

# QCD lecture 14

January 28

# Goldstone bosons

We have shown that in QCD axial  $SU(3)$  symmetry is spontaneously broken, and this implies the existence of eight Goldstone bosons. What is the effective lagrangian? Natural choice for example:

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi_a^\dagger \partial^\mu \phi_a - V(\phi_a^\dagger \phi_a) \quad \phi'_a = [e^{-i\theta_c T_{\text{adj}}^c}]_{ab} \phi_b = \phi_a - i\theta_c (T_{\text{adj}}^c)_{ab} \phi_b + \dots$$

This lagrangian is invariant under  $SU_V(3)$  but it is not clear how it transforms under  $SU_A(3)$ . We will show, that we can write a lagrangian which is much more "powerfull" (infinte series in powers of field derivatives) and takes explicitly into account  $SU_A(3)$  breaking. For this we will need a bit of mathematics.

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Consider a hamiltonian  $\hat{H}$  (note a "hat"! ) which is invariant under a compact Lie group  $G$ . Moreover, the ground state is invariant only under a subgroup  $H$ . We have therefore  $n = n_G - n_H$  Goldstone bosons  $\phi_i$ , which are continous, real functions on Minkowski space  $M^4$ . Define vector space

$$M_1 \equiv \{ \Phi : M^4 \rightarrow R^n \mid \phi_i : M^4 \rightarrow R \text{ continuous} \}$$

and find its elements.

based on: Stefan Scherer *Introduction to Chiral Perturbation Theory*, hep-ph/0210398v1

# Goldstone bosons

$$M_1 \equiv \{\Phi : M^4 \rightarrow R^n | \phi_i : M^4 \rightarrow R \text{ continuous}\}$$

Define a mapping that associates with each pair  $(g, \Phi) \in G \times M_1$   
 $g$  – group element,  
 $\Phi$  –  $n$  component vector with elements  $\phi_i$

an element  $\varphi(g, \Phi) \in M_1$  such that

$$\varphi(e, \Phi) = \Phi \quad \forall \Phi \in M_1, \quad e \text{ identity of } G,$$

$$\varphi(g_1, \varphi(g_2, \Phi)) = \varphi(g_1 g_2, \Phi) \quad \forall g_1, g_2 \in G, \quad \forall \Phi \in M_1$$

This is nothing but definition of an operation of  $G$  on  $M_1$ . This mapping is not necessarily linear:

$$\varphi(g, \lambda\Phi) \neq \lambda\varphi(g, \Phi)$$

Vacuum ("origin" of  $M_1$ )  $\Phi = 0$  We require that all elements of  $G$   $h \in H$  map the origin onto itself (little group of  $\Phi = 0$ )

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# Goldstone bosons

- $H$  is not empty, because identity maps the origin onto itself
- If  $\varphi(h_1, 0) = \varphi(h_2, 0) = 0$  then  $\varphi(h_1 h_2, 0) = \varphi(h_1, \varphi(h_2, 0)) = \varphi(h_1, 0) = 0$  which means that  $h_1 h_2 \in H$
- Inverse element is also in  $H$ :  $\varphi(h^{-1}, 0) = \varphi(h^{-1}, \varphi(h, 0)) = \varphi(h^{-1} h, 0) = \varphi(e, 0)$  which means that  $h^{-1} \in H$

Define left coset  $gH = \{gH | g \in G\}$  ( $g$  is fixed) We will establish a connection between the set of all left cosets  $G/H$  with the Goldstone boson fields.

We will check now that all elements of a given coset map the origin onto the same vector in  $R^n$

$$\varphi(gh, 0) = \varphi(g, \varphi(h, 0)) = \varphi(g, 0) \quad \forall g \in G \text{ and } h \in H$$

These vectors are different if  $g$  and  $g'$  are "different":  $\varphi(g, 0) \neq \varphi(g', 0)$  if  $g' \notin gH$   
This means that mapping  $\varphi$  is injective with respect to the cosets.

# Goldstone bosons

Proof proceeds by negation of the thesis. Assume  $\varphi(g, 0) = \varphi(g', 0)$

Then

$$0 = \varphi(e, 0)$$

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However, this implies  $g^{-1}g' \in H$  or  $g' \in gH$ , which contradicts our assumption.

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We will now discuss transformations of  $\Phi$ . To each  $\Phi$  corresponds a coset  $\tilde{g}H$

with  $\tilde{g}$  fixed:

$$\Phi = \varphi(f, 0) = \varphi(\tilde{g}h, 0) \quad \longleftarrow$$

Consider transformation of  $\Phi$  with  $\varphi(g)$

$$\varphi(g, \Phi) = \varphi(g, \varphi(\tilde{g}h, 0)) = \varphi(g\tilde{g}h, 0) = \varphi(f', 0) = \Phi' \quad f' \in g(\tilde{g}H)$$

To obtain transformed  $\Phi'$  from  $\Phi$  we need to multiply the left coset  $\tilde{g}H$  representing  $\Phi$  by  $g$  to obtain a new left coset representing  $\Phi'$ .

# Goldstone bosons in QCD

Symmetry group of QCD

$$G = \text{SU}_{\text{L}}(N) \times \text{SU}_{\text{R}}(N) = \{(L, R) | L \in \text{SU}_{\text{L}}(N), R \in \text{SU}_{\text{R}}(N)\}$$

and little group  $H = \{(V, V) | V \in \text{SU}(N)\}$  (which is isomorphic to  $\text{SU}(N)$ )

Left coset  $\tilde{g}H = \{(\tilde{L}V, \tilde{R}V) | V \in \text{SU}(N)\}$  is uniquely characterized by  $U = \tilde{R}\tilde{L}^\dagger$

Indeed:

$$(\tilde{L}V, \tilde{R}V) = (\tilde{L}V, \tilde{R}\tilde{L}^\dagger\tilde{L}V) = (1, \tilde{R}\tilde{L}^\dagger) \underbrace{(\tilde{L}V, \tilde{L}V)}_{\in H}, \quad \text{i.e.} \quad \tilde{g}H = (1, \tilde{R}\tilde{L}^\dagger)H.$$

(because  $\varphi(gh, 0) = \varphi(g, \varphi(h, 0)) = \varphi(g, 0) \forall g \in G$  and  $h \in H$ )

Therefore matrix  $U = \tilde{R}\tilde{L}^\dagger$  is isomorphic to  $\Phi$ .

# Goldstone bosons in QCD

Now, we will find transformation law for  $U$ . Recall  $\Phi = \varphi(f, 0) = \varphi(\tilde{g}h, 0)$

and  $\varphi(f', 0) = \Phi'$  where  $f' = g\tilde{g}h$  or  $f' \in g(\tilde{g}H)$ . This means, that transformation

of  $U$  under  $g = (L, R) \in G$  is (recall  $\tilde{g}H = (1, \tilde{R}\tilde{L}^\dagger)H$ )

$$g\tilde{g}H = (L, R\tilde{R}\tilde{L}^\dagger)H = (1, R\tilde{R}\tilde{L}^\dagger L^\dagger) \underbrace{(L, L)H}_{= H} = (1, R(\tilde{R}\tilde{L}^\dagger)L^\dagger)H$$

Hence we have  $U = \tilde{R}\tilde{L}^\dagger \mapsto U' = R(\tilde{R}\tilde{L}^\dagger)L^\dagger = RUL^\dagger$

where we have to reintroduce space-time dependence

$$U(x) \mapsto RU(x)L^\dagger$$

We now see, how the symmetry is broken. Vacuum corresponds to  $U \sim 1$  and the symmetry of vacuum is  $R = L$ .

# Nonlinear realization of $SU(N) \times SU(N)$

We can parametrize  $SU(N)$  matrix as  $U(x) = \exp\left(i\frac{\phi(x)}{F_0}\right)$

where for  $SU(2)$

$$\phi(x) = \sum_{i=1}^3 \tau_i \phi_i(x) = \begin{pmatrix} \phi_3 & \phi_1 - i\phi_2 \\ \phi_1 + i\phi_2 & -\phi_3 \end{pmatrix} \equiv \begin{pmatrix} \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -\pi^0 \end{pmatrix}$$

or for  $SU(3)$

$$\phi(x) = \sum_{a=1}^8 \lambda_a \phi_a(x) = \begin{pmatrix} \phi_3 + \frac{1}{\sqrt{3}}\phi_8 & \phi_1 - i\phi_2 & \phi_4 - i\phi_5 \\ \phi_1 + i\phi_2 & -\phi_3 + \frac{1}{\sqrt{3}}\phi_8 & \phi_6 - i\phi_7 \\ \phi_4 + i\phi_5 & \phi_6 + i\phi_7 & -\frac{2}{\sqrt{3}}\phi_8 \end{pmatrix}$$

$$\equiv \begin{pmatrix} \pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}\pi^+ & \sqrt{2}K^+ \\ \sqrt{2}\pi^- & -\pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}K^0 \\ \sqrt{2}K^- & \sqrt{2}\bar{K}^0 & -\frac{2}{\sqrt{3}}\eta \end{pmatrix},$$

[there exist different conventions  
for signs of particle fields]

# Nonlinear realization of $SU(N) \times SU(N)$

**Define**  $M_3 \equiv \left\{ U : M^4 \rightarrow SU(N) \mid U(x) = \exp \left( i \frac{\phi(x)}{F_0} \right) \right\}$

**The homomorphism**

$$\varphi : G \times M_3 \rightarrow M_3 \quad \text{with} \quad \varphi[(L, R), U](x) \equiv RU(x)L^\dagger$$

**defines an operation of  $G$  on  $M_3$**

1.  $RU(x)L^\dagger \in M_3$ , since  $U \in M_3$  and  $R, L^\dagger \in SU(N)$ .
2.  $\varphi[(1_{N \times N}, 1_{N \times N}), U](x) = 1_{N \times N}U(x)1_{N \times N} = U(x)$ .
3. Let  $g_i = (L_i, R_i) \in G$  and thus  $g_1g_2 = (L_1L_2, R_1R_2) \in G$ .

$$\begin{aligned} \varphi[g_1, \varphi[g_2, U]](x) &= \varphi[g_1, (R_2UL_2^\dagger)](x) = R_1R_2U(x)L_2^\dagger L_1^\dagger, \\ \varphi[g_1g_2, U](x) &= R_1R_2U(x)(L_1L_2)^\dagger = R_1R_2U(x)L_2^\dagger L_1^\dagger. \end{aligned}$$

**all group requirements are fulfilled. This mapping is called nonlinear because  $M_3$  is not a vector space (sum of two  $U$  matrices is not a unitary matrix).**

# Nonlinear realization of $SU(N) \times SU(N)$

The origin (vacuum) corresponds to  $\phi(x) = 0$ , i.e.  $U_0 = 1$

Indeed

$$\begin{aligned}\varphi[g = (V, V), 1] &= VV^\dagger = 1 \\ \varphi[g = (A, A^\dagger), 1] &= A^\dagger A^\dagger \neq 1\end{aligned}$$

Axial symmetry is broken, left and right fermions must be transformed the same way.

Transformation of fields  $\phi(x)$

$$U = 1 + i\frac{\phi}{F_0} - \frac{\phi^2}{2F_0^2} + \dots$$

and transformation matrix  $V = \exp\left(-i\Theta_a^V \frac{\lambda_a}{2}\right)$  give

$$\phi = \lambda_b \phi_b \quad h \in SU(3)_V \quad \xrightarrow{\quad} \quad V\phi V^\dagger = \phi - i\Theta_a^V \left[ \frac{\lambda_a}{2}, \phi_b \lambda_b \right] + \dots = \phi + \underbrace{f_{abc} \Theta_a^V \phi_b \lambda_c}_{\phi_b i f_{abc} \lambda_c} + \dots$$

Fields  $\phi(x)$  transform according to the adjoint rep. of  $SU(3)$  (like gauge fields...)

# Effective lagrangian

Matrix  $U$  is our "building block". Lagrangian must be symmetric under global

$$\text{SU}(3)_L \times \text{SU}(3)_R \times \text{U}(1)_V \quad U(x) \mapsto RU(x)L^\dagger \quad U(x) = \exp\left(i\frac{\phi(x)}{F_0}\right)$$

The most general lagrangian with two derivatives (Weinberg lagrangian)

$$\mathcal{L}_{\text{eff}} = \frac{F_0^2}{4} \text{Tr} (\partial_\mu U \partial^\mu U^\dagger)$$

where (experimentally)  $F_0 \approx 93 \text{ MeV}$  can be deduced from  $\pi^+ \rightarrow \mu^+ \nu_\mu$

Invariance:

$$U \mapsto RUL^\dagger \quad \partial_\mu U \mapsto R\partial_\mu UL^\dagger \quad U^\dagger \mapsto LU^\dagger R^\dagger \quad \partial_\mu U^\dagger \mapsto L\partial_\mu U^\dagger R^\dagger$$

$$\mathcal{L}_{\text{eff}} \mapsto \frac{F_0^2}{4} \text{Tr} \left( R\partial_\mu U \underbrace{L^\dagger L}_1 \partial^\mu U^\dagger R^\dagger \right) = \frac{F_0^2}{4} \text{Tr} \left( \underbrace{R^\dagger R}_1 \partial_\mu U \partial^\mu U^\dagger \right) = \mathcal{L}_{\text{eff}}$$

# Effective lagrangian

Expanding  $U = 1 + i\phi/F_0 + \dots$        $\partial_\mu U = i\partial_\mu\phi/F_0 + \dots$

$$\begin{aligned}\mathcal{L}_{\text{eff}} &= \frac{F_0^2}{4} \text{Tr} \left[ \frac{i\partial_\mu\phi}{F_0} \left( -\frac{i\partial^\mu\phi}{F_0} \right) \right] + \dots = \frac{1}{4} \text{Tr}(\lambda_a \partial_\mu\phi_a \lambda_b \partial^\mu\phi_b) + \dots \\ &= \frac{1}{4} \partial_\mu\phi_a \partial^\mu\phi_b \text{Tr}(\lambda_a \lambda_b) + \dots = \frac{1}{2} \partial_\mu\phi_a \partial^\mu\phi_a + \mathcal{L}_{\text{int}}\end{aligned}$$

we get usual lagrangian plus interactions that proceed only through derivatives (momenta). For small momenta higher derivative terms are small. Interactions are even in  $\phi_a$  Parity

$$\phi_a(\vec{x}, t) \mapsto -\phi_a(-\vec{x}, t) \quad U(\vec{x}, t) \mapsto U^\dagger(-\vec{x}, t)$$

This lagrangian is unique up to total derivatives. E.g.:

$$\text{Tr}[(\partial_\mu \partial^\mu U)U^\dagger] = \partial_\mu [\text{Tr}(\partial^\mu U U^\dagger)] - \text{Tr}(\partial^\mu U \partial_\mu U^\dagger)$$

Single derivatives vanish under trace  $\text{Tr}(\partial_\mu U U^\dagger) = 0$

# Currents

Left currents. Set  $\Theta_a^R = 0$  and make left transformation space-time dependent:

$$\Theta_a^L = \Theta_a^L(x)$$

Then 
$$U \mapsto U' = RUL^\dagger = U \left( 1 + i\Theta_a^L \frac{\lambda_a}{2} \right)$$

$$\partial_\mu U \mapsto \partial_\mu U' = \partial_\mu U \left( 1 + i\Theta_a^L \frac{\lambda_a}{2} \right) + U i \partial_\mu \Theta_a^L \frac{\lambda_a}{2}$$

$$\partial_\mu U^\dagger \mapsto \partial_\mu U'^\dagger = \left( 1 - i\Theta_a^L \frac{\lambda_a}{2} \right) \partial_\mu U^\dagger - i \partial_\mu \Theta_a^L \frac{\lambda_a}{2} U^\dagger$$

and:

$$\begin{aligned} \delta \mathcal{L}_{\text{eff}} &= \frac{F_0^2}{4} \text{Tr} \left[ U i \partial_\mu \Theta_a^L \frac{\lambda_a}{2} \partial^\mu U^\dagger + \partial_\mu U \left( -i \partial^\mu \Theta_a^L \frac{\lambda_a}{2} U^\dagger \right) \right] \\ &= \frac{F_0^2}{4} i \partial_\mu \Theta_a^L \text{Tr} \left[ \frac{\lambda_a}{2} (\partial^\mu U^\dagger U - U^\dagger \partial^\mu U) \right] \quad \leftarrow \partial^\mu U^\dagger U = -U^\dagger \partial^\mu U \\ &= \frac{F_0^2}{4} i \partial_\mu \Theta_a^L \text{Tr} (\lambda_a \partial^\mu U^\dagger U). \end{aligned}$$

Left current:

$$J_L^{\mu,a} = \frac{\partial \delta \mathcal{L}_{\text{eff}}}{\partial \partial_\mu \Theta_a^L} = i \frac{F_0^2}{4} \text{Tr} (\lambda_a \partial^\mu U^\dagger U)$$

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Right current:

$$J_R^{\mu,a} = \frac{\partial \delta \mathcal{L}_{\text{eff}}}{\partial \partial_\mu \Theta_a^R} = -i \frac{F_0^2}{4} \text{Tr} (\lambda_a U \partial^\mu U^\dagger)$$

# Currents

We can now calculate vector and axial currents:

$$J_V^{\mu,a} = J_R^{\mu,a} + J_L^{\mu,a} = -i \frac{F_0^2}{4} \text{Tr} (\lambda_a [U, \partial^\mu U^\dagger]),$$

$$J_A^{\mu,a} = J_R^{\mu,a} - J_L^{\mu,a} = -i \frac{F_0^2}{4} \text{Tr} (\lambda_a \{U, \partial^\mu U^\dagger\})$$

Internal parity:

$$J_V^{\mu,a} \quad \begin{array}{l} \phi \mapsto -\phi \\ \mapsto \end{array} \quad -i \frac{F_0^2}{4} \text{Tr} [\lambda_a (U^\dagger \partial^\mu U - \partial^\mu U U^\dagger)]$$

$$\partial^\mu U^\dagger U = -U^\dagger \partial^\mu U \quad \longrightarrow \quad = \quad -i \frac{F_0^2}{4} \text{Tr} [\lambda_a (-\partial^\mu U^\dagger U + U \partial^\mu U^\dagger)] = J_V^{\mu,a}$$

$$J_A^{\mu,a} \quad \begin{array}{l} \phi \mapsto -\phi \\ \mapsto \end{array} \quad -i \frac{F_0^2}{4} \text{Tr} [\lambda_a (U^\dagger \partial^\mu U + \partial^\mu U U^\dagger)]$$

$$= \quad i \frac{F_0^2}{4} \text{Tr} [\lambda_a (\partial^\mu U^\dagger U + U \partial^\mu U^\dagger)] = -J_A^{\mu,a}$$

# Matrix element of axial current

**Axial current**  $J_A^{\mu,a} = -i \frac{F_0^2}{4} \text{Tr} (\lambda_a \{U, \partial^\mu U^\dagger\})$

**expanding:**  $J_A^{\mu,a} = -i \frac{F_0^2}{4} \text{Tr} \left( \lambda_a \left\{ 1 + \dots, -i \frac{\lambda_b \partial^\mu \phi_b}{F_0} + \dots \right\} \right) = -F_0 \partial^\mu \phi_a + \dots$

**Matrix element of axial current between GB and vacuum:**

$$\langle 0 | J_A^{\mu,a}(x) | \phi^b(p) \rangle = -F_0 \langle 0 | \partial^\mu \phi^a(x) | \phi^b(p) \rangle$$

# Reminder

In the case of quantum fields there are different conventions. Here we follow:  
T-P. Cheng and L-F. Li *Gauge theory of elementary particle physics*

$$\phi_a(\mathbf{x}, t) = \int \frac{d^3k}{(2\pi)^{3/2}} \frac{1}{\sqrt{2E_k}} [a_a(\mathbf{k})e^{-ik \cdot x} + a_a^\dagger(\mathbf{k})e^{+ik \cdot x}]$$

$$[a_a(\mathbf{k}), a_a^\dagger(\mathbf{k}')] = \delta^{(3)}(\mathbf{k} - \mathbf{k}')$$

Fock state:  $|\phi_a(k)\rangle = \sqrt{(2\pi)^3 2E_k} a_a^\dagger(\mathbf{k}) |0\rangle$

# Matrix element of axial current

**Axial current**  $J_A^{\mu,a} = -i \frac{F_0^2}{4} \text{Tr} (\lambda_a \{U, \partial^\mu U^\dagger\})$

**expanding:**  $J_A^{\mu,a} = -i \frac{F_0^2}{4} \text{Tr} \left( \lambda_a \left\{ 1 + \dots, -i \frac{\lambda_b \partial^\mu \phi_b}{F_0} + \dots \right\} \right) = -F_0 \partial^\mu \phi_a + \dots$

**Matrix element of axial current between GB and vacuum:**

$$\begin{aligned}
 \langle 0 | J_A^{\mu,a}(x) | \phi^b(p) \rangle &= -F_0 \langle 0 | \partial^\mu \phi^a(x) | \phi^b(p) \rangle \\
 &= -F_0 \sqrt{(2\pi)^3 2E_p} \int \frac{d^3 \mathbf{k}}{\sqrt{(2\pi)^3 2E_k}} \partial_\mu e^{-ikx} \langle 0 | a_a(\mathbf{k}) a_b^\dagger(\mathbf{p}) | 0 \rangle \\
 &= iF_0 \sqrt{E_p} \int \frac{d^3 \mathbf{k}}{\sqrt{E_k}} k_\mu e^{-ikx} \delta^{(3)}(\mathbf{k} - \mathbf{p}) \delta^{ab} \\
 &= ip_\mu e^{-ipx} F_0 \delta^{ab}.
 \end{aligned}$$

**This agrees with previous result from QCD**

$$\langle 0 | A_\mu^a(0) | \phi^b(p) \rangle = ip_\mu F_0 \delta^{ab}$$

# Mass term

In QCD

$$\mathcal{L}_M = -\bar{q}_R M q_L - \bar{q}_L M^\dagger q_R, \quad M = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix}$$

This would be invariant if  $M \mapsto R M L^\dagger$

What is the effective lagrangian that respects this would be symmetry? To the lowest order in  $M$

$$\mathcal{L}_{\text{s.b.}} = \frac{F_0^2 B_0}{2} \text{Tr}(M U^\dagger + U M^\dagger)$$

where  $B_0$  is a new parameter. This means that the ground state ( $U = 1$ ) energy density is

$$\langle \mathcal{H}_{\text{eff}} \rangle = -F_0^2 B_0 (m_u + m_d + m_s)$$

In QCD

$$\left. \frac{\partial \langle 0 | \mathcal{H}_{\text{QCD}} | 0 \rangle}{\partial m_q} \right|_{m_u=m_d=m_s=0} = \frac{1}{3} \langle 0 | \bar{q} q | 0 \rangle_0 = \frac{1}{3} \langle \bar{q} q \rangle$$

and we have

$$3F_0^2 B_0 = -\langle \bar{q} q \rangle$$

# Mass term

$$\mathcal{L}_{\text{s.b.}} = \frac{F_0^2 B_0}{2} \text{Tr}(MU^\dagger + UM^\dagger) \quad 3F_0^2 B_0 = -\langle \bar{q}q \rangle$$

Constant  $B_0$  has dimension 1 (energy).

Expanding  $U(x) = \exp\left(i\frac{\phi(x)}{F_0}\right)$  gives  $\mathcal{L}_{\text{s.b.}} = -\frac{B_0}{2} \text{Tr}(\phi^2 M) + \dots$

Using  $\phi(x) = \sum_{a=1}^8 \lambda_a \phi_a(x) \equiv \begin{pmatrix} \pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}\pi^+ & \sqrt{2}K^+ \\ \sqrt{2}\pi^- & -\pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}K^0 \\ \sqrt{2}K^- & \sqrt{2}\bar{K}^0 & -\frac{2}{\sqrt{3}}\eta \end{pmatrix}$

one gets

$$\begin{aligned} \text{Tr}(\phi^2 M) = & 2(m_u + m_d)\pi^+\pi^- + 2(m_u + m_s)K^+K^- + 2(m_d + m_s)K^0\bar{K}^0 \\ & + (m_u + m_d)\pi^0\pi^0 + \frac{2}{\sqrt{3}}(m_u - m_d)\pi^0\eta + \frac{m_u + m_d + 4m_s}{3}\eta^2. \end{aligned}$$

mixing

# Mass term

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**Isospin symmetric limit**  $m_u = m_d = m$

$$\mathcal{L}_{\text{s.b}} = -\frac{B_0}{2}\text{Tr}(\phi^2 M) \quad \text{implies the following meson masses}$$

$$M_\pi^2 = 2B_0m,$$

$$M_K^2 = B_0(m + m_s), \quad \text{where } B_0 = -\langle\bar{q}q\rangle/(3F_0^2)$$

$$M_\eta^2 = \frac{2}{3}B_0(m + 2m_s)$$

**Gell-Mann – Okubo mass relation (does not depend on  $B_0$ )**

$$4M_K^2 = 4B_0(m + m_s) = 2B_0(m + 2m_s) + 2B_0m = 3M_\eta^2 + M_\pi^2$$

$$L = 4 \times 494^2 = 976\,144 \text{ MeV}^2 \quad R = 3 \times 548^2 + 138^2 = 919\,956 \text{ MeV}^2$$

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$$\frac{L - R}{L + R} = 3\%$$

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# PCAC

## partially conserved axial current

Let's calculate

$$\begin{aligned}
 \langle 0 | \phi^a(x) | \phi^b(p) \rangle &= \int \frac{d^3k}{(2\pi)^{3/2}} \frac{1}{\sqrt{2E_k}} e^{-ik \cdot x} \langle 0 | a_a(\mathbf{k}) \sqrt{(2\pi)^3 2E_p} a_b^\dagger(\mathbf{p}) | 0 \rangle \\
 &= \int d^3k \sqrt{\frac{E_p}{E_k}} e^{-ik \cdot x} \langle 0 | a_a(\mathbf{k}) a_b^\dagger(\mathbf{p}) | 0 \rangle \\
 &= \delta^{ab} e^{-ip \cdot x}
 \end{aligned}$$

Every field that has this property is called *interpolating field*.

Let's consider isospin subgroup of SU(3). Axial current matrix element

$$\langle 0 | A_i^\mu(x) | \pi_j(q) \rangle = iq^\mu F_0 e^{-iq \cdot x} \delta_{ij}$$

Let's take its divergence

$$\langle 0 | \partial_\mu A_i^\mu(x) | \pi_j(q) \rangle = iq^\mu F_0 \partial_\mu e^{-iq \cdot x} \delta_{ij} = M_\pi^2 F_0 e^{-iq \cdot x} \delta_{ij} = 2m_q B_0 F_0 e^{-iq \cdot x} \delta_{ij}$$

This means that divergence of the axial current, up to a constant, is itself the pion interpolating field. On the other hand  $\partial_\mu A_i^\mu = i m_q (\bar{q} \tau_i \gamma_5 q) = m_q P_i$  so pseudoscalar density is also a pion interpolating field.

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$$\frac{2 m_q \langle \bar{q}q \rangle}{3 F_0}$$

Let's take its divergence

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