

QCD Lecture 2a

October 15, 2025

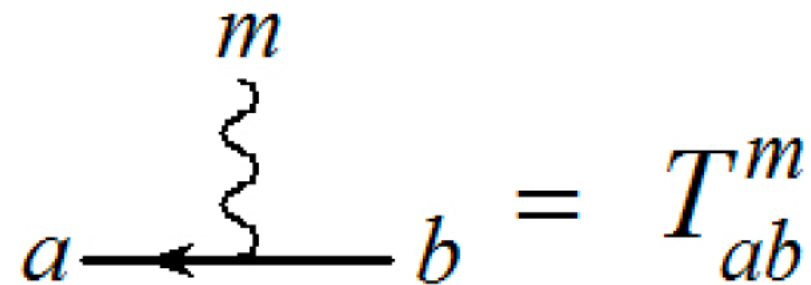
Color factors

each Feynman diagram is a product of a momentum-Dirac structure (like in QED) and a **color factor**

to calculate color factors it is very practical to use the **graphical notation**

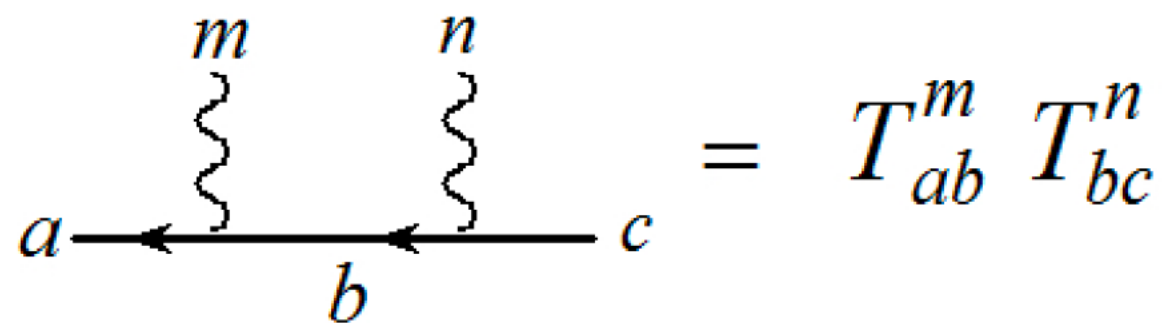
fundamental generator:

$$m, n = 1, 2, \dots, N^2 - 1, \quad a, b = 1, 2, \dots, N$$



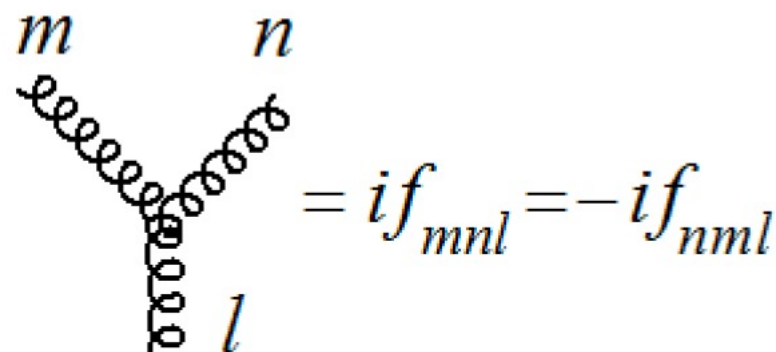
$$a \xleftarrow{\quad} b \quad \text{with wavy line } m \text{ attached} = T_{ab}^m$$

multiplication:



$$a \xleftarrow{\quad} b \quad \text{with wavy lines } m \text{ and } n \text{ attached} = T_{ab}^m T_{bc}^n$$

adjoint generator:



$$\text{Vertex with wavy lines } m, n, l = if_{mnl} = -if_{nml}$$

Color factors

Kroneker deltas and traces:

$$a \overleftarrow{\quad} b = \delta_{ab} \quad \text{[circle with arrow]} = N$$

$$m \text{ [wavy line]} n = \delta_{mn} \quad \text{[sun-like shape]} = N^2 - 1$$

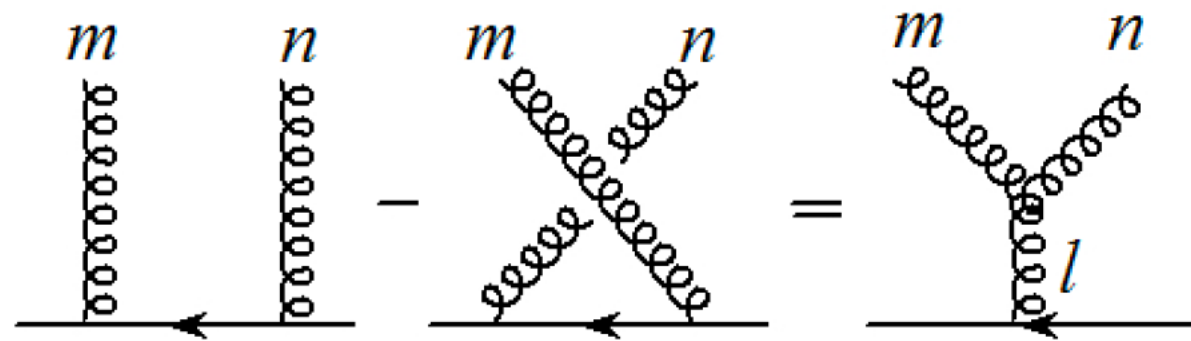
generators are traceless and normalized to 1/2

$$\text{[wavy line]} \text{ [circle with arrow]} = 0 \quad m \text{ [wavy line]} \text{ [circle with arrow]} \text{ [wavy line]} n = \frac{1}{2} m \text{ [wavy line]} n \quad \text{Tr}(T_m T_n) = \frac{1}{2} \delta_{mn}$$

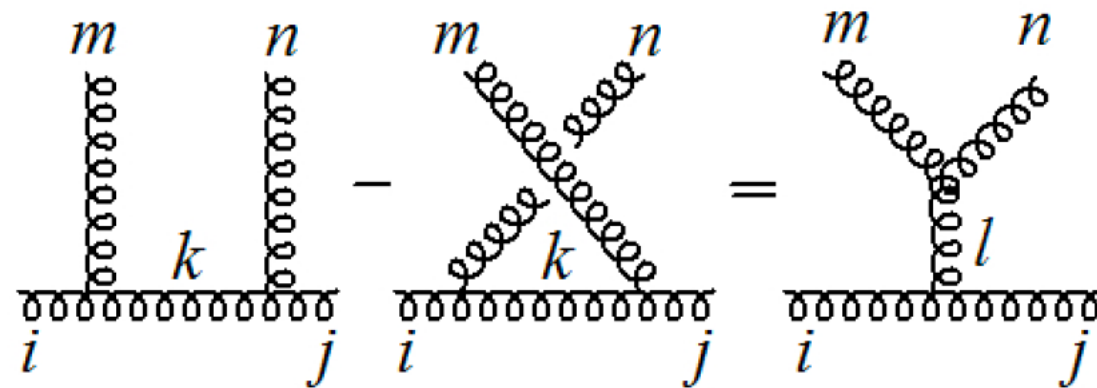
Color factors

commutation relations: $[T_m, T_n] = i f_{mnl} T_l$

fundamental:



adjoint:



Color factors

Example:

Casimir operator for the fundamental representation

quadratic Casimir operator is the sum over all generators squared and it is proportional to unity multiplied by a number, which is simply called “Casimir”

$$\sum_n (T^n)^2 = C_F \mathbf{1}$$

In SU(2) for any representation of spin s it is equal to

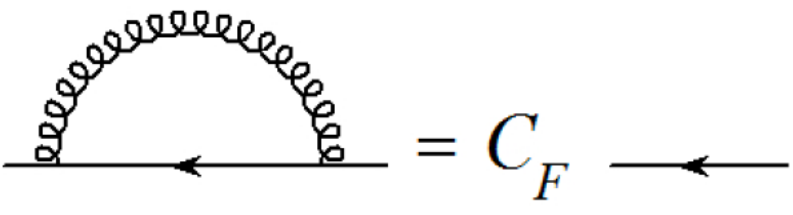
$$\sum_n \hat{S}_n^2 = s(s + 1) \mathbf{1}$$

Color factors

Example:

Casimir operator for the fundamental representation

$$\sum_n (T^n)^2 = C_F \mathbf{1}$$

$$\sum_n (T^n)^2 = \text{gluon loop} = C_F \text{ fermion line}$$
A Feynman diagram illustrating the Casimir operator for the fundamental representation. It shows a horizontal fermion line with an arrow pointing to the left. A gluon loop, represented by a semi-circular line with loops, is attached to the fermion line. The diagram is equated to the Casimir factor C_F multiplied by the same fermion line.

Color factors

Example:

Casimir operator for the fundamental representation

$$\sum_n (T^n)^2 = C_F \mathbf{1}$$

$$\sum_n (T^n)^2 = \text{diagram with gluon loop on a fermion line} = C_F \text{diagram with straight fermion line}$$

contract fermion line:

$$\text{diagram with gluon loop on a closed fermion loop} = C_F \text{diagram with straight closed fermion loop}$$

use:

$$\text{diagram with gluon loop on a fermion line with external lines } m \text{ and } n = \frac{1}{2} \text{diagram with two gluon lines } m \text{ and } n$$

$$\text{diagram with a fermion loop} = N$$

$$\text{diagram with a gluon loop} = N^2 - 1$$

Color factors

Example:

Casimir operator for the fundamental representation

$$\sum_n (T^n)^2 = C_F \mathbf{1}$$

$$\sum_n (T^n)^2 = \text{gluon loop on fermion line} = C_F \text{ fermion line}$$

contract fermion line:

$$\text{gluon loop on contracted fermion line} = C_F \text{ contracted fermion line}$$

use:

$$\text{fermion loop with external wavy lines } m, n = \frac{1}{2} \text{ wavy lines } m, n$$

$$\text{fermion loop} = N$$

$$\text{gluon loop} = N^2 - 1$$

$$\frac{1}{2} \text{ gluon loop} = C_F N$$

Color factors

Example:

Casimir operator for the fundamental representation

$$\sum_n (T^n)^2 = C_F \mathbf{1}$$

$$\sum_n (T^n)^2 = \text{gluon loop on fermion line} = C_F \text{ fermion line}$$

contract fermion line:

$$\text{gluon loop on contracted fermion line} = C_F \text{ contracted fermion line}$$

use:

$$\text{fermion loop} = \frac{1}{2} \text{fermion line}$$

$$\frac{1}{2} \text{gluon loop} = C_F N$$

$$\text{fermion loop} = N$$

$$\text{gluon loop} = N^2 - 1$$

$$C_F = \frac{N^2 - 1}{2N} = \begin{cases} \frac{3}{4} & \text{SU(2)} \\ \frac{4}{3} & \text{SU(3)} \end{cases}$$