\nu \frac{1}{(\not p - \not k_1) - m}\gamma_\mu\right] + (\mu \longleftrightarrow \nu, k_1 \leftrightarrow k_2)\,.$$

large p limit

$$\frac{1}{p^6}\mathrm{Tr}\left[\not p\gamma_\lambda\gamma_5\not p\gamma_\nu\not p\gamma_\mu\right]$$

apply expansion

$$\Delta_{\mu\nu\lambda}(a) = -\frac{i}{(2\pi)^4} 2\pi^2 a^\sigma \lim_{P\to\infty} P^3 \frac{P_\sigma}{P} \mathrm{Tr}\left[\not P\gamma_\lambda\gamma_5\not P\gamma_\nu\not P\gamma_\mu\right]\frac{1}{P^6}$$

$$+ (\mu \longleftrightarrow \nu, k_1 \leftrightarrow k_2)$$

calculate Trace

$$\mathrm{Tr}\left[\not P\gamma_\lambda\gamma_5\not P\gamma_\nu\not P\gamma_\mu\right] = 4iP^2\varepsilon_{\alpha\mu\nu\lambda}P^\alpha$$

Shift in full amplitude

We arrive at: $$\Delta_{\mu\nu\lambda}(a) = \frac{1}{(2\pi)^4} 8\pi^2 \varepsilon_{\mu\nu\lambda\alpha}\, a_\sigma \lim_{P\to\infty} \frac{P^\sigma P^\alpha}{P^2} + (\mu \longleftrightarrow \nu, k_1 \leftrightarrow k_2)$$

Shift in full amplitude

We arrive at:

$$\Delta_{\mu\nu\lambda}(a) = \frac{1}{(2\pi)^4} 8\pi^2 \varepsilon_{\mu\nu\lambda\alpha}\, a_\sigma \lim_{P\to\infty} \frac{P^\sigma P^\alpha}{P^2} + (\mu \longleftrightarrow \nu, k_1 \leftrightarrow k_2)$$

take symmetric limit:

$$\lim_{P\to\infty} \frac{P^\sigma P^\alpha}{P^2} = \frac{1}{4} g^{\sigma\alpha}$$

recall: $a = \alpha k_1 + (\alpha - \beta) k_2$

Shift in full amplitude

We arrive at: $$\Delta_{\mu\nu\lambda}(a) = \frac{1}{(2\pi)^4} 8\pi^2 \varepsilon_{\mu\nu\lambda\alpha}\, a_\sigma \lim_{P\to\infty} \frac{P^\sigma P^\alpha}{P^2} + (\mu \longleftrightarrow \nu, k_1 \leftrightarrow k_2)$$

take symmetric limit: $$\lim_{P\to\infty} \frac{P^\sigma P^\alpha}{P^2} = \frac{1}{4} g^{\sigma\alpha}$$

recall: $a = \alpha k_1 + (\alpha - \beta) k_2$

Final result:
$$\begin{aligned} \Delta_{\mu\nu\lambda}(a) &= \frac{1}{8\pi^2} \varepsilon_{\alpha\mu\nu\lambda} a^\alpha + (\mu \longleftrightarrow \nu, k_1 \leftrightarrow k_2) \\ &= \frac{1}{8\pi^2} \varepsilon_{\alpha\mu\nu\lambda} \left(\alpha k_1^\alpha + (\alpha - \beta) k_2^\alpha - \alpha k_2^\alpha - (\alpha - \beta) k_1^\alpha\right) \\ &= \frac{\beta}{8\pi^2} \varepsilon_{\alpha\mu\nu\lambda} \left(k_1 - k_2\right)^\alpha . \end{aligned}$$

depends on β, there is an ambiguity, which we have to fix demanding that vector current is conserved.

Axial current, cont.

Recall:

$$
\begin{aligned}
q^\lambda T_{\mu\nu\lambda}(a) &= q^\lambda \left(T_{\mu\nu\lambda}(a) - T_{\mu\nu\lambda}(0)\right) + q^\lambda T_{\mu\nu\lambda}(0) \\
&= q^\lambda \Delta_{\mu\nu\lambda}(a) + 2mT_{\mu\nu} + \Delta^{(1)}_{\mu\nu} + \Delta^{(2)}_{\mu\nu}
\end{aligned}
$$

calculated — finite — needs to be computed

Axial current, cont.

Recall:

$$
\begin{aligned}
q^\lambda T_{\mu\nu\lambda}(a) &= q^\lambda \left(T_{\mu\nu\lambda}(a) - T_{\mu\nu\lambda}(0)\right) + q^\lambda T_{\mu\nu\lambda}(0) \\
&= q^\lambda \Delta_{\mu\nu\lambda}(a) + 2mT_{\mu\nu} + \Delta^{(1)}_{\mu\nu} + \Delta^{(2)}_{\mu\nu}
\end{aligned}
$$

calculated finite needs to be computed

Let's calculate

$$
\Delta^{(1)}_{\mu\nu} = \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr}\left[\frac{i}{\not{p} - m}\gamma_5\gamma_\nu \frac{i}{(\not{p} - \not{k}_1) - m}\gamma_\mu - \frac{i}{(\not{p} - \not{k}_2) - m}\gamma_5\gamma_\nu \frac{i}{(\not{p} - \not{q}) - m}\gamma_\mu\right]
$$

Axial current, cont.

Recall:

$$\begin{aligned} q^\lambda T_{\mu\nu\lambda}(a) &= q^\lambda \left(T_{\mu\nu\lambda}(a) - T_{\mu\nu\lambda}(0)\right) + q^\lambda T_{\mu\nu\lambda}(0) \\ &= q^\lambda \Delta_{\mu\nu\lambda}(a) + 2mT_{\mu\nu} + \Delta^{(1)}_{\mu\nu} + \Delta^{(2)}_{\mu\nu} \end{aligned}$$

calculated finite needs to be computed

Let's calculate (multiply *i*'s and change order)

$$\begin{aligned} \Delta^{(1)}_{\mu\nu} &= \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr}\left[\frac{i}{\not p - m}\gamma_5\gamma_\nu \frac{i}{(\not p - \not k_1) - m}\gamma_\mu - \frac{i}{(\not p - \not k_2) - m}\gamma_5\gamma_\nu \frac{i}{(\not p - \not q) - m}\gamma_\mu\right] \\ &= \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr}\left[\frac{1}{(\not p - \not k_2) - m}\gamma_5\gamma_\nu \frac{1}{(\not p - \not k_2 - \not k_1) - m}\gamma_\mu - \frac{1}{\not p - m}\gamma_5\gamma_\nu \frac{1}{(\not p - \not k_1) - m}\gamma_\mu\right] \end{aligned}$$

We can use the same trick as previously $p \to p - k_2$ where $a = -k_2$

Axial current, cont.

Recall:

$$
\begin{aligned}
q^\lambda T_{\mu\nu\lambda}(a) &= q^\lambda \left(T_{\mu\nu\lambda}(a) - T_{\mu\nu\lambda}(0)\right) + q^\lambda T_{\mu\nu\lambda}(0) \\
&= q^\lambda \Delta_{\mu\nu\lambda}(a) + 2mT_{\mu\nu} + \Delta^{(1)}_{\mu\nu} + \Delta^{(2)}_{\mu\nu}
\end{aligned}
$$

calculated finite needs to be computed

Let's calculate

$$
\begin{aligned}
\Delta^{(1)}_{\mu\nu} &= \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr}\left[\frac{i}{\not p - m}\gamma_5\gamma_\nu \frac{i}{(\not p - \not k_1) - m}\gamma_\mu - \frac{i}{(\not p - \not k_2) - m}\gamma_5\gamma_\nu \frac{i}{(\not p - \not q) - m}\gamma_\mu\right] \\
&= \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr}\left[\frac{1}{(\not p - \not k_2) - m}\gamma_5\gamma_\nu \frac{1}{(\not p - \not k_2 - \not k_1) - m}\gamma_\mu - \frac{1}{\not p - m}\gamma_5\gamma_\nu \frac{1}{(\not p - \not k_1) - m}\gamma_\mu\right]
\end{aligned}
$$

We can use the same trick as previously $p \to p - k_2$ where $a = -k_2$

$$
\Delta^{(1)}_{\mu\nu} = -\frac{1}{(2\pi)^4} 2i\pi^2 k_2^\rho \lim_{P\to\infty} \frac{P_\rho}{P^2} \mathrm{Tr}\left[\not P \gamma_5 \gamma_\nu (\not P - \not k_1)\gamma_\mu\right]
$$

keep k_1, because Tr with 2 P's is zero

Axial current, cont.

$$\Delta^{(1)}_{\mu\nu} = -\frac{1}{(2\pi)^4} 2i\pi^2 k_2^\rho \lim_{P\to\infty} \frac{P_\rho}{P^2} \operatorname{Tr}\left[\not{P}\gamma_5\gamma_\nu(\not{P} - \not{k}_1)\gamma_\mu\right]$$

We have

$$\begin{aligned}
\Delta^{(1)}_{\mu\nu} &= \frac{1}{(2\pi)^4} 2i\pi^2 k_2^\rho k_1^\sigma \lim_{P\to\infty} \frac{P_\rho P^\alpha}{P^2} \operatorname{Tr}\left[\gamma_\alpha\gamma_5\gamma_\nu\gamma_\sigma\gamma_\mu\right] \\
&= \frac{1}{(2\pi)^4} 2i\pi^2 k_2^\rho k_1^\sigma \, \frac{1}{4}(-)\underbrace{\operatorname{Tr}\left[\gamma_5\gamma_\rho\gamma_\nu\gamma_\sigma\gamma_\mu\right]}_{-4i\varepsilon_{\rho\nu\sigma\mu}} \\
&= -\frac{1}{8\pi^2}\varepsilon_{\mu\nu\sigma\rho}\, k_1^\sigma k_2^\rho.
\end{aligned}$$

We obtain $\Delta^{(2)}_{\mu\nu}$ by $\mu \longleftrightarrow \nu, k_1 \leftrightarrow k_2$, hence

$$\Delta^{(1)}_{\mu\nu} = \Delta^{(2)}_{\mu\nu}$$

Axial current, final

$$
\begin{aligned}
q^\lambda T_{\mu\nu\lambda}(a) &= q^\lambda \left(T_{\mu\nu\lambda}(a) - T_{\mu\nu\lambda}(0)\right) + q^\lambda T_{\mu\nu\lambda}(0) \\
&= 2mT_{\mu\nu} + \Delta^{(1)}_{\mu\nu} + \Delta^{(2)}_{\mu\nu} + q^\lambda \Delta_{\mu\nu\lambda}(a) \\
&= 2mT_{\mu\nu} - \frac{1}{4\pi^2}\varepsilon_{\mu\nu\sigma\rho}\, k_1^\sigma k_2^\rho + (k_1 + k_2)^\lambda \frac{\beta}{8\pi^2}\varepsilon_{\alpha\mu\nu\lambda}\,(k_1 - k_2)^\alpha \\
&= 2mT_{\mu\nu} - \frac{1-\beta}{4\pi^2}\varepsilon_{\mu\nu\sigma\rho}\, k_1^\sigma k_2^\rho
\end{aligned}
$$

Vector current

We shall use the same trick to calculate the divergence of a vecor current

$$\Delta_{\mu\nu\lambda}(a)$$

$$\begin{aligned} k_1^\mu T_{\mu\nu\lambda}(a) &= k_1^\mu \left(T_{\mu\nu\lambda}(a) - T_{\mu\nu\lambda}(0)\right) + k_1^\mu T_{\mu\nu\lambda}(0) \\ &= k_1^\mu T_{\mu\nu\lambda}(0) + k_1^\mu \frac{\beta}{8\pi^2} \varepsilon_{\alpha\mu\nu\lambda} \left(k_1 - k_2\right)^\alpha \\ &= k_1^\mu T_{\mu\nu\lambda}(0) + \frac{\beta}{8\pi^2} \varepsilon_{\nu\lambda\sigma\rho} k_1^\sigma k_2^\rho . \end{aligned}$$

We ned the first piece

Vector current

We shall use the same trick to calculate the divergence of a vecor current

$$\begin{aligned} k_1^\mu T_{\mu\nu\lambda}(a) &= k_1^\mu \left(T_{\mu\nu\lambda}(a) - T_{\mu\nu\lambda}(0)\right) + k_1^\mu T_{\mu\nu\lambda}(0) \\ &= k_1^\mu T_{\mu\nu\lambda}(0) + k_1^\mu \frac{\beta}{8\pi^2}\varepsilon_{\alpha\mu\nu\lambda}\left(k_1 - k_2\right)^\alpha \\ &= k_1^\mu T_{\mu\nu\lambda}(0) + \frac{\beta}{8\pi^2}\varepsilon_{\nu\lambda\sigma\rho}k_1^\sigma k_2^\rho . \end{aligned}$$

We ned the first piece

$$\begin{aligned} k_1^\mu T_{\mu\nu\lambda} &= -\int \frac{d^4p}{(2\pi)^4} \\ & \left\{ \mathrm{Tr}\left[\gamma_\lambda\gamma_5 \frac{1}{(\not{p} - \not{a}) - m}\gamma_\nu \frac{1}{(\not{p} - \not{k}_1) - m}\right] - \mathrm{Tr}\left[\gamma_\lambda\gamma_5 \frac{1}{(\not{p} - \not{k}_2) - m}\gamma_\nu \frac{1}{\not{p} - m}\right]\right\} \end{aligned}$$

Vector current

We shall use the same trick to calculate the divergence of a vecor current

$$
\begin{aligned}
k_1^\mu T_{\mu\nu\lambda}(a) &= k_1^\mu \left(T_{\mu\nu\lambda}(a) - T_{\mu\nu\lambda}(0)\right) + k_1^\mu T_{\mu\nu\lambda}(0) \\
&= k_1^\mu T_{\mu\nu\lambda}(0) + k_1^\mu \frac{\beta}{8\pi^2} \varepsilon_{\alpha\mu\nu\lambda} \left(k_1 - k_2\right)^\alpha \\
&= k_1^\mu T_{\mu\nu\lambda}(0) + \frac{\beta}{8\pi^2} \varepsilon_{\nu\lambda\sigma\rho} k_1^\sigma k_2^\rho .
\end{aligned}
$$

We ned the first piece

$$
\begin{aligned}
k_1^\mu T_{\mu\nu\lambda} = & -\int \frac{d^4 p}{(2\pi)^4} \\
& \left\{ \mathrm{Tr}\left[\gamma_\lambda \gamma_5 \frac{1}{(\not{p} - \not{a}) - m} \gamma_\nu \frac{1}{(\not{p} - \not{k}_1) - m}\right] - \mathrm{Tr}\left[\gamma_\lambda \gamma_5 \frac{1}{(\not{p} - \not{k}_2) - m} \gamma_\nu \frac{1}{\not{p} - m}\right] \right\}
\end{aligned}
$$

$$
\left\{ \mathrm{Tr}\left[\gamma_\lambda \gamma_5 \frac{1}{(\not{p} - \not{k}_2 - \not{k}_1)) - m} \gamma_\nu \frac{1}{(\not{p} - \not{k}_1) - m}\right] - \mathrm{Tr}\left[\gamma_\lambda \gamma_5 \frac{1}{(\not{p} - \not{k}_2) - m} \gamma_\nu \frac{1}{\not{p} - m}\right] \right\}
$$

Vector current

We shall use the same trick to calculate the divergence of a vecor current

$$\begin{aligned} k_1^\mu T_{\mu\nu\lambda}(a) &= k_1^\mu \left(T_{\mu\nu\lambda}(a) - T_{\mu\nu\lambda}(0)\right) + k_1^\mu T_{\mu\nu\lambda}(0) \\ &= k_1^\mu T_{\mu\nu\lambda}(0) + k_1^\mu \frac{\beta}{8\pi^2}\varepsilon_{\alpha\mu\nu\lambda}\left(k_1 - k_2\right)^\alpha \\ &= k_1^\mu T_{\mu\nu\lambda}(0) + \frac{\beta}{8\pi^2}\varepsilon_{\nu\lambda\sigma\rho}k_1^\sigma k_2^\rho . \end{aligned}$$

We ned the first piece

$$k_1^\mu T_{\mu\nu\lambda} = -\int \frac{d^4p}{(2\pi)^4} \left\{ \mathrm{Tr}\left[\gamma_\lambda\gamma_5 \frac{1}{(\not{p}-\not{a})-m}\gamma_\nu \frac{1}{(\not{p}-\not{k}_1)-m}\right] - \mathrm{Tr}\left[\gamma_\lambda\gamma_5 \frac{1}{(\not{p}-\not{k}_2)-m}\gamma_\nu \frac{1}{\not{p}-m}\right]\right\}$$

$$\left\{ \mathrm{Tr}\left[\gamma_\lambda\gamma_5 \frac{1}{(\not{p}-\not{k}_2-\not{k}_1))-m}\gamma_\nu \frac{1}{(\not{p}-\not{k}_1)-m}\right] - \mathrm{Tr}\left[\gamma_\lambda\gamma_5 \frac{1}{(\not{p}-\not{k}_2)-m}\gamma_\nu \frac{1}{\not{p}-m}\right]\right\}$$

$$k_1^\mu T_{\mu\nu\lambda} = -\frac{1}{(2\pi)^4} 2i\pi^2 (-) k_1^\sigma \lim_{P\to\infty} \frac{P_\sigma}{P^2}\,\mathrm{Tr}\left[\gamma_\lambda\gamma_5(\not{P}-\not{k}_2)\gamma_\nu \not{P}\right]$$

Vector current

$$
\begin{aligned}
k_1^\mu T_{\mu\nu\lambda} &= -\frac{1}{(2\pi)^4} 2i\pi^2 (-) k_1^\sigma \lim_{P\to\infty} \frac{P_\sigma}{P^2} \mathrm{Tr}\left[\gamma_\lambda \gamma_5 (\not{P} - \not{k}_2) \gamma_\nu \not{P}\right] \\
&= -\frac{1}{8\pi^2} i \frac{1}{4} \mathrm{Tr}\left[\gamma_\lambda \gamma_5 \gamma_\rho \gamma_\nu \gamma_\sigma\right] k_1^\sigma k_2^\rho \\
&= \frac{1}{8\pi^2} \varepsilon_{\nu\lambda\sigma\rho} k_1^\sigma k_2^\rho .
\end{aligned}
$$

Recall

$$
k_1^\mu T_{\mu\nu\lambda}(a) = k_1^\mu T_{\mu\nu\lambda}(0) + \frac{\beta}{8\pi^2} \varepsilon_{\nu\lambda\sigma\rho} k_1^\sigma k_2^\rho = \frac{1+\beta}{8\pi^2} \varepsilon_{\nu\lambda\sigma\rho} k_1^\sigma k_2^\rho
$$

We need to choose $\beta = -1$ to have vector current conserved!

Axial anomaly

Summarizing:

$$q^\lambda T_{\mu\nu\lambda}(a) \quad = \quad 2mT_{\mu\nu} - \frac{1-\beta}{4\pi^2}\varepsilon_{\mu\nu\sigma\rho}\, k_1^\sigma k_2^\rho$$

$$k_1^\mu T_{\mu\nu\lambda}(a) = \frac{1+\beta}{8\pi^2}\varepsilon_{\nu\lambda\sigma\rho} k_1^\sigma k_2^\rho.$$

Choose $\beta = -1$

$$q^\lambda T_{\mu\nu\lambda} = 2mT_{\mu\nu} - \frac{1}{2\pi^2}\varepsilon_{\mu\nu\sigma\rho}\, k_1^\sigma k_2^\rho$$

Axial current is anomalous

This can be translated to the configurations space

$$\partial^\lambda J_\lambda^5(x) = \frac{1}{(4\pi)^2}\varepsilon_{\mu\nu\sigma\rho} F^{\mu\nu}(x) F^{\sigma\rho}(x) + \mathcal{O}(m)$$

Axial anomaly

Summarizing:

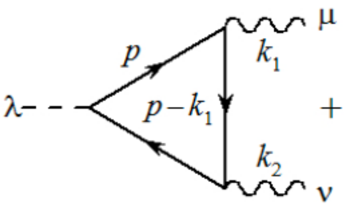
$$q^\lambda T_{\mu\nu\lambda}(a) \;=\; 2mT_{\mu\nu} - \frac{1-\beta}{4\pi^2}\varepsilon_{\mu\nu\sigma\rho}\,k_1^\sigma k_2^\rho$$

$$k_1^\mu T_{\mu\nu\lambda}(a) = \frac{1+\beta}{8\pi^2}\varepsilon_{\nu\lambda\sigma\rho}k_1^\sigma k_2^\rho.$$

Choose $\beta = -1$

$$q^\lambda T_{\mu\nu\lambda} = 2mT_{\mu\nu} - \frac{1}{2\pi^2}\varepsilon_{\mu\nu\sigma\rho}\,k_1^\sigma k_2^\rho$$

- Anomaly is mass independent
- Adler-Bardeen theorem (69): no higher order correctoons
- name: Adler-Bardeen-Jackiw anomaly
- Fujikawa (79) path integral formulation
- In non-Abelian case one can nullify anomaly Tr(…)=0



Axial current is anomalous

This can be translated to the configurations space

$$\partial^\lambda J^5_\lambda(x) = \frac{1}{(4\pi)^2}\varepsilon_{\mu\nu\sigma\rho}F^{\mu\nu}(x)F^{\sigma\rho}(x) + \mathcal{O}(m)$$

Anomaly in the light quark sector

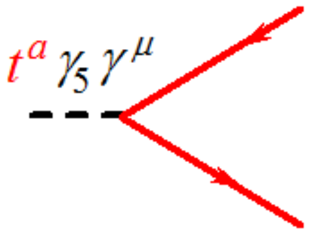
Recall Noether theorem:
global symmetry implies conserved current(s)
To calculate conserved currents promote the symmetry to the local one, calculate the change of action (as disscused on previous slide).

Consider SU(2) chiral transformation:

$$\mathcal{U}(x) = \exp\left(i\gamma^5 \alpha^a(x)\, t^a\right) \qquad \psi = \begin{bmatrix} u \\ d \end{bmatrix}$$

Conserved current:

$$J_5^{\mu\, a} = \bar{\psi}\gamma^5\gamma^\mu t^a \psi$$



Anomaly in the light quark sector

Consider diagonal (neutral) axial current generated by matrix

$$t = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

in falovor space and unit matrix in the color (gauge) space. Then:

$$\mathrm{tr}\left(t^a t^b t\right) = \mathrm{tr}_{\mathrm{colour}}\left(t^a t^b\right) \times \underbrace{\mathrm{tr}_{\mathrm{flavour}}\left(t\right)}_{1-1=0} = 0$$

Anomaly vanishes. Physically up quark contribution is cancelled by d quark.

Anomaly in the light quark sector

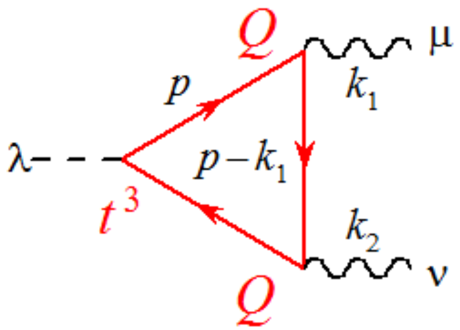
Consider diagonal (neutral) axial current generated by matrix

$$t = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

coupled to QED. In flavor space $\psi = \begin{bmatrix} u \\ d \end{bmatrix}$ electric charge is a matrix

$$Q \equiv \begin{pmatrix} \frac{2}{3} & 0 \\ 0 & -\frac{1}{3} \end{pmatrix}$$

therefore anomaly is proportional to



$$\mathrm{tr}_{\mathrm{flavour}}\left(Q^2 t\right) \times \mathrm{tr}_{\mathrm{colour}}\left(\mathbf{1}_{\mathrm{colour}}\right) = \frac{N_c}{3}$$