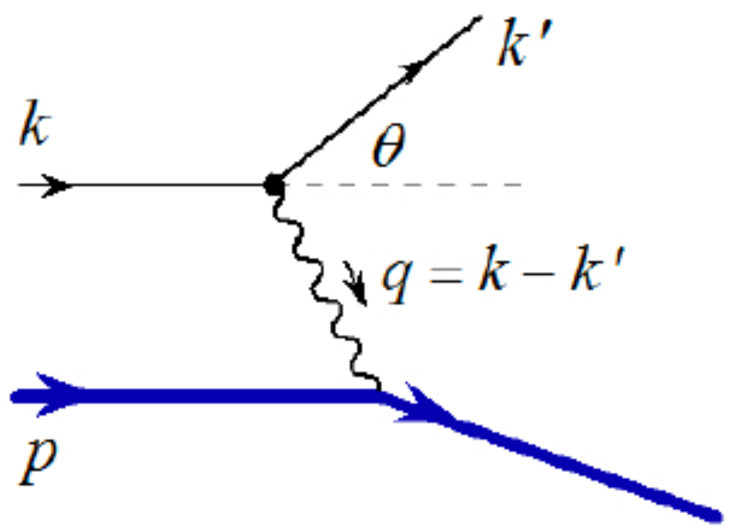
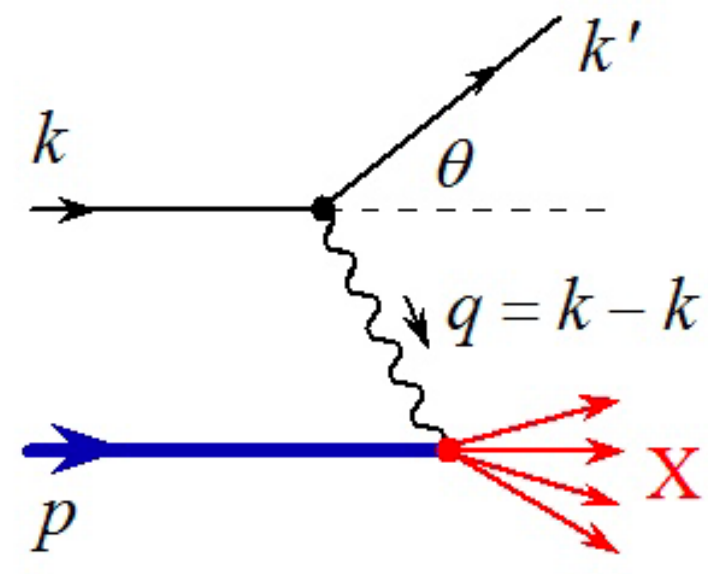
QCD Lecture 3

October 22

Deep Inelastic Scattering (DIS)





$$
\begin{aligned}
p &= M(1, 0, 0, 0), \\
k &= \omega(1, 0, 0, 1), \\
k' &= \omega'(1,\ \sin\theta \sin\varphi,\ \sin\theta\cos\varphi,\ \cos\theta) \\
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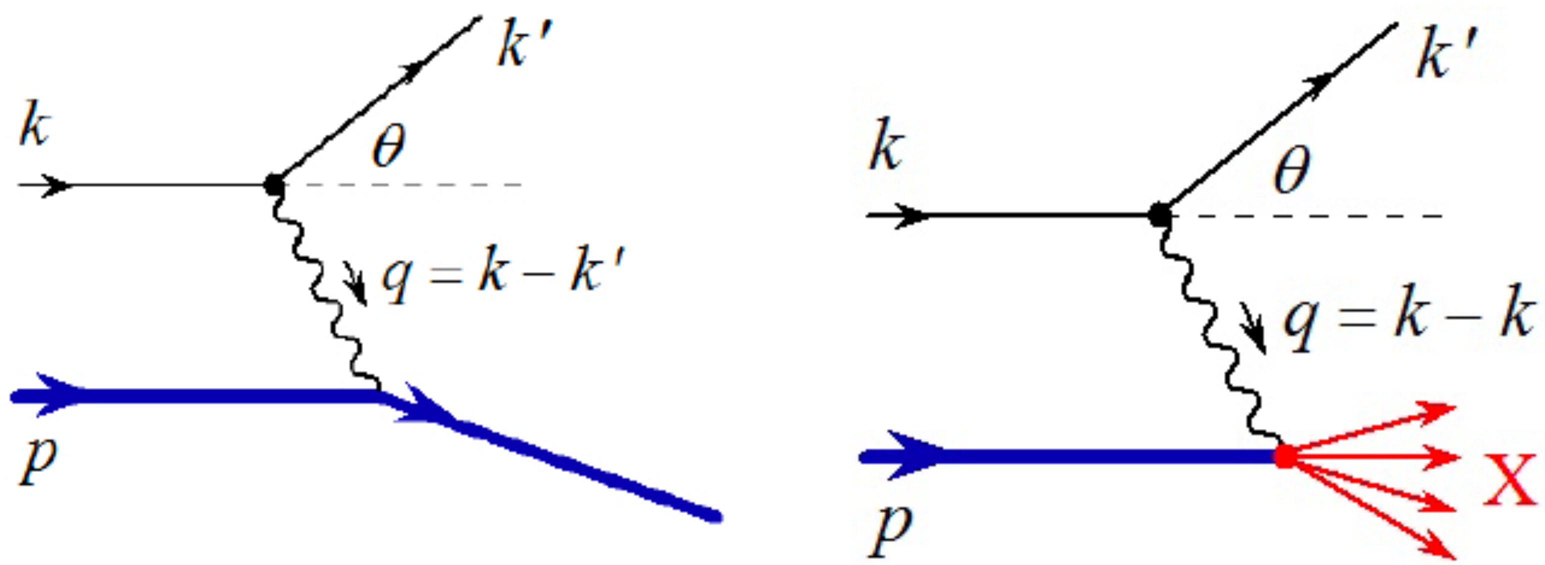
Deep Inelastic Scattering (DIS)



4-momentum transfer and energy transfer

$$q^2 = -2\omega\omega'(1-\cos\theta) = -4\omega\omega'\sin^2\frac{\theta}{2}, \qquad \nu = \omega - \omega'$$

on mass-shell condition for scattered proton (not present in the inelastic case):

$$Q^2 = -q^2 \qquad \delta((p+q)^2 - M^2) = \delta(2M\nu - Q^2) = \frac{1}{2M}\delta\left(\nu - \frac{Q^2}{2M}\right)$$

Elastic cross-section

on elementary fermion

$$\frac{d\sigma}{dQ^2} = \frac{\pi\alpha^2}{4\omega^2 \sin^4\frac{\theta}{2}} \frac{e_p^2}{\omega\omega'} \left\{ \cos^2\frac{\theta}{2} + \frac{Q^2}{2M^2}\sin^2\frac{\theta}{2} \right\} \qquad \alpha = \frac{e^2}{4\pi}$$

Inelastic cross-section

on non-elementary fermion (proton)

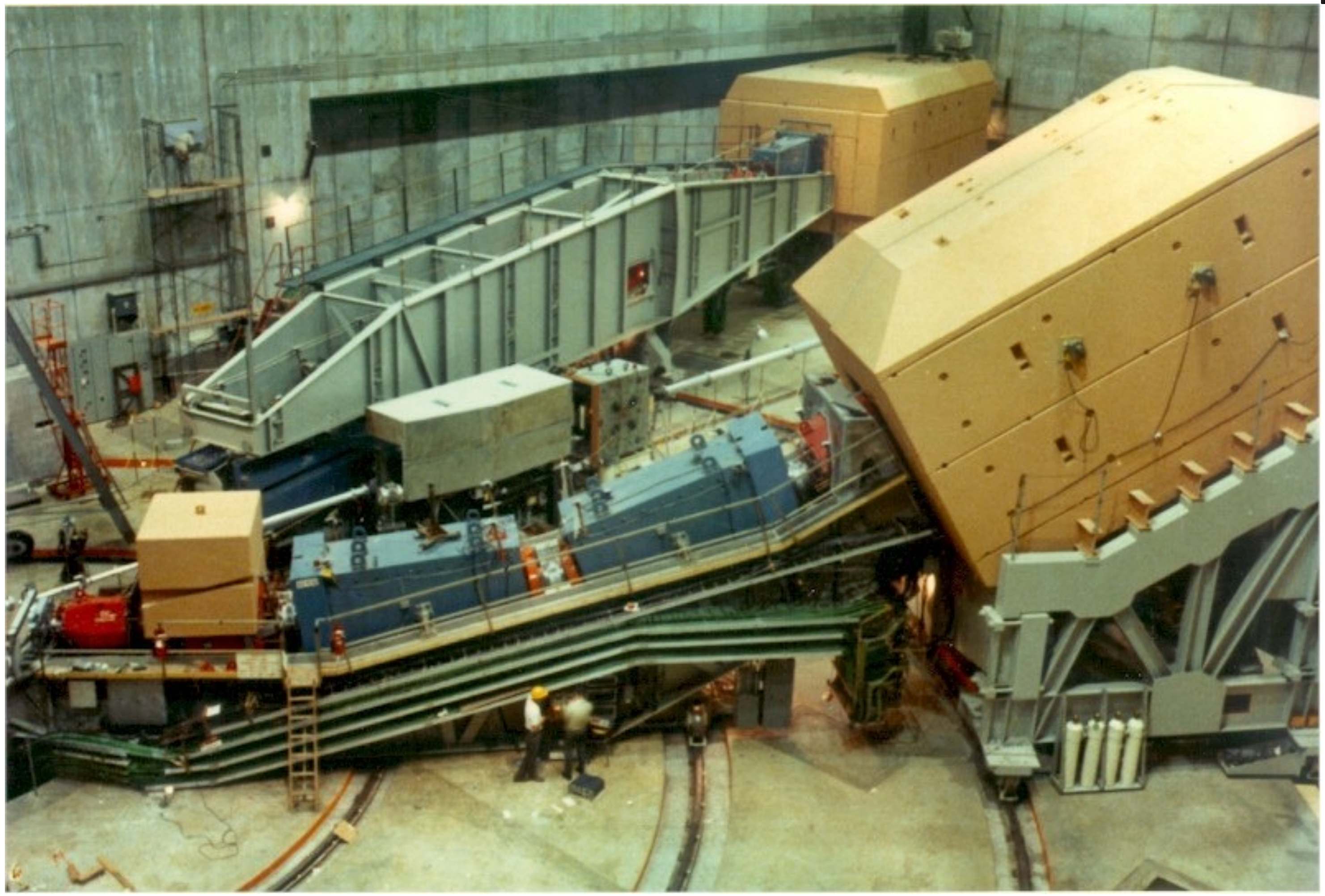
$$\frac{d\sigma}{dQ^2 d\nu} = \frac{\pi\alpha^2}{4\omega^3\omega' \sin^4\frac{\theta}{2}} \left\{ W_2(Q^2,\nu)\cos^2\frac{\theta}{2} + 2W_1(Q^2,\nu)\sin^2\frac{\theta}{2} \right\}$$

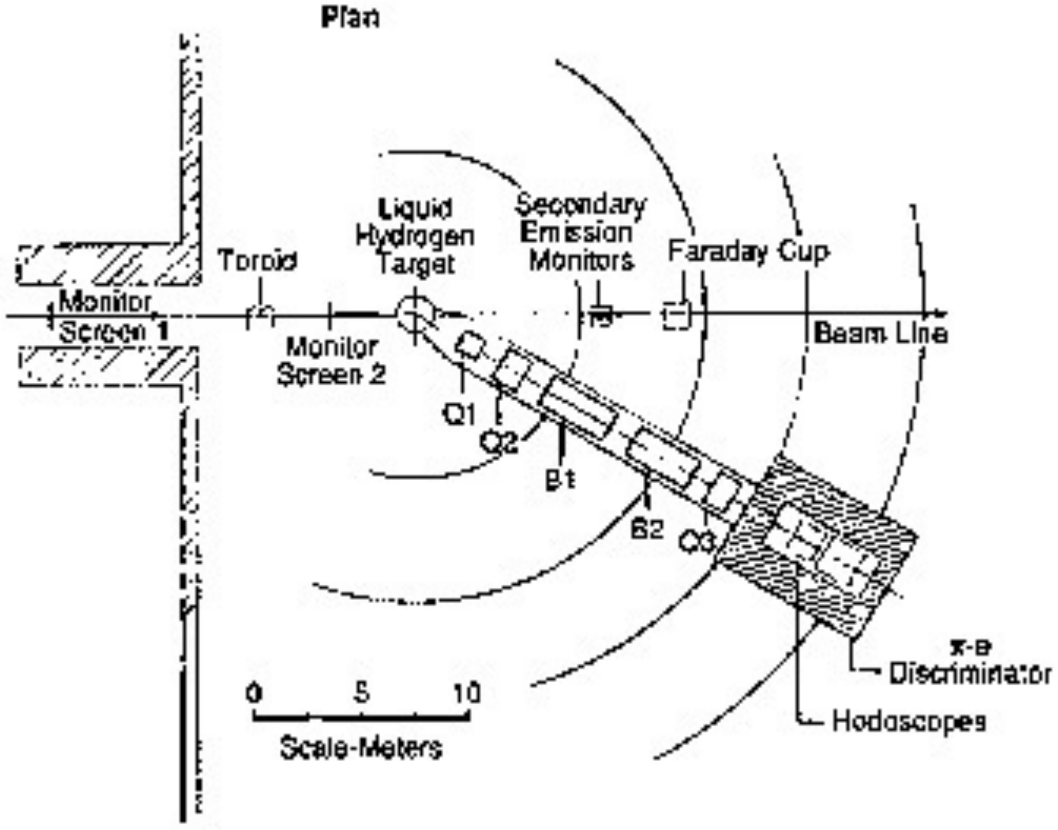
$$MW_1(Q^2,\nu) = F_1$$

$$\nu W_2(Q^2,\nu) = F_2$$

SLAC

SLAC built in1967
Length ~ 2 miles
Energy: 20 GeV





1968: convinced by James Bjorker analysis of DIS has been made

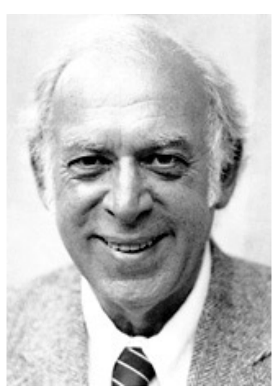
Interpretation was given by Richard Feynman

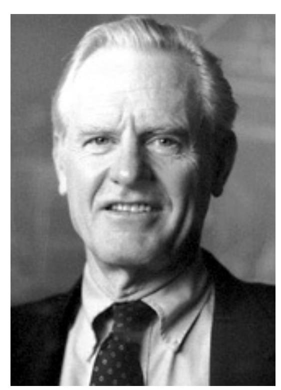
Nobel 1990:

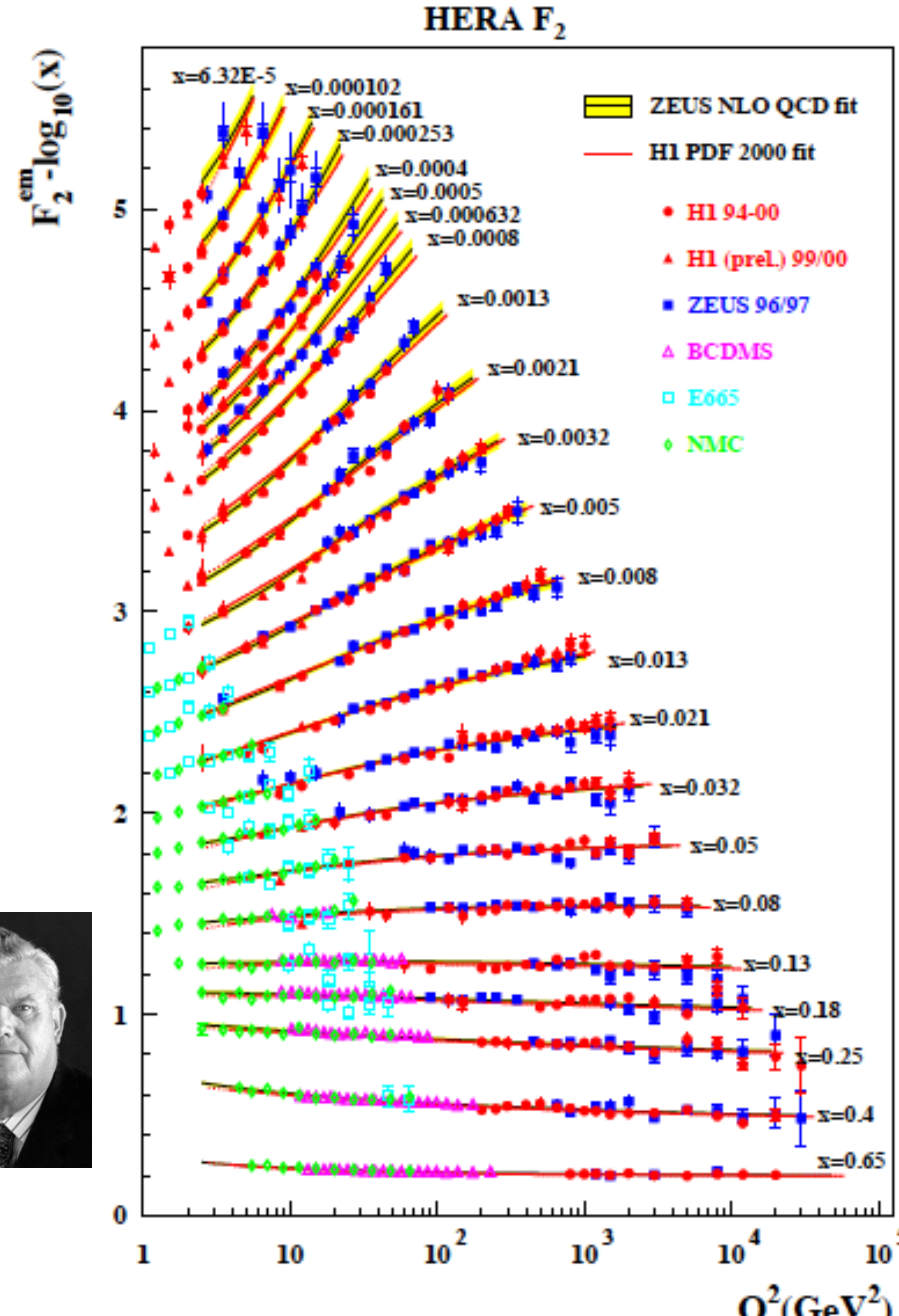
Jerome Friedman (MIT)

Henry Kendall (MIT)

Richard Taylor (SLAC)







Bjorken Scaling

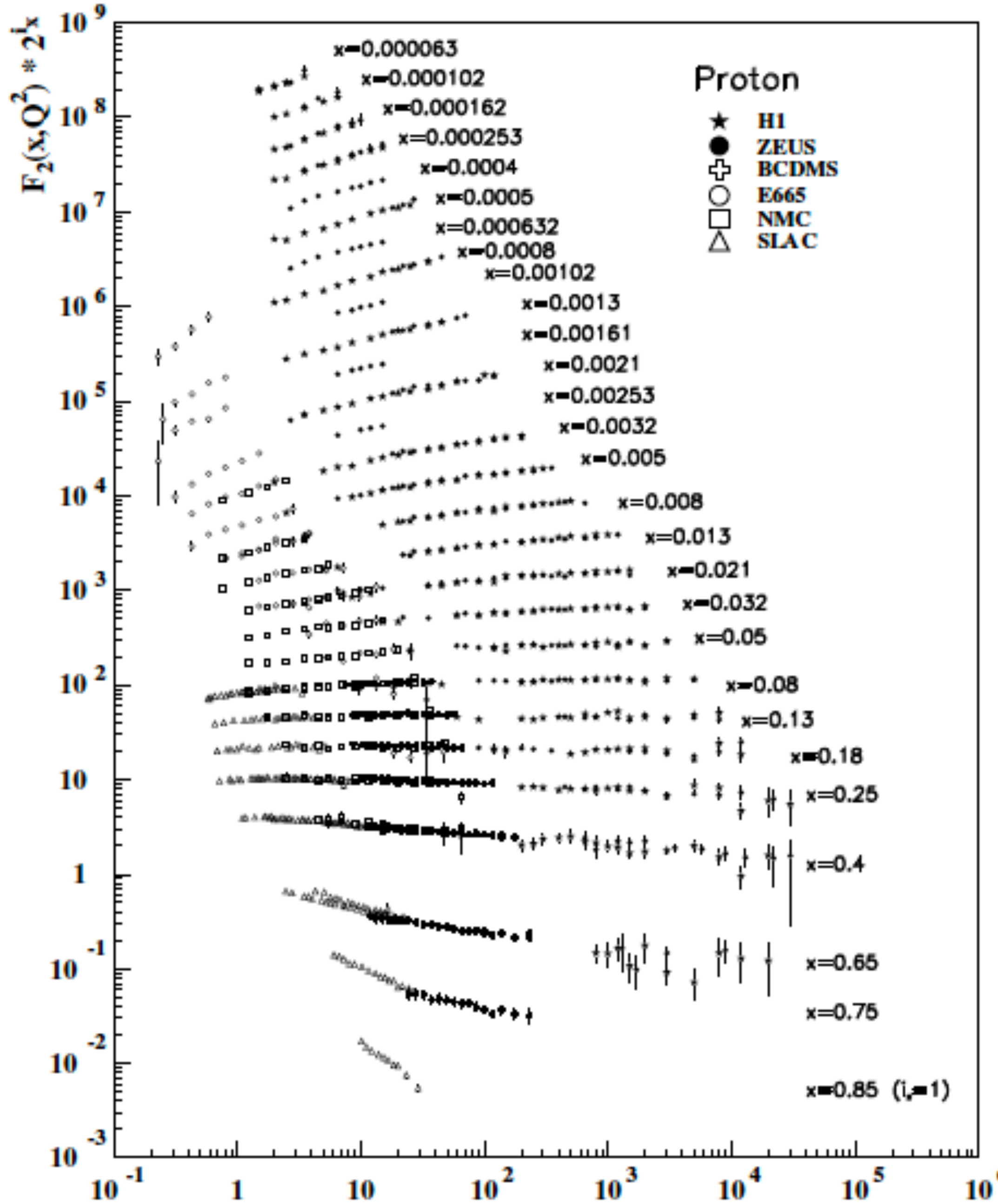
Bjorken limit:

$$Q^2, \nu \to \infty \qquad Q^2/\nu$$

$$MW_1(Q^2, \nu) = F_1(x)$$
$$\nu W_2(Q^2, \nu) = F_2(x)$$

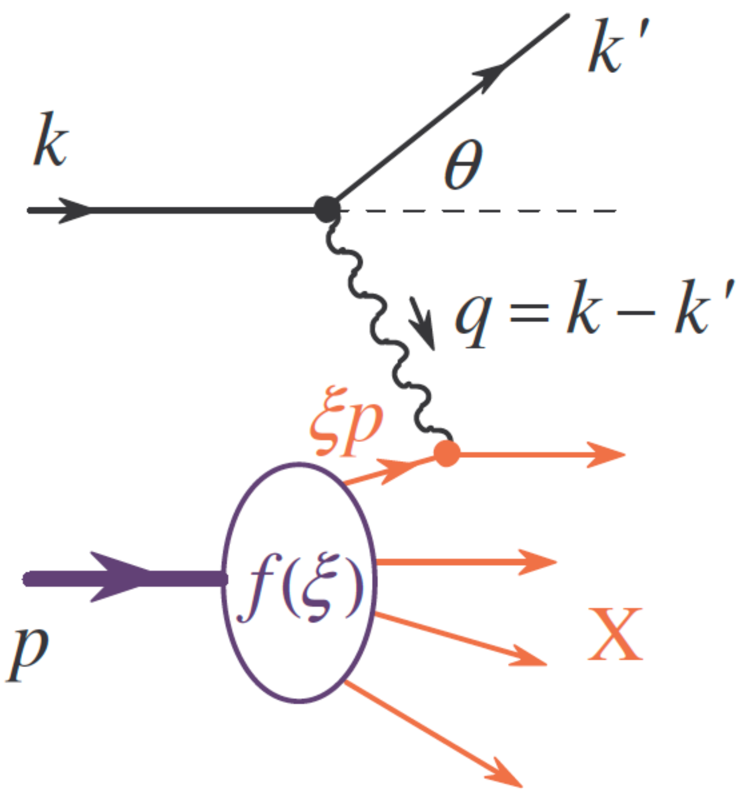
where:

$$x = \frac{Q^2}{2M\nu}$$



Feynman Parton Model

Inelastic scattering on proton
is a sum of elastic scattrings on partons
that are parallel to p
and carry momentum fraction ξ



In the proton rest frame we have to assume that parton mass is

$$m_\xi = \xi M$$

then the on-shell condition for the struck parton reads

$$(\xi p + q)^2 = m_\xi^2$$

$$\xi^2 M^2 + 2\xi M\nu - Q^2 = \xi^2 M^2 \;\rightarrow\; \xi = \frac{Q^2}{2M\nu} = x$$

ξ is the same as Bjorken x !

Reminder: S matrix and cross-section

In far past we prepare an initial state: $\lim_{t\to-\infty} |\psi(t)\rangle = |i\rangle$

and ask, what is the probability amplitude that this state evolves into some final state in far future:

$$S_{fi} \stackrel{\text{df}}{=} \langle f|\, S\, |i\rangle$$

There exists formal expression for operator S

$$S = T e^{-i\int d^4x' \mathcal{H}_1(x')}$$

where T denotes time ordering and $\mathcal{H}_1$ is an interaction hamiltonian

Reminder: S matrix and cross-section

$$S_{fi} = \delta_{fi} - i(2\pi)^4 \delta^{(4)}(P_f - P_i)\, M_{fi}$$

We have separated non-interacting part and energy-momentum conservation.

In order to define the cross-section we have to close the system
In a box: $V = L^3$ and finite time T

Then momenta are quantized $\vec{p} = \frac{2\pi}{L}(n_1, n_2, n_3)$

Reminder: S matrix and cross-section

In a box:

$$\delta^{(3)}(p - q) \to \frac{1}{(2\pi)^3}\int_V d^3x\, e^{i(\vec{p}-\vec{q})\cdot\vec{x}} = \frac{V}{(2\pi)^3}\delta_{p,q}$$

$$\delta(E_p - E_q) \to \frac{1}{2\pi}\int_0^T dt\; e^{i(E_p - E_q)t} = \frac{T}{2\pi}\delta_{p,q}$$

and a state with particle having momentum p

$$\left|p^{\mathrm{box}}\right\rangle = \left[\frac{(2\pi)^3}{V}\right]^{1/2} |p\rangle$$

$$\langle q|p\rangle = \delta^{(3)}(\vec{p} - \vec{q}) \longrightarrow \left\langle q^{\mathrm{box}}|p^{\mathrm{box}}\right\rangle = \delta_{q,p}$$

Reminder: S matrix and cross-section

Transition probability for $f \neq i$

$$P\left(i \longrightarrow f\right) = \left|S_{fi}^{\text{box}}\right|^2 = (2\pi)^8 \left[\delta_{\text{box}}^{(4)}\left(P_f - P_i\right)\right]^2 \left|M_{fi}^{\text{box}}\right|^2$$

Square of Dirac delta is poorly defined, but in a box there is no problem

$$P\left(i \longrightarrow f\right) = VT\,(2\pi)^4\, \delta_{\text{box}}^{(4)}\left(P_f - P_i\right) \left[\frac{(2\pi)^3}{V}\right]^{N_f + N_i} \left|M_{fi}\right|^2$$

where $N_{i,f}$ denotes number of initial and final particles

Reminder: S matrix and cross-section

Differential transition probability

$$dP\left(i \longrightarrow f\right) = P\left(i \longrightarrow f\right)\, dN_f$$

Where

$$dN_f = \frac{V d^3 q_1}{\left(2\pi\right)^3} \cdots \frac{V d^3 q_{N_f}}{\left(2\pi\right)^3}$$

and

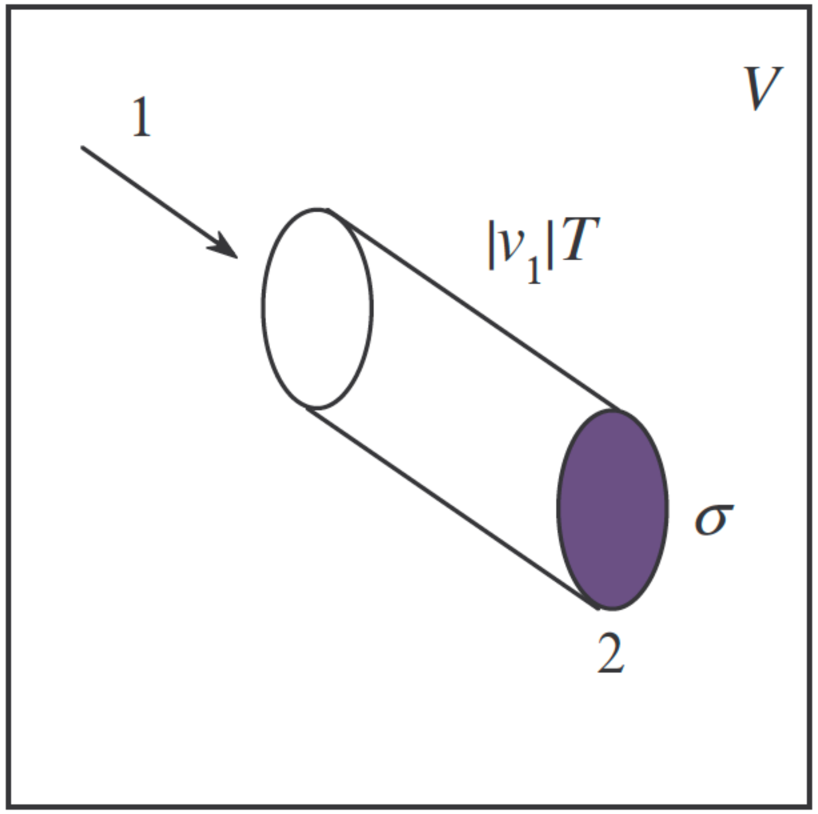
$$dP\left(i \longrightarrow f\right) = VT\left(2\pi\right)^4 \delta^{(4)}\left(P_f - P_i\right)\left[\frac{\left(2\pi\right)^3}{V}\right]^{N_i} \left|M_{fi}\right|^2 d^3 q_1 \ldots d^3 q_{N_f}$$

Cross-section

Consider $p_1 + p_2 \to q_1 + q_2 + \ldots + q_n$

and define "effective area" σ
Probability to hit σ is equal to the ratio of volumes

$$P(1+2\to n) = \frac{|\vec{v}_1|\, T\sigma_{1+2\to n}}{V}$$



Matrix element can be computed using Feynman rules provided we define

$$M_{fi} = \left[\frac{1}{(2\pi)^3 2E(p_1)} \cdots \frac{1}{(2\pi)^3 2E(p_{N_i})} \frac{1}{(2\pi)^3 2E(q_1)} \cdots \frac{1}{(2\pi)^3 2E(q_{N_f})}\right]^{1/2} \mathcal{M}_{fi}$$

Cross-section

$$\sigma_{1+2\to n} = \frac{V}{|\vec{v}_1|\,T} VT \frac{1}{2VE(p_1)} \frac{1}{2VE(p_2)} \left|\mathcal{M}_{fi}\right|^2$$

$$\times (2\pi)^4 \delta^{(4)}(P_f - P_i) \frac{d^3 q_1}{(2\pi)^3 2E(q_1)} \cdots \frac{d^3 q_{N_f}}{(2\pi)^3 2E\left(q_{N_f}\right)}$$

Note that time T and volume V cancel. From initial particles we get flux factor: $\frac{1}{4E_1 E_2 |\vec{v}_1|}$

where

$$F = 4E_1 E_2 |\vec{v}_1| = 4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}$$

Cross-section 1+2 → n

$$d\sigma_{1+2\to n} = \frac{1}{4\sqrt{(p_1\cdot p_2)^2 - M_1 M_2}} \int \prod_{i=1}^{n} \left(\frac{d^4 k_i}{(2\pi)^3} \delta_+(k_i^2 - m_i^2) \right) \overline{|\mathcal{M}_{fi}|^2} \, (2\pi)^4 \delta^{(4)} \left(p_1 + p_2 - \sum_i k_i \right)$$

flux — final state integration with on-shell cond. — matrix element — mom.-energy conservation

$$\overline{|\mathcal{M}_{fi}|^2} = \frac{1}{(2s_1+1)(2s_2+1)} \sum_{pol} |\mathcal{M}_{fi}|^2$$

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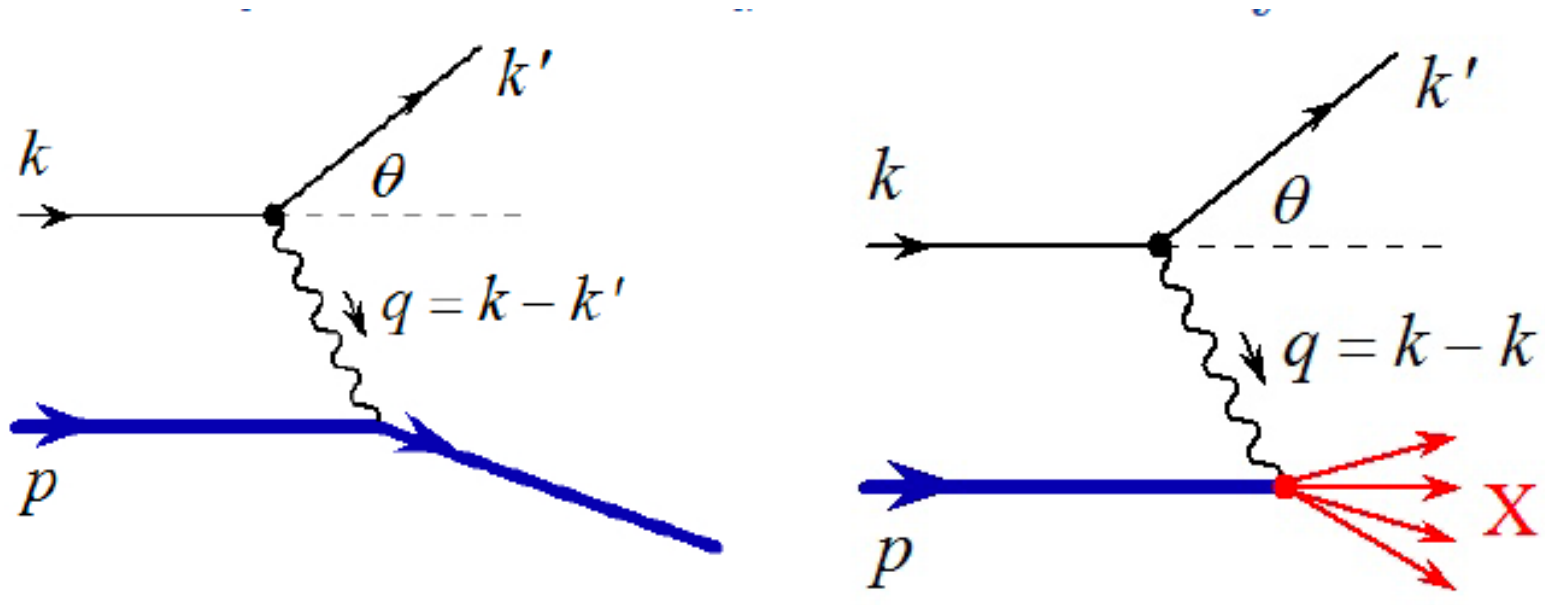

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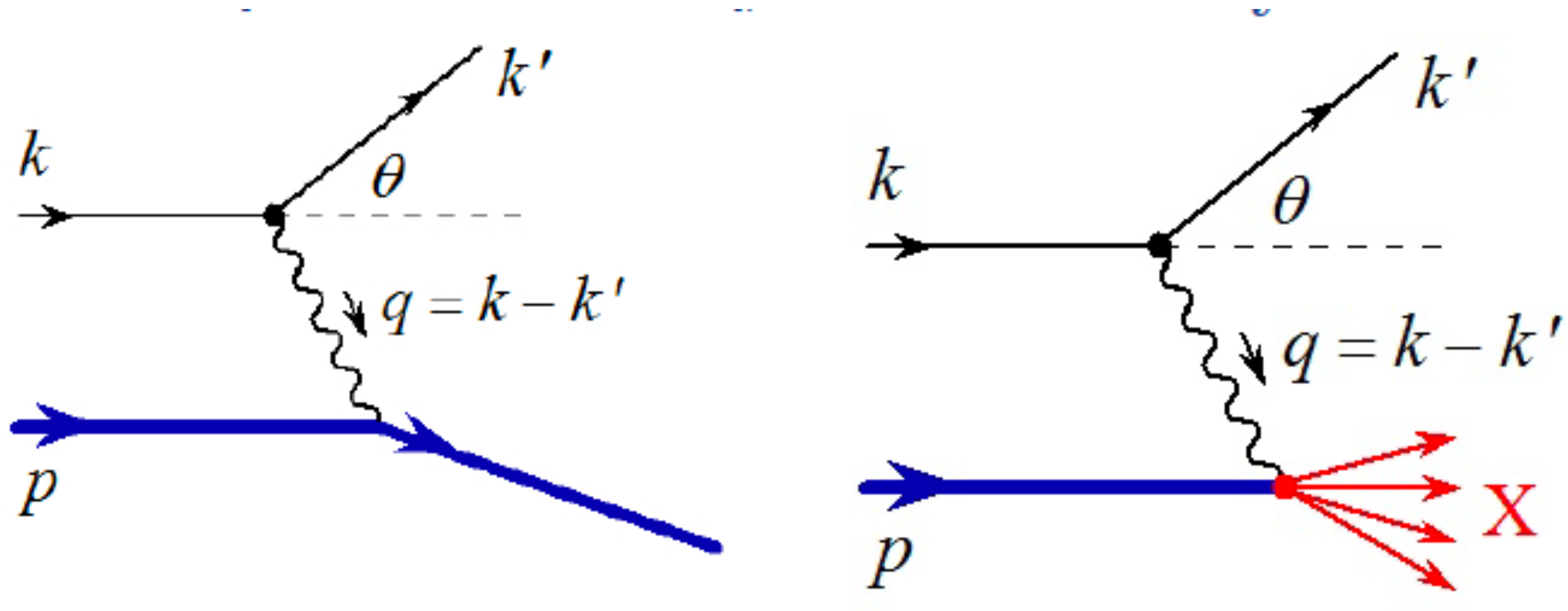


4-momentum transfer and energy transfer

$$q^2 = -2\omega\omega'(1-\cos\theta) = -4\omega\omega'\sin^2\frac{\theta}{2}, \qquad \nu = \omega - \omega'$$

on mass-shell condition for scattered proton (not present in the inelastic case):

$$\delta((p+q)^2 - M^2) = \delta(2M\nu - Q^2) = \frac{1}{2M}\delta\left(\nu - \frac{Q^2}{2M}\right)$$

Elastic cross-section

$$d\sigma = \frac{1}{4M\omega} \int \frac{1}{4} \sum_{\text{pol}} |\mathcal{M}_{fi}|^2 \, \frac{d^4k'}{(2\pi)^3} \delta(k'^2) \, \frac{d^4p'}{(2\pi)^3} \delta(p'^2 - M^2) \, (2\pi)^4 \delta(k + p - k' - p')$$

flux — matrix element — final state integration with on-shell cond. — mom.-energy conservation

perform dp' integration first, then

$$d\sigma = \frac{1}{4M\omega} \frac{1}{(2\pi)^2} \int \frac{1}{4} \sum_{\text{pol}} |\mathcal{M}_{fi}|^2 \, \underbrace{d\omega' d^3\boldsymbol{k}' \delta(\omega'^2 - k'^2)}_{=I} \, \delta((p+q)^2 - M^2)$$

compute I

Elastic cross-section

$$I = \int d\omega' d^3\boldsymbol{k}'\delta(\omega'^2 - \boldsymbol{k}'^2) = \int \boldsymbol{k}'^2 d|\boldsymbol{k}|' d\varphi\, d\cos\theta\, d\omega'\delta(\omega'^2 - \boldsymbol{k}'^2)$$

$$= 2\pi \int d\cos\theta \frac{\omega'^2 d\omega'}{2\omega'} = \pi \int \omega' d\omega' d\cos\theta.$$

We have assumed that the matrix element does not depend on φ

Change of variables:

$$Q^2 = -q^2 = 2\omega\omega'(1 - \cos\theta)$$
$$\nu = \omega - \omega',$$

Jacobian:

$$d\omega' d\cos\theta = \left|\frac{d(\omega', \cos\theta)}{d(\nu, Q^2)}\right| dQ^2 d\nu = \frac{1}{2\omega\omega'} dQ^2 d\nu$$

$$I = \frac{\pi}{2\omega} \int dQ^2 d\nu$$

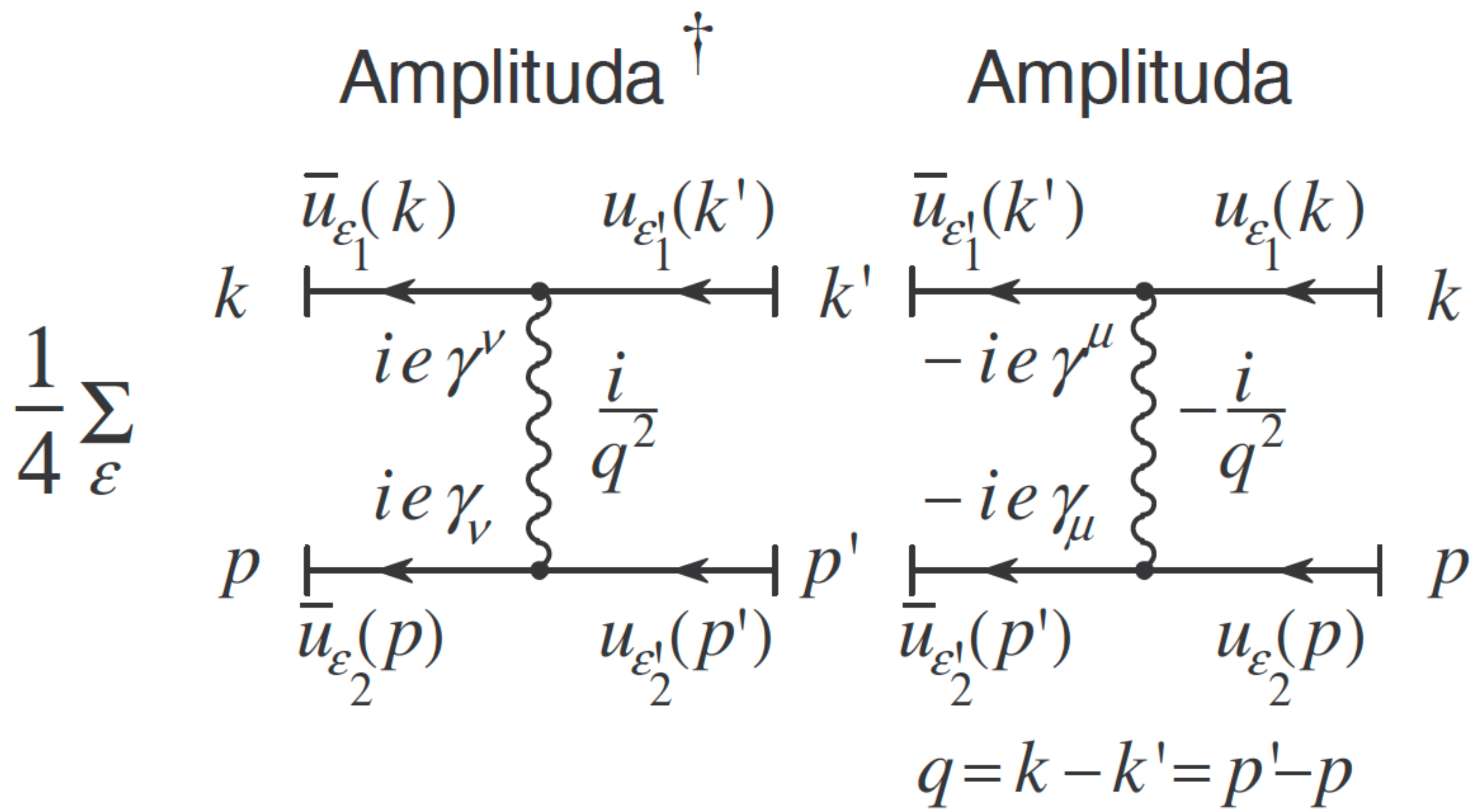
Elastic cross-section

$$\delta((p+q)^2 - M^2) = \delta(2M\nu - Q^2) = \frac{1}{2M}\delta\left(\nu - \frac{Q^2}{2M}\right)$$

$$\frac{d\sigma}{dQ^2} = \frac{1}{16M^2\omega^2}\frac{1}{4\pi}\int d\nu\,\frac{1}{4}\sum_{\text{pol}}|\mathcal{M}_{fi}|^2\,\delta\left(\nu - \frac{Q^2}{2M}\right)$$

Next: calculate amplitude squared averaged over initial polarizations and summed over final polarizations

Amplitude squared

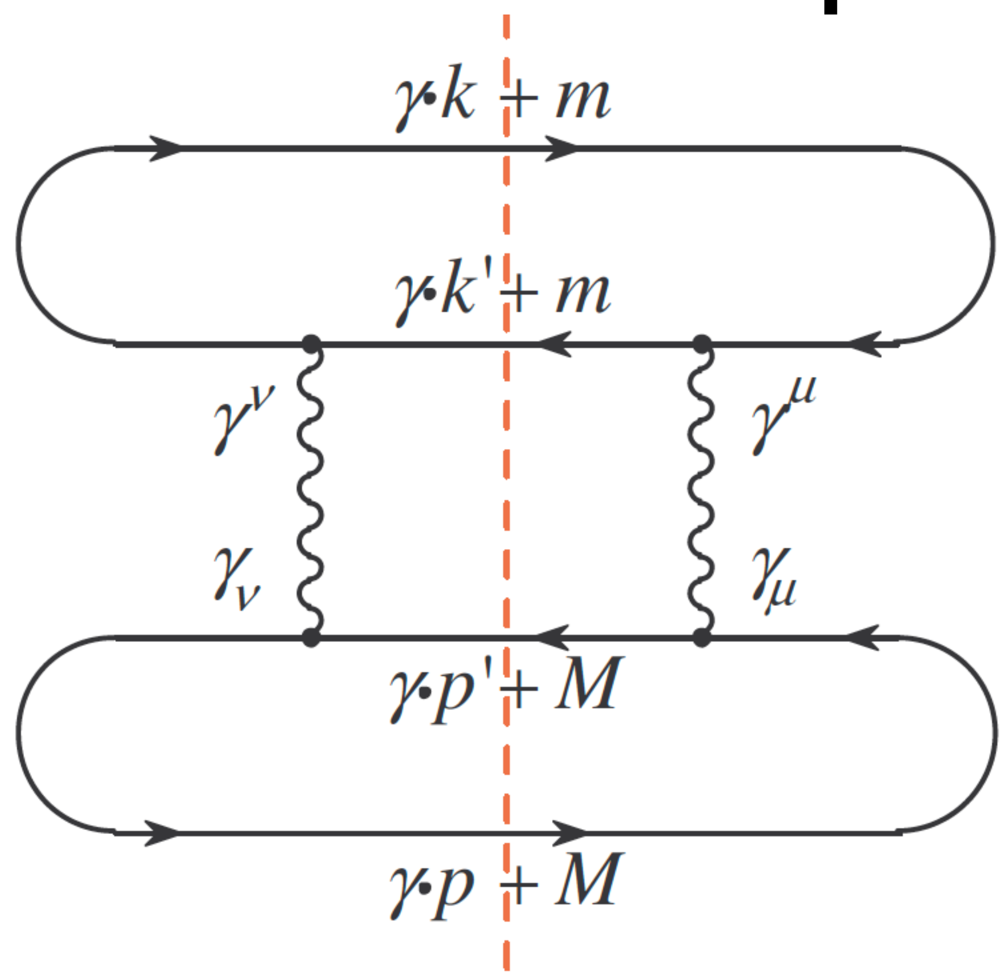


using the following identity identity

$$\sum_{\varepsilon} [u_{\varepsilon}(p)]_{\alpha} [\overline{u}_{\varepsilon}(p)]_{\beta} = (\gamma \cdot p + m)_{\alpha\beta}$$

we get:

Amplitude squared



$$\frac{1}{4}\sum_{\varepsilon} A^\dagger A = \frac{e^4 e_M^2}{4q^4} \underbrace{\mathrm{Tr}\left[\gamma^\mu(\gamma\cdot k + m)\gamma^\nu(\gamma\cdot k' + m)\right]}_{2L^{\mu\nu}(k,k')}$$

$$\underbrace{\mathrm{Tr}\left[\gamma_\mu(\gamma\cdot p + M)\gamma_\nu(\gamma\cdot p' + M)\right]}_{2L_{\mu\nu}(p,p')} = \frac{e^4 e_M^2}{q^4} L^{\mu\nu}(k,k') L_{\mu\nu}(p,p')$$

Calculating traces

$$L_{\mu\nu}(p,p') = \frac{1}{2}\mathrm{Tr}\left[\gamma_\mu(\gamma\cdot p + M)\gamma_\nu(\gamma\cdot p' + M)\right]$$
$$= 2\left[p_\mu p'_\nu + p'_\mu p_\nu + \frac{q^2}{2}g_{\mu\nu}\right]$$

Gauge invariance

$$q^\mu L_{\mu\nu}(p,p') = q^\nu L_{\mu\nu}(p,p') = 0$$

Check:

$$p\cdot p' = M^2 - q^2/2$$

$$q^\mu L_{\mu\nu} = 2\left[(p'-p)\cdot p\, p'_\nu + (p'-p)\cdot p'\, p_\nu + \frac{q^2}{2}(p'-p)_\nu\right]$$
$$= 2\left[\left(M^2 - \frac{q^2}{2} - M^2\right)p'_\nu + \left(M^2 + \frac{q^2}{2} - M^2\right)p_\nu + \frac{q^2}{2}(p'-p)_\nu\right] = 0$$

Invariant form

$$L_{\mu\nu}(p,q) = 4\left(p_\mu - \frac{p\cdot q}{q^2}q_\mu\right)\left(p_\nu - \frac{p\cdot q}{q^2}q_\nu\right) - q^2\left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right)$$

Two structures are separately gauge invariant.

Treat two coefficients as free paramteters

$$L^{\mu\nu}(p,q) = \mathcal{A}\left(p_\mu - \frac{p\cdot q}{q^2}q_\mu\right)\left(p_\nu - \frac{p\cdot q}{q^2}q_\nu\right) - \mathcal{B}\left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right)$$

and skip q_μ terms because of gauge invariance

$$\rightarrow \mathcal{A}p^\mu p^\nu + \mathcal{B}g^{\mu\nu}$$

where for elastic scattering on an elementary fermion

$$\boxed{\mathcal{A} = 4,\ \mathcal{B} = q^2 = -Q^2}$$

Squared amplitude

$$\frac{1}{4}\sum_{\varepsilon} A^\dagger A = \frac{e^4 e_M^2}{q^4}\left\{\mathcal{A}\, p^\mu p^\nu L_{\mu\nu}(k,k') + \mathcal{B}\, g^{\mu\nu} L_{\mu\nu}(k,k')\right\}$$

Compute in our kinematics:

$$p^\mu p^\nu L_{\mu\nu}(k,k') = 2\left[2\,(p\cdot k)\,(p\cdot k') - \frac{Q^2}{2}M^2\right] = 4M^2\omega\omega'\left(1-\sin^2\frac{\theta}{2}\right) = 4M^2\omega\omega'\cos^2\frac{\theta}{2}$$

$$g^{\mu\nu}L_{\mu\nu}(k,k') = 2\left[2\,(k\cdot k') - 2Q^2\right] = -2Q^2 = -8\omega\omega'\sin^2\frac{\theta}{2}$$

which finally gives

$$\frac{1}{4}\sum_{\varepsilon} A^\dagger A = \frac{M^2}{\omega\omega'}\frac{e^4 e_M^2}{\sin^4\frac{\theta}{2}}\left\{\frac{\mathcal{A}}{4}\cos^2\frac{\theta}{2} - \frac{\mathcal{B}}{4M^2}\,2\sin^2\frac{\theta}{2}\right\}$$

Elastic cross-section:

$$\frac{d\sigma}{dQ^2} = \frac{\pi\alpha^2}{4\omega^2\sin^4\frac{\theta}{2}}\int \frac{e_p^2}{\omega\omega'}\left\{\frac{\mathcal{A}}{4}\cos^2\frac{\theta}{2} - \frac{\mathcal{B}}{2M^2}\sin^2\frac{\theta}{2}\right\} d\nu\,\delta\left(\nu - \frac{Q^2}{2M}\right)$$

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Recall:

$$\frac{1}{4}\sum_{\text{pol}}|\mathcal{M}_{fi}|^2 = \frac{e_1^2 e_2^2}{(q^2)^2} L^{\nu\mu}(k,k')\,L_{\nu\mu}(p,p')$$

$$L_{\nu\mu}(p,q) = 4\underbrace{\left(p_\nu - \frac{p\cdot q}{q^2}q_\nu\right)\left(p_\mu - \frac{p\cdot q}{q^2}q_\mu\right)}_{\mathcal{A}} + q^2\underbrace{\left(g_{\nu\mu} - \frac{q_\nu q_\mu}{q^2}\right)}_{\mathcal{B}}$$

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$$= \frac{\pi\alpha^2}{4\omega^2 \sin^4\frac{\theta}{2}} \frac{e_p^2}{\omega\omega'} \left\{ \cos^2\frac{\theta}{2} + \frac{Q^2}{2M^2} \sin^2\frac{\theta}{2} \right\}.$$

Recall:

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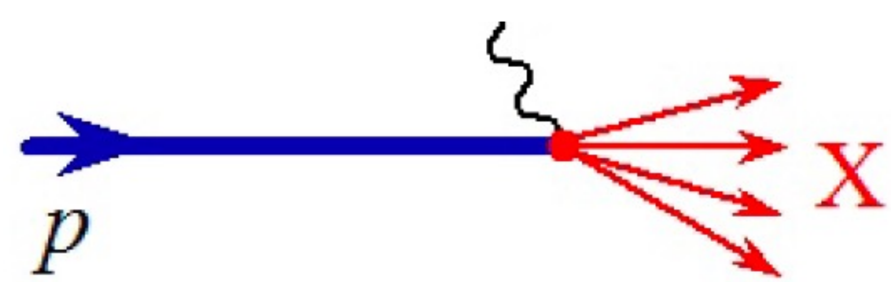
Inelastic case:

1) ν not fixed (X not mesured)

2) proton is not elementary



$$W_{\mu\nu}(p,q) = \underbrace{4W_2}_{\mathcal{A}} \left(p_\mu - \frac{p\cdot q}{q^2} q_\mu\right)\left(p_\nu - \frac{p\cdot q}{q^2} q_\nu\right) + \underbrace{4M^2 W_1}_{-\mathcal{B}} \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right)$$

Inelastic cross-section:

$$\frac{d\sigma}{dQ^2 d\nu} = \frac{\pi\alpha^2}{4\omega^3\omega' \sin^4\frac{\theta}{2}} \left\{ \frac{\mathcal{A}}{4}\cos^2\frac{\theta}{2} - \frac{\mathcal{B}}{4M^2} 2\sin^2\frac{\theta}{2} \right\}$$
$$= \frac{\pi\alpha^2}{4\omega^3\omega' \sin^4\frac{\theta}{2}} \left\{ W_2(Q^2,\nu)\cos^2\frac{\theta}{2} + 2W_1(Q^2,\nu)\sin^2\frac{\theta}{2} \right\}$$

Two unknown functions describing the proton structure: W_1 and W_2 depending on two independent variables: Q^2 and ν

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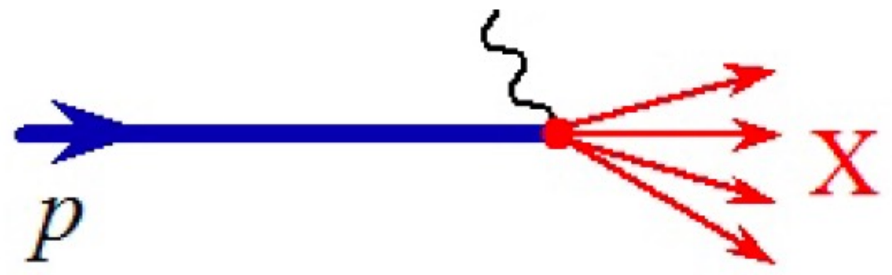

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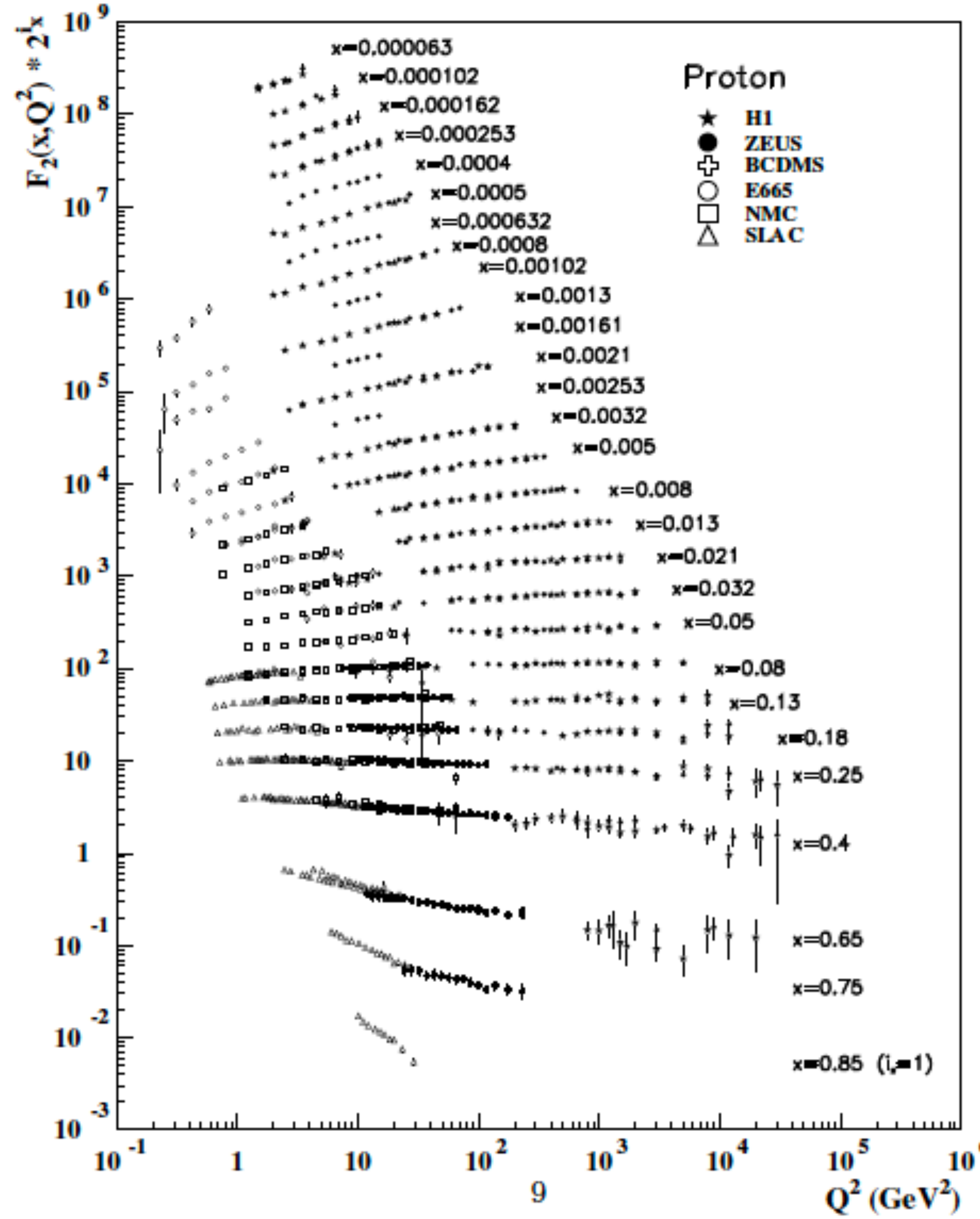

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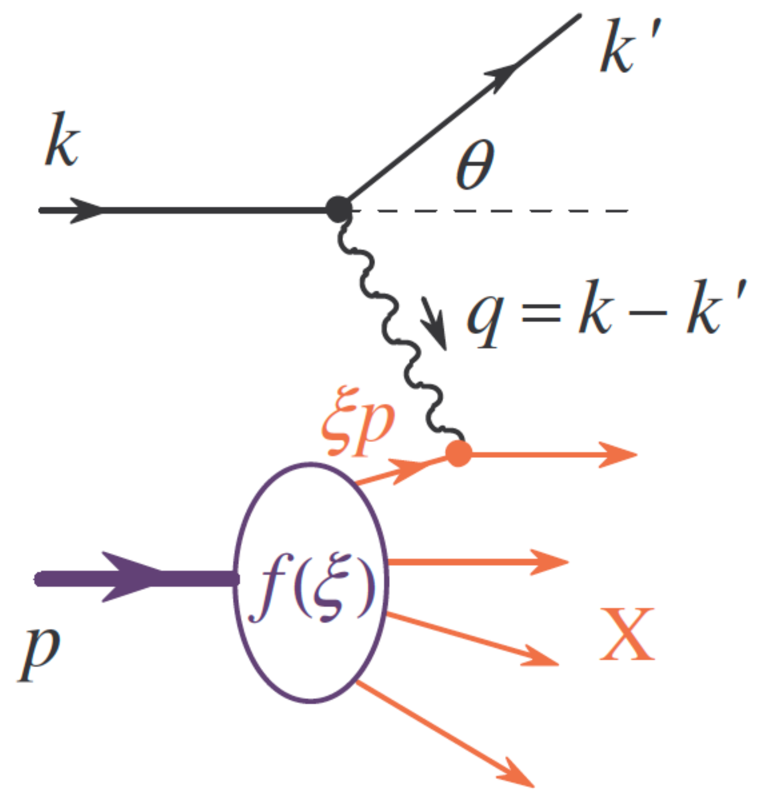

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parton elastic cross-section with proton mas M replaced by $\xi_i M$
and proton charge replaced by parton charge e_i

$$\left.\frac{d\sigma_i}{dQ^2 d\nu}\right|_{\text{parton}} = \frac{\pi\alpha^2 e_i^2}{4\omega^3\omega'\sin^4\frac{\theta}{2}}\left\{\cos^2\frac{\theta}{2} + \frac{Q^2}{4\xi_i^2 M^2}2\sin^2\frac{\theta}{2}\right\}\delta\left(\nu - \frac{1}{\xi_i}\frac{Q^2}{2M}\right)$$

parton elastic cross-section with proton mas M replaced by $\xi_i M$
and proton charge replaced by parton charge e_i

$$\left.\frac{d\sigma_i}{dQ^2 d\nu}\right|_{\text{parton}} = \frac{\pi\alpha^2 e_i^2}{4\omega^3\omega' \sin^4\frac{\theta}{2}} \left\{\cos^2\frac{\theta}{2} + \frac{Q^2}{4\xi_i^2 M^2} 2\sin^2\frac{\theta}{2}\right\} \delta\left(\nu - \frac{1}{\xi_i}\frac{Q^2}{2M}\right)$$

multiply by probabilty of finding parton i in the proton,
sum over all partons and integrate over $d\xi_i$ and you get the inelastic cross-section on the proton

$$\frac{d\sigma}{dQ^2 d\nu} = \sum_i \int d\xi_i \, f_i(\xi_i) \left.\frac{d\sigma_i}{dQ^2 d\nu}\right|_{\text{parton}}$$

parton elastic cross-section with proton mas M replaced by $\xi_i M$
and proton charge replaced by parton charge e_i

$$\left.\frac{d\sigma_i}{dQ^2 d\nu}\right|_{\text{parton}} = \frac{\pi\alpha^2 e_i^2}{4\omega^3\omega' \sin^4\frac{\theta}{2}} \left\{\cos^2\frac{\theta}{2} + \frac{Q^2}{4\xi_i^2 M^2} 2\sin^2\frac{\theta}{2}\right\} \delta\left(\nu - \frac{1}{\xi_i}\frac{Q^2}{2M}\right)$$

multiply by probabilty of finding parton i in the proton,
sum over all partons and integrate over $d\xi_i$ and you get the inelastic cross-section on the proton expresed in terms of the Bjorken functions $W_{1,2}$

$$\frac{d\sigma}{dQ^2 d\nu} = \sum_i \int d\xi_i\, f_i(\xi_i) \left.\frac{d\sigma_i}{dQ^2 d\nu}\right|_{\text{parton}} = \frac{\pi\alpha^2}{4\omega^3\omega' \sin^4\frac{\theta}{2}} \left\{W_2 \cos^2\frac{\theta}{2} + 2W_1 \sin^2\frac{\theta}{2}\right\}$$

parton elastic cross-section with proton mas M replaced by $\xi_i M$
and proton charge replaced by parton charge e_i

$$\left.\frac{d\sigma_i}{dQ^2 d\nu}\right|_{\rm parton} = \frac{\pi\alpha^2 e_i^2}{4\omega^3\omega' \sin^4\frac{\theta}{2}}\left\{\cos^2\frac{\theta}{2} + \frac{Q^2}{4\xi_i^2 M^2} 2\sin^2\frac{\theta}{2}\right\}\ \delta\left(\nu - \frac{1}{\xi_i}\frac{Q^2}{2M}\right)$$

multiply by probabilty of finding parton i in the proton,
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$$\frac{d\sigma}{dQ^2 d\nu} = \sum_i \int d\xi_i\, f_i(\xi_i) \left.\frac{d\sigma_i}{dQ^2 d\nu}\right|_{\rm parton} = \frac{\pi\alpha^2}{4\omega^3\omega' \sin^4\frac{\theta}{2}}\left\{W_2\cos^2\frac{\theta}{2} + 2W_1\sin^2\frac{\theta}{2}\right\}$$

we can now immediately calculate $W_{1,2}$ in terms of $f(\xi)$

$$W_2 = \sum_i e_i^2 \int d\xi\ f_i(\xi)\delta\left(\nu - \nu\frac{x}{\xi}\right) = \sum_i e_i^2 \int d\xi\ f_i(\xi)\frac{\xi^2}{\nu x}\delta\left(\xi - x\right) = \frac{1}{\nu}\sum_i e_i^2 x\, f_i(x)$$

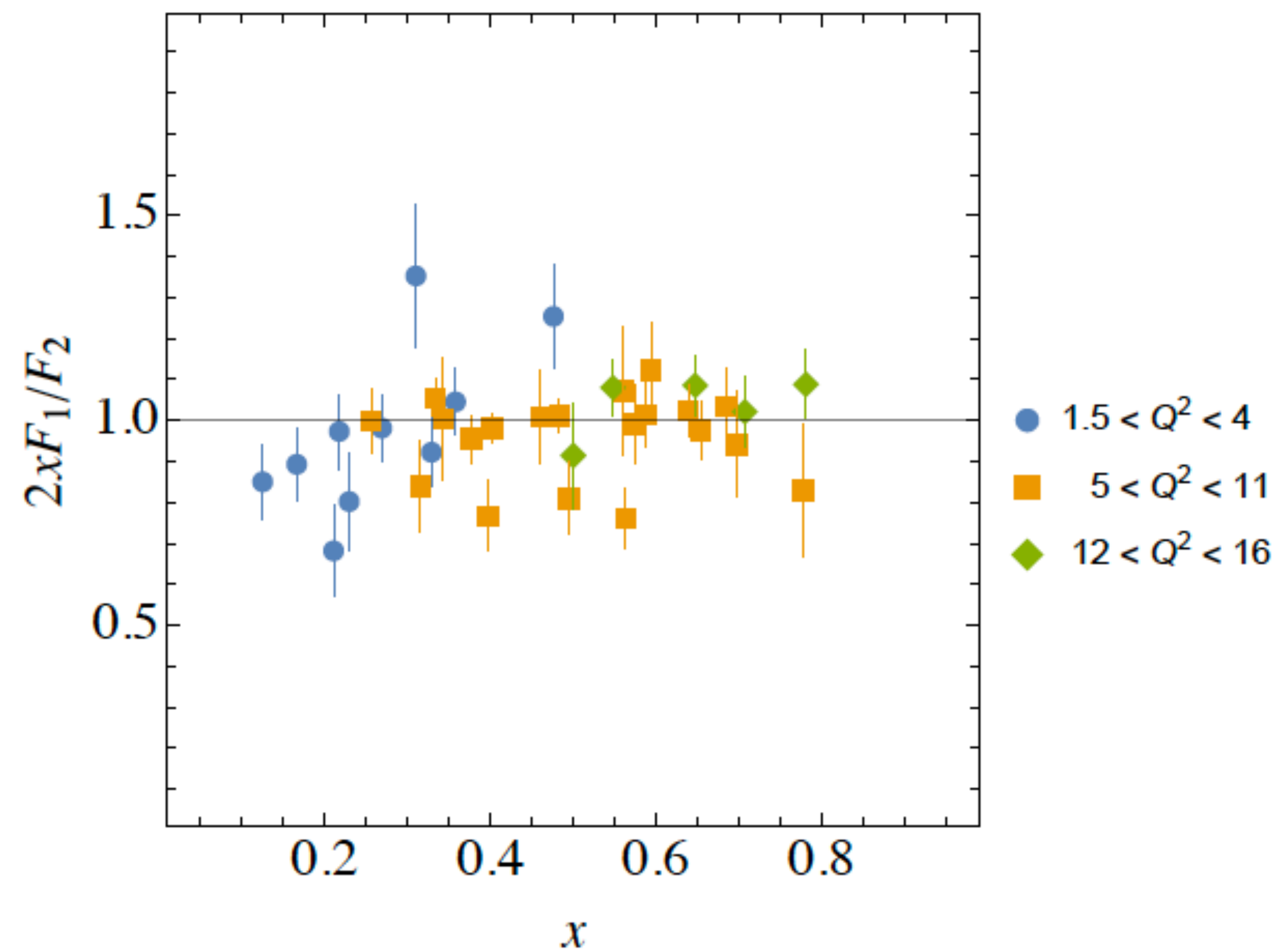
$$W_1 = \sum_i e_i^2 \int d\xi\ f_i(\xi)\frac{Q^2}{4\xi^2 M^2}\frac{\xi^2}{\nu x}\delta\left(\xi - x\right) = \frac{1}{2M}\sum_i e_i^2\, f_i(x). \qquad x = \frac{Q^2}{2M\nu}$$

Bjorken Scaling vs. Parton Model

$$F_2(x) = \nu W_2 = x \sum_i e_i^2 f_i(x)$$

$$F_1(x) = M W_1 = \frac{1}{2} \sum_i e_i^2 f_i(x)$$

$$F_2(x) = 2x F_1(x)$$



in parton model structure functions are related: Callan-Gross relation

Quarks as Partons

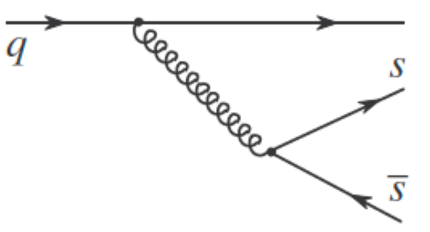
$$F_2^{\mathrm{p}}(x) = \frac{4}{9}x\left[u_{\mathrm{p}}(x) + \overline{u}_{\mathrm{p}}(x)\right] + \frac{1}{9}x\left[d_{\mathrm{p}}(x) + \overline{d}_{\mathrm{p}}(x) + s_{\mathrm{p}}(x) + \overline{s}_{\mathrm{p}}(x)\right]$$

$$F_2^{\mathrm{n}}(x) = \frac{4}{9}x\left[u_{\mathrm{n}}(x) + \overline{u}_{\mathrm{n}}(x)\right] + \frac{1}{9}x\left[d_{\mathrm{n}}(x) + \overline{d}_{\mathrm{n}}(x) + s_{\mathrm{n}}(x) + \overline{s}_{\mathrm{n}}(x)\right]$$

assuming isospin symmetry:

$$u_{\mathrm{p}} = d_{\mathrm{n}} = u, \quad d_{\mathrm{p}} = u_{\mathrm{n}} = d, \quad s_{\mathrm{p}} = s_{\mathrm{n}} = s$$

no strangeness in the nucleon: $\int dx(s(x) - \overline{s}(x)) = 0$



Quarks as Partons

proton and neutron charges

$$q_{\rm p} = \int dx \left[\frac{2}{3}(u(x) - \overline{u}(x)) - \frac{1}{3}(d(x) - \overline{d}(x)) - \frac{1}{3}(s(x) - \overline{s}(x))\right] = 1$$

$$\updownarrow = 0$$

$$q_{\rm n} = \int dx \left[\frac{2}{3}(d(x) - \overline{d}(x)) - \frac{1}{3}(u(x) - \overline{u}(x)) - \frac{1}{3}(s(x) - \overline{s}(x))\right] = 0$$

imply constraints on the parton distributions (PDF's):

$$\int dx (u(x) - \overline{u}(x)) = 2, \quad \int dx (d(x) - \overline{d}(x)) = 1, \quad \int dx (s(x) - \overline{s}(x)) = 0$$

valence and sea quarks: $u = u_v + q_s, \quad d = d_v + q_s, \quad \overline{u} = \overline{d} = \overline{s} = s = q_s$

total momenum – for typical parametrizations

$$\int dx\, x (u(x) + \overline{u}(x) + d(x) + \overline{d}(x) + s(x) + \overline{s}(x)) = 1 - \varepsilon$$

$\varepsilon \sim 45\%$

there must be other partons that do not inteact electromagnetically: gluons