QCD lecture 5

November 5, 2025

Infrared divergences

$$S_F^R = \frac{i}{\not p}\left(1 + \frac{\alpha(\mu^2)}{4\pi}C_F\left(\ln\left(\frac{-p^2}{\bar{\mu}^2}\right) - 1\right)\right)$$

Divergent for $p^2 = 0$. This is infrared divergence (from the lower int. limit).
It can be regularized by going to the number of dimensions higher than 4.
Before expansion, change $\varepsilon \to -\kappa$

$$S_F^R(p) = \frac{i}{\not p}\left(1 - \frac{\alpha_s}{4\pi}C_F\left(\frac{\bar{\mu}^2}{-p^2}\right)^{\varepsilon}\left(\frac{1}{\varepsilon} + 1\right) + \frac{\alpha_s}{4\pi}C_F\frac{1}{\varepsilon}\right)$$

Infrared divergences

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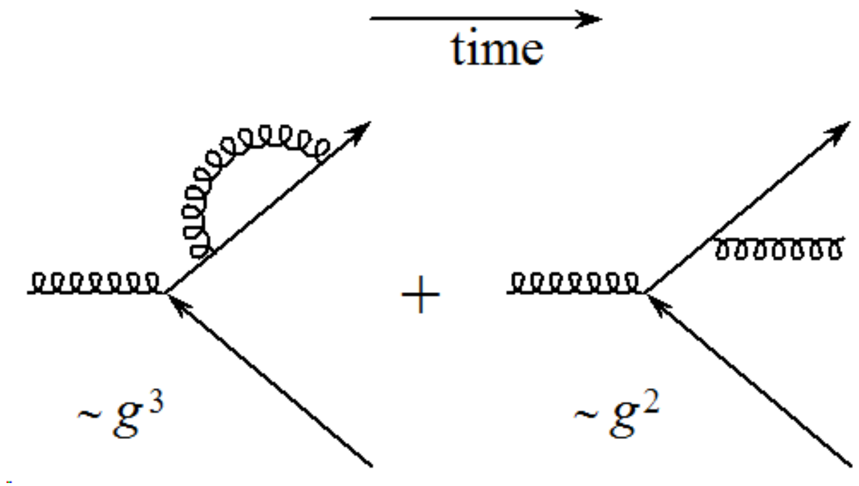
$$\begin{aligned} S_F^R(p) &= \frac{i}{\not p}\left(1 - \frac{\alpha_s}{4\pi} C_F \left(\frac{-p^2}{\bar{\mu}^2}\right)^{\kappa} \left(-\frac{1}{\kappa} + 1\right) - \frac{\alpha_s}{4\pi} C_F \frac{1}{\kappa}\right) \\ &\underset{p^2=0}{=} \frac{i}{\not p}\left(1 - \frac{\alpha_s}{4\pi} C_F \frac{1}{\kappa}\right). \end{aligned}$$

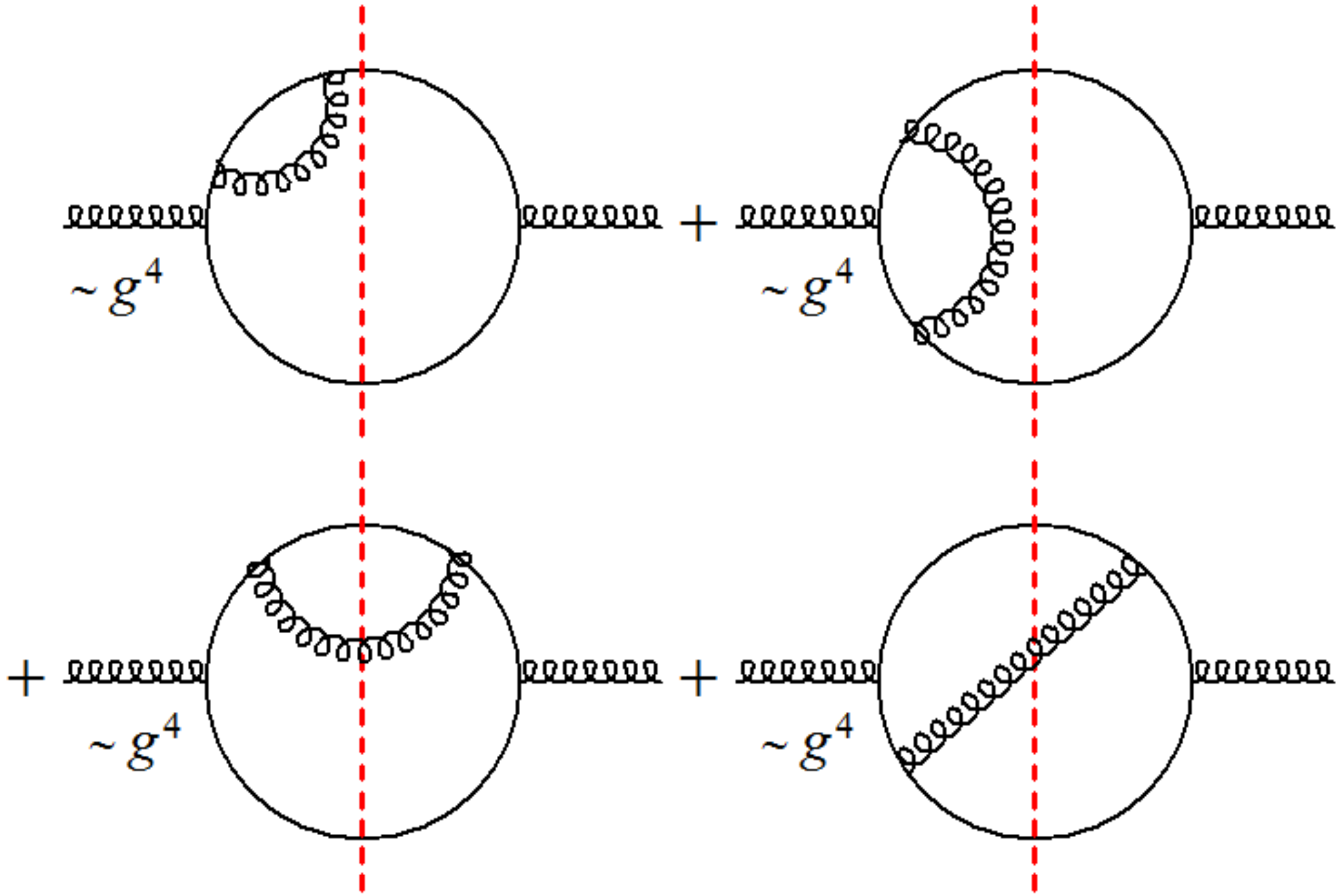
Infrared divergencies



One cannot distinguish a single electron from an electron accompanied by a zero energy foton or a collinear foton (for massless fermion).

One has to sum over such degenerate states.

Infrared divergencies



Here IR singularities cancel out

Infrared singularities

IR singulariteis arise when the theory has massless particles (photon, gluon)

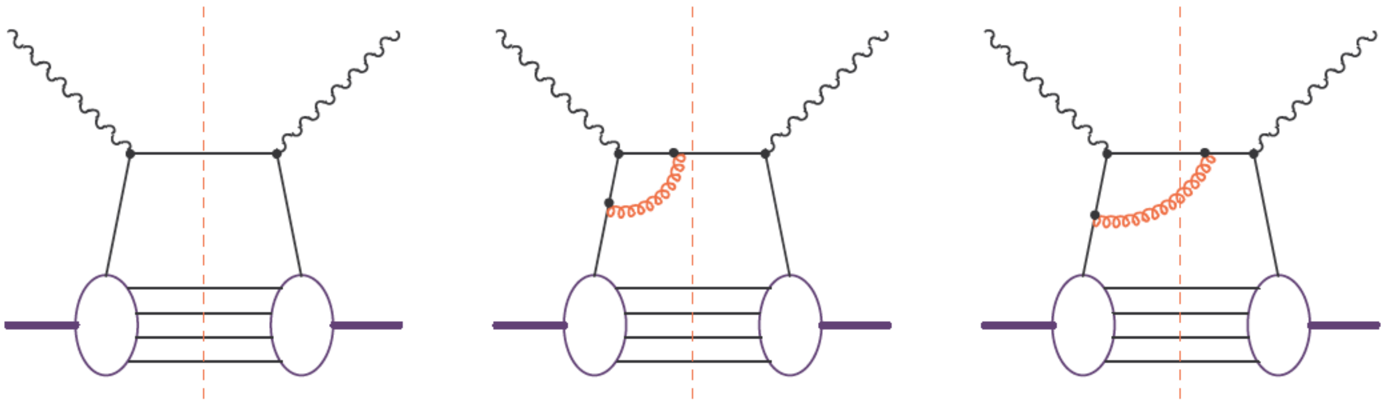
- when energy of photon (gluon) is small – soft singularity
- when for massless fermion photon (gluon) is parallel to that fermion – collinear singularity

Bloch – Nordsieck theorem (basically derived for QED)
Kinoshita – Lee – Nauenberg theorem (generalized to QCD)

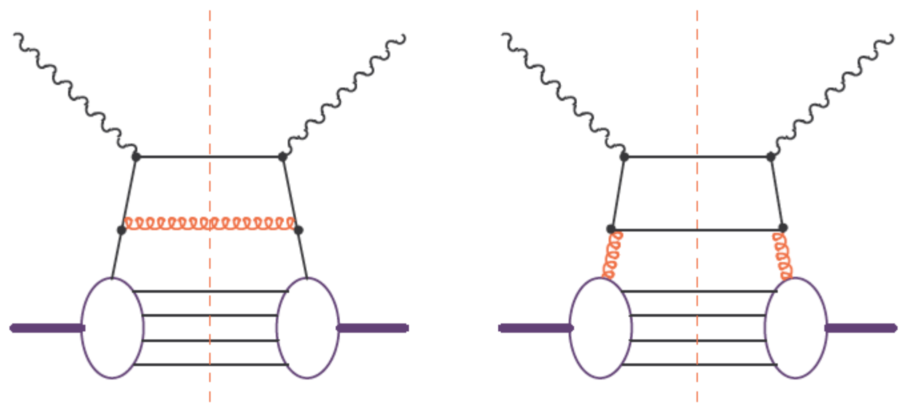
Kinoshita-Lee-Nauenberg (KLN) theorem assures that a summation over degenerate initial and final states removes all infrared (IR) divergences in QCD.

This very broad topic, beyond the scope of this lecture

QCD corrections to parton model



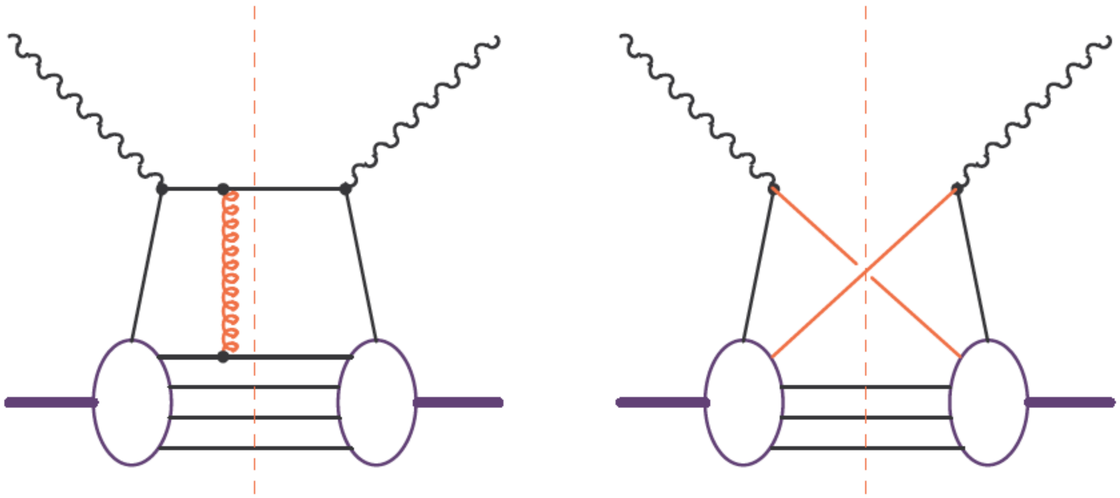
Leading corrections
not suppressed by $1/Q^2$



photon scatters off the gluon

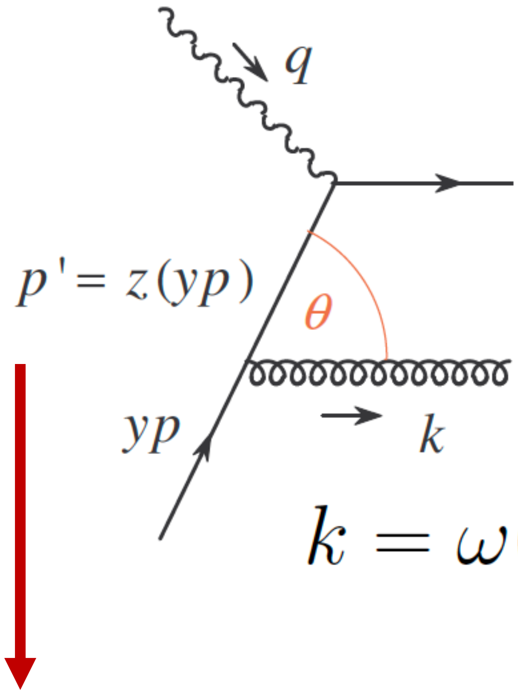
QCD corrections to parton model

Non-leading corrections suppressed by $1/Q^2$



QCD corrections to parton model

$$yp = E(1, 0, 0, 1)$$



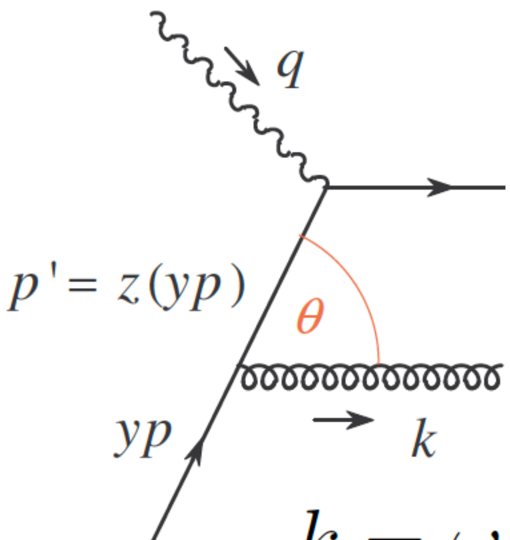
$$d^4k\,\delta(k^2) \sim \frac{d^3\boldsymbol{k}}{\omega} \sim \omega d\omega d\cos\theta d\varphi$$

$$k = \omega(1,\ \sin\theta\sin\varphi,\ \sin\theta\cos\varphi,\ \cos\theta)$$

$$\frac{1}{p'^{\,2}} = \frac{1}{(yp-k)^2} = \frac{1}{2yp\,k} = \frac{1}{2E\omega(1-\cos\theta)}$$

Ignoring overall minus sign

QCD corrections to parton model



$$yp = E(1,0,0,1)$$

$$d^4k\,\delta(k^2) \sim \frac{d^3\boldsymbol{k}}{\omega} \sim \omega d\omega d\cos\theta d\varphi$$

$$k = \omega(1,\ \sin\theta\sin\varphi,\ \sin\theta\cos\varphi,\ \cos\theta)$$

$$|\mathcal{M}|^2\, d^4k\,\delta(k^2) \sim \sin^2\theta \frac{\omega d\omega\, d\cos\theta}{\omega^2(1-\cos\theta)^2} \sim \frac{d\omega}{\omega}\frac{d\theta^2}{\theta^2}$$

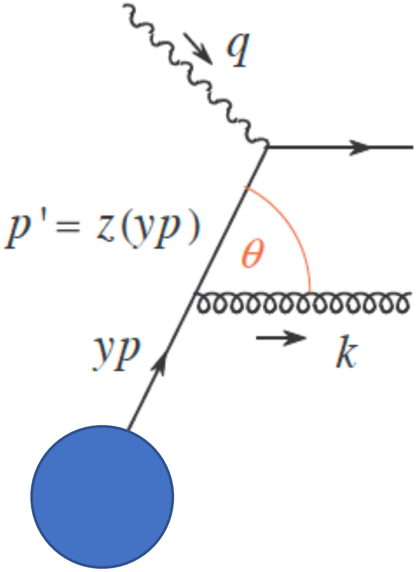
QCD corrections to parton model

$$\frac{d\omega}{\omega}\frac{d\theta^2}{\theta^2}$$

- soft (cancel) $\omega \longrightarrow 0$
- collinear (remain) $\theta \longrightarrow 0$

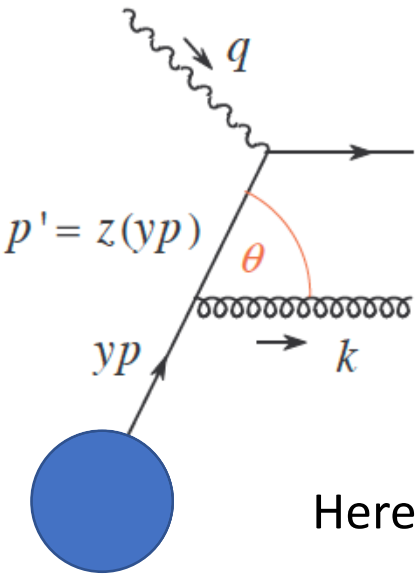
In dimensional regularization:

$$\left(\frac{Q^2}{\mu^2}\right)^{\kappa}\frac{1}{\kappa} = \frac{1}{\kappa} + \log\left(\frac{Q^2}{\mu^2}\right)$$



Poles can be absorbed into bare parton densities.
Logs can be summed up to all orders. Factrozation.
Coefficients of the poles are universal functions of z

Altarelli-Parisi probabilities



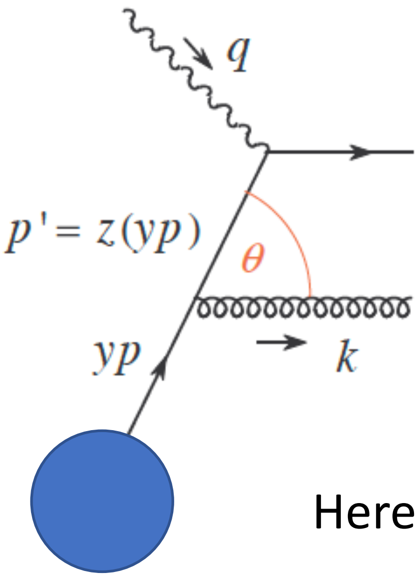
$$\log\left(\frac{Q^2}{\mu^2}\right) P_{qq}(z)$$

It turns out that potentially large logs are multiplied by universal functions of the momentum fraction *z* (with respect to the emitting parton)

Here $P_{qq}(z) = P_{q\leftarrow q}(z)$ is a probability of "finding"

a quark of the longitudinal momentum fraction z in initial quark

Altarelli-Parisi probabilities



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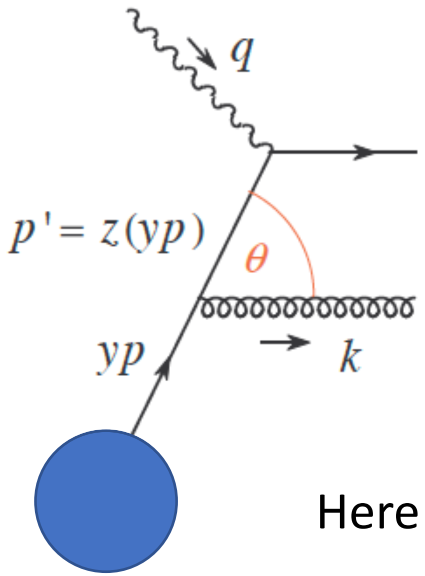
$$P_{qq}(z) = C_F\left(\frac{1+z^2}{1-z}\right)_+$$

Altarelli-Parisi probabilities



sample diagram

$$\log\left(\frac{Q^2}{\mu^2}\right) P_{qq}(z)$$

It turns out that potentially large logs are multiplied by universal functions of the momentum fraction *z* (with respect to the emitting parton)

Here $P_{qq}(z) = P_{q\leftarrow q}(z)$ is a probability of "finding"

a quark of the longitudinal momentum fraction z in initial quark

$$P_{qq}(z) = C_F\left(\frac{1+z^2}{1-z}\right)_+$$

"Plus" distribution:

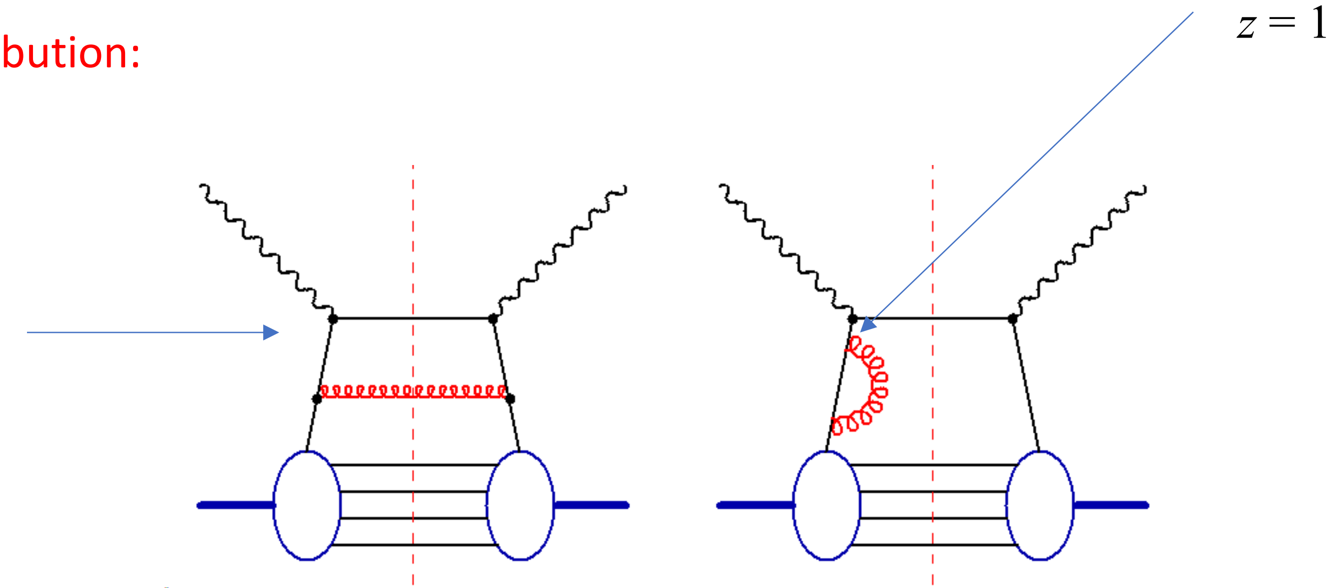
$$\int_0^1 dz\,(\ldots)_+\,g(z) = \int_0^1 dz\,(\ldots)\,[g(z)-g(1)]$$

appears because of the virtual diagram for which $z = 1$

Altarelli-Parisi probabilities

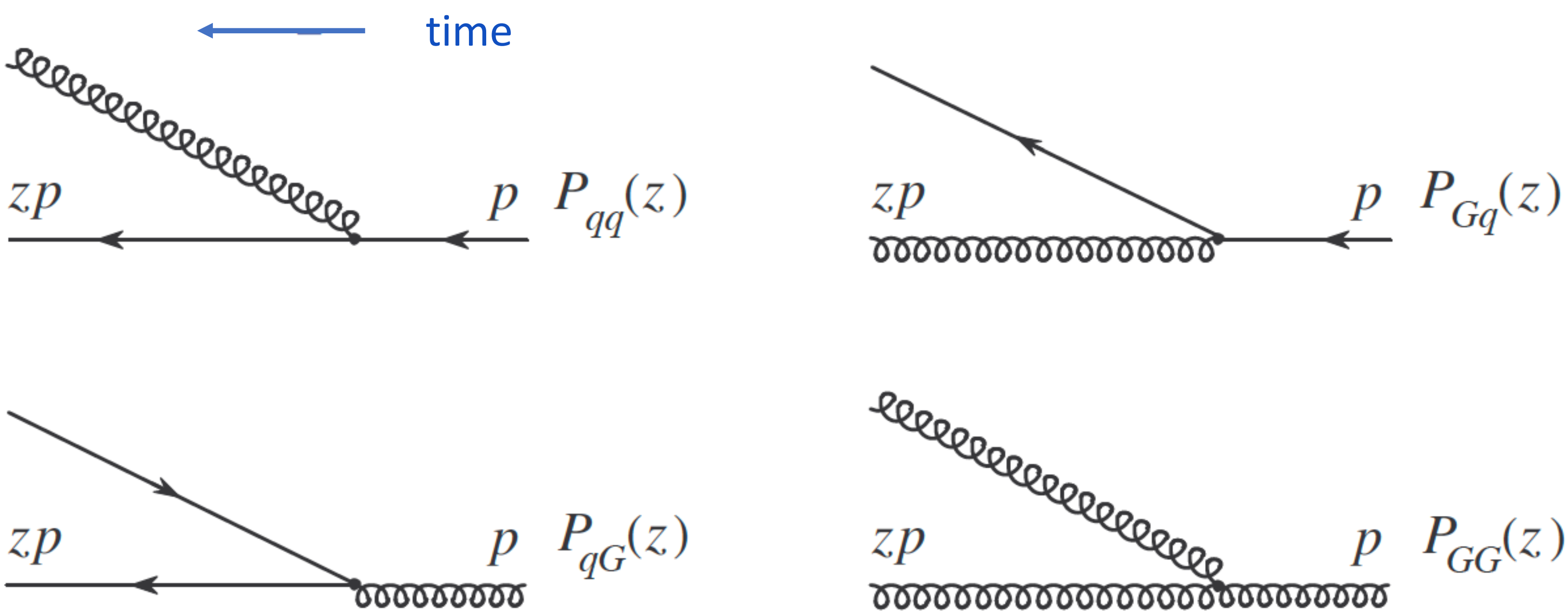
“Plus” distribution:

$\int^{1} dz$



Different diagrams give extra contribution at $z = 1$ in different gauges.
The result is the same: no singularity at $z = 1$.

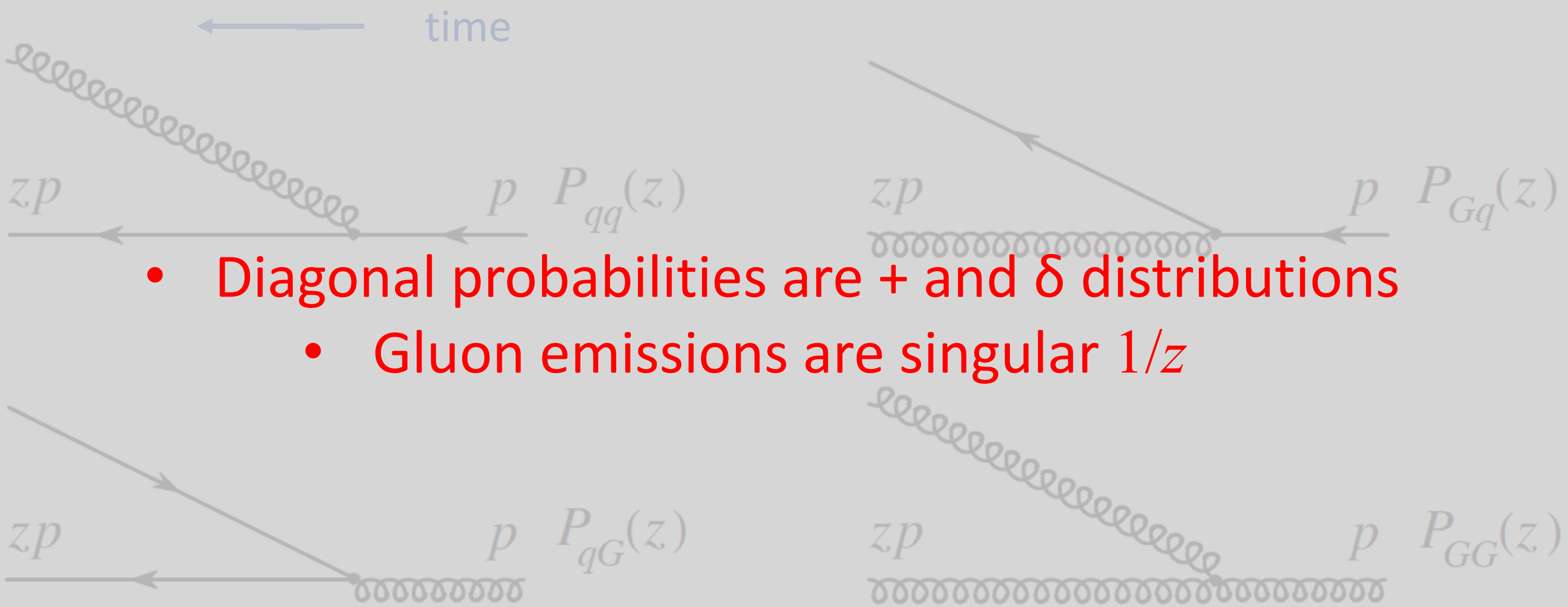
Altarelli-Parisi probabilities



$$P_{qq}(z) = C_F \left(\frac{1+z^2}{1-z}\right)_+ , \quad P_{Gq}(z) = C_F \frac{1+(1-z)^2}{z}, \quad P_{qG}(z) = \frac{1}{2}\left[z^2 + (1-z)^2\right]$$

$$P_{GG}(z) = 2C_A \left[\frac{z}{(1-z)_+} + \frac{1-z}{z} + z(1-z)\right] + \frac{1}{2}\left(\frac{11}{3}C_A - \frac{2}{3}n_f\right)\delta(1-z)$$

Altarelli-Parisi probabilities



- Diagonal probabilities are + and δ distributions
  - Gluon emissions are singular $1/z$

$$P_{qq}(z) = C_F \left(\frac{1+z^2}{1-z} \right)_+ , \quad P_{Gq}(z) = C_F \frac{1+(1-z)^2}{z}, \quad P_{qG}(z) = \frac{1}{2} \left[z^2 + (1-z)^2 \right]$$

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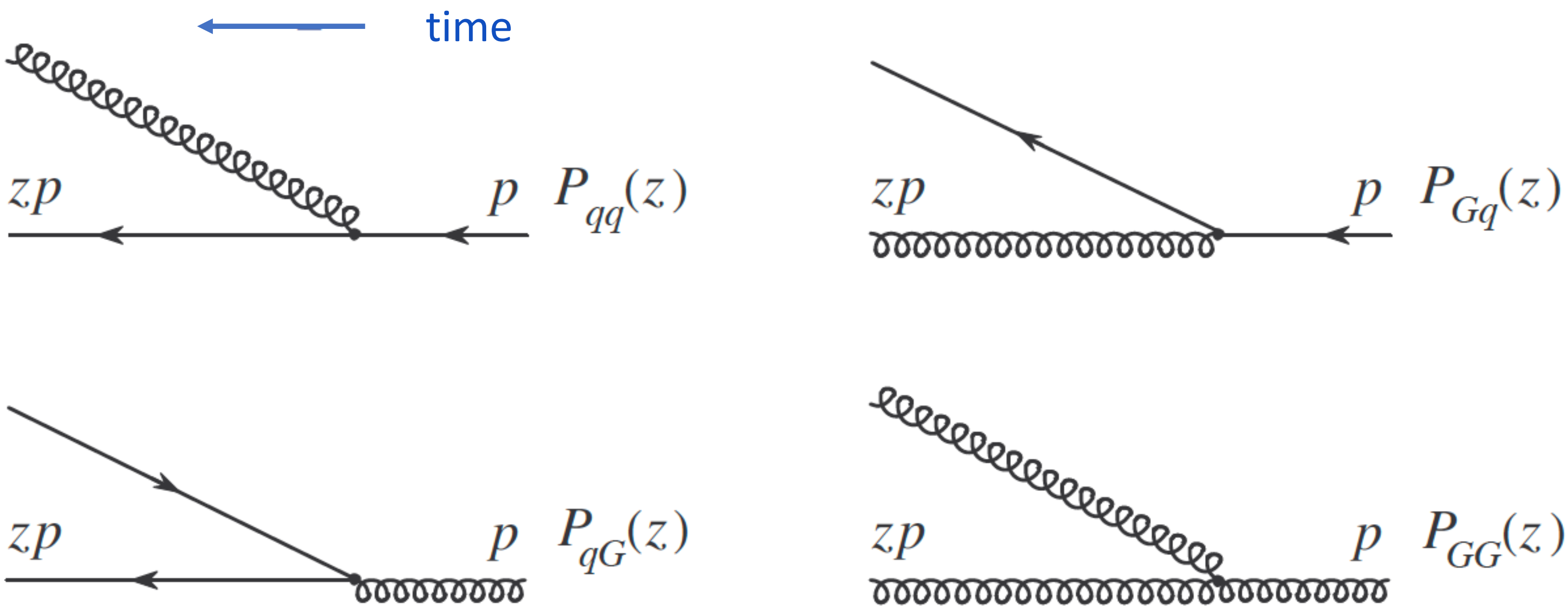
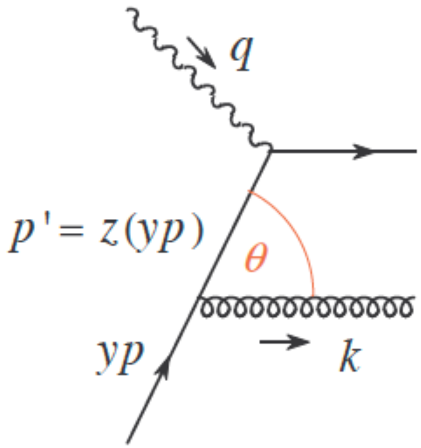
Altarelli-Parisi probabilities



$$P_{qG}(z) = P_{\overline{q}G}(z), \qquad P_{Gq}(z) = P_{G\overline{q}}(z),$$

$$P_{qq}(z) = P_{Gq}(1-z), \; P_{GG}(z) = P_{GG}(1-z), \quad P_{qG}(z) = P_{qG}(1-z)$$

QCD corrections to parton model

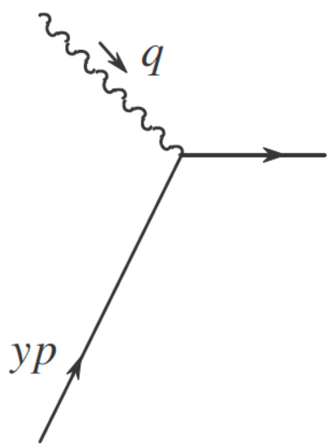


on-shell condition

$$0 = (zyp + q)^2 = 2zy\, pq + q^2 = 2M\nu\, zy - Q^2$$

$$zy = \frac{Q^2}{2M\nu} = x$$

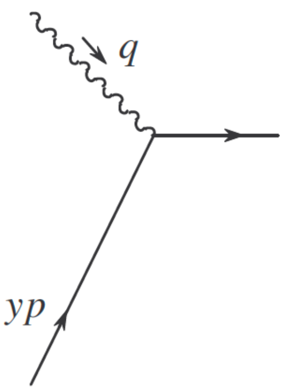
Recall F_1: $$F_1(x) = \frac{1}{2}\sum_i e_i^2\, f_i(x)$$



Lowest order diagram:

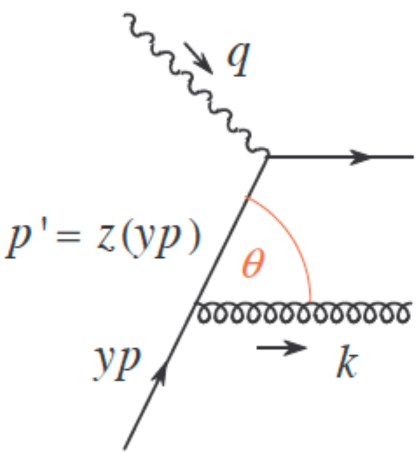
$$2F_1(x) = e_q^2 \int_0^1 dy q(y)\delta(y - x)$$

QCD corrections to parton model



Recall F_1:

$$2F_1(x) = e_q^2 \int_0^1 dy q(y) \delta(y - x)$$

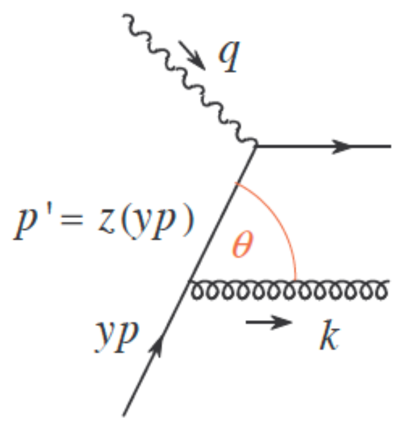


$$2\Delta F_1(x) = e_q^2 \frac{\alpha_s}{2\pi} \int_0^1 dy q(y) \int_0^1 dz \delta(zy - x) \left[P_{qq}(z) \ln \frac{Q^2}{\mu^2} + \cancel{C(z)} \right]$$

Correction to F_1 large logs

$$q(x,Q^2) = q(x,\mu^2) + \frac{\alpha_s}{2\pi}\ln\frac{Q^2}{\mu^2}\int_x^1 \frac{dy}{y} P_{qq}\left(\frac{x}{y}\right) q(y,\mu^2) + \ldots$$

$$= q(x,\mu^2) + \frac{\alpha_s}{2\pi}\ln\frac{Q^2}{\mu^2} P_{qq} \otimes q(\mu^2)$$ -



Convolution:

$$P_{qq} \otimes q = \int_0^1 dz \int_0^1 dy\, \delta(zy - x) P_{qq}(z) q(y)$$

Integration over $d\theta$ gave a pole

DGLAP Evolution Equation

$$\frac{d}{d\ln Q^2} = Q^2 \frac{d}{dQ^2} \quad \Rightarrow \quad q(x, Q^2) = q(x, \mu^2) + \frac{\alpha_s}{2\pi} \ln \frac{Q^2}{\mu^2} P_{qq} \otimes q(\mu^2) + \ldots$$

Evolution eq.
Dokshitzer,
Gribov, Lipatov
Altarelli, Parisi

$$\frac{d}{d\ln Q^2} q(x, Q^2) = \frac{\alpha_s}{2\pi} P_{qq} \otimes q(Q^2)$$

Such an equation sums up all powers $\frac{\alpha_s}{2\pi} \ln \frac{Q^2}{\mu^2}$.

Leading Log Approximation (LLA)

DGLAP Evolution Equations

Full set of DGLAP equations:

$$Q^2 \frac{d}{dQ^2} q_i(x, Q^2) = \frac{\alpha_s(Q^2)}{2\pi} \left[P_{qq} \otimes q_i(Q^2) + P_{qG} \otimes G(Q^2) \right]$$

$$Q^2 \frac{d}{dQ^2} G(x, Q^2) = \frac{\alpha_s(Q^2)}{2\pi} \left[P_{Gq} \otimes \sum_i q_i(Q^2) + P_{GG} \otimes G(Q^2) \right]$$

We need an input at one scale Q_0^2 and then we can evolve them up to some other Q^2
note that index i runs over quarks and antiquarks
when we construct a difference, called non-singlet, gluons cancel

$$q_i^{NS}(x, Q^2) = q_i(x, Q^2) - \overline{q}_i(x, Q^2)$$

DGLAP Evolution Equations

Define:

singlet

$$q^S(x, Q^2) = \sum_i \left(q_i(x, Q^2) + \overline{q}_i(x, Q^2)\right)$$

nonsinglet

$$q_i^{NS}(x, Q^2) = q_i(x, Q^2) - \overline{q}_i(x, Q^2)$$

DGLAP Evolution Equations

$$Q^2 \frac{d}{dQ^2} q^{NS}(x, Q^2) = \frac{\alpha_s(Q^2)}{2\pi} P_{qq} \otimes q^{NS}(Q^2)$$

$$Q^2 \frac{d}{dQ^2} q^{S}(x, Q^2) = \frac{\alpha_s(Q^2)}{2\pi} \left[P_{qq} \otimes q^{S}(Q^2) + 2n_f P_{qG} \otimes G(Q^2) \right]$$

$$Q^2 \frac{d}{dQ^2} G(x, Q^2) = \frac{\alpha_s(Q^2)}{2\pi} \left[P_{Gq} \otimes q^{S}(Q^2) + P_{GG} \otimes G(Q^2) \right]$$

DGLAP for Mellin moments

Moments of the convolution

$$M_n = \int_0^1 dx\, x^{n-1} P \otimes f = \int_0^1 dx\, x^{n-1} \int_0^1 dz \int_0^1 dy \delta(zy - x) P(z) f(y)$$

$$= \int_0^1 dz\, z^{n-1} P(z) \int_0^1 dy\, y^{n-1} f(y) = P_n\, f_n = \gamma^n f_n$$

γ^n anomalous dimension

convolution is replaced by a product

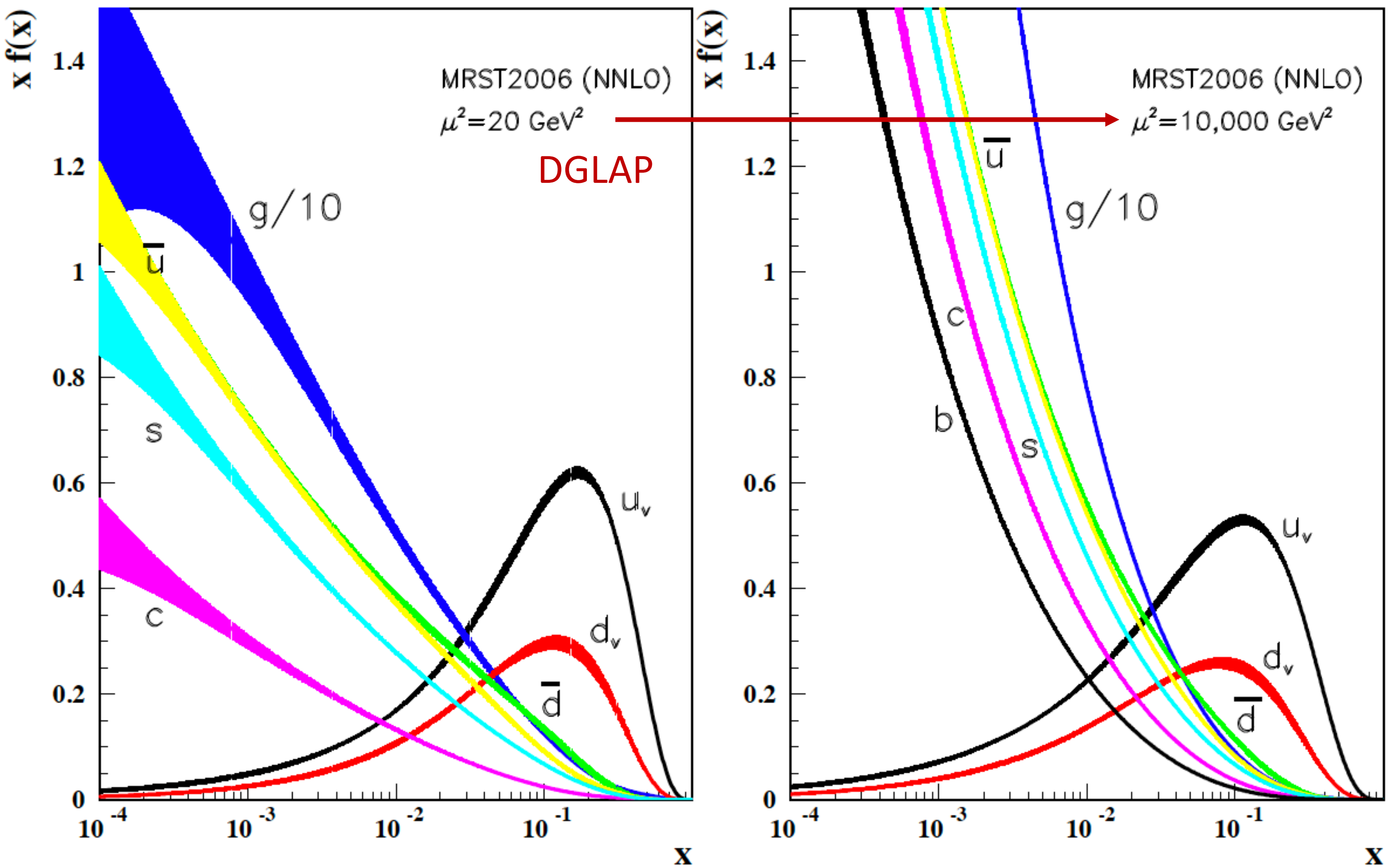
DGLAP for Mellin moments

$$t = \log Q^2 \qquad \frac{dq_n^{NS}(t)}{dt} = \frac{\alpha_s(t)}{2\pi}\gamma_{qq}^n q_n^{NS}(t)$$

$$\frac{d}{dt}\begin{bmatrix} q_n^S(t) \\ G_n(t) \end{bmatrix} = \frac{\alpha_s(t)}{2\pi}\begin{bmatrix} \gamma_{qq}^n & 2n_f\gamma_{qG}^n \\ \gamma_{Gq}^n & \gamma_{GG}^n \end{bmatrix}\begin{bmatrix} q_n^S(t) \\ G_n(t) \end{bmatrix}$$

$$\frac{\alpha_s(t)}{2\pi} = 2\,a_s(t) = 2\,\frac{1}{\beta_0 t}$$

Numerical solutions



HERA F_2: data vs. theory

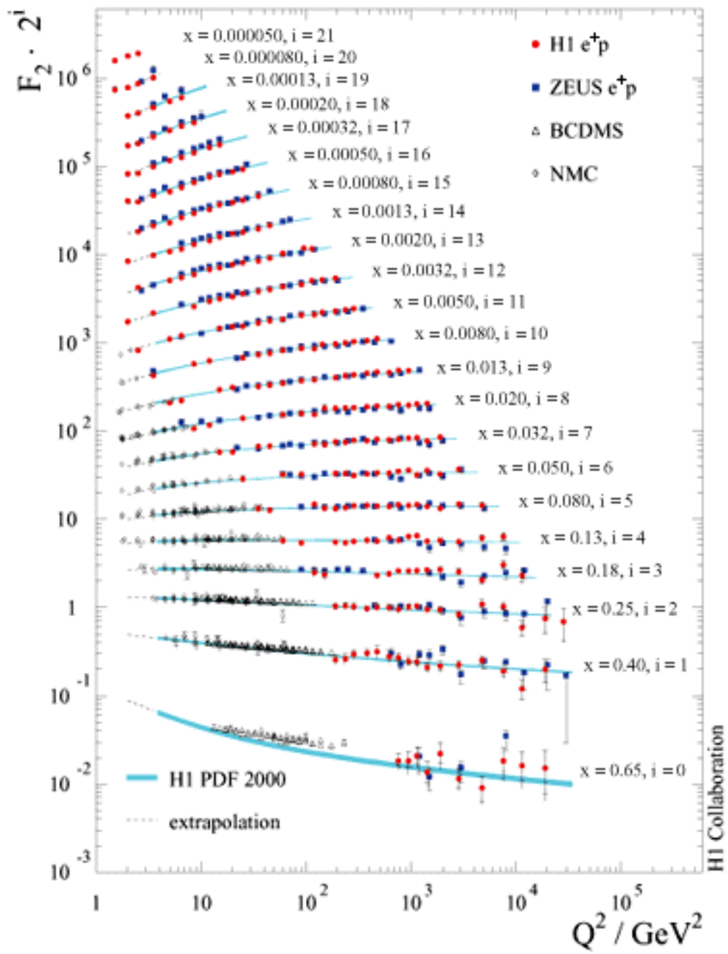


FIG. 2: Structure function F_2 as a function of Q^2 based on HERA-I measurements of H1 [2, 3] and ZEUS [4] collaboration compared to results from fixed target experiments BCDMS [5] and NMC [6].