

QCD lecture 13

January 21

QCD charges

$$\begin{aligned}Q_L^a(t) &= \int d^3x q_L^\dagger(\vec{x}, t) \frac{\lambda^a}{2} q_L(\vec{x}, t), \quad a = 1, \dots, 8, \\Q_R^a(t) &= \int d^3x q_R^\dagger(\vec{x}, t) \frac{\lambda^a}{2} q_R(\vec{x}, t), \quad a = 1, \dots, 8, \\Q_V(t) &= \int d^3x \left[q_L^\dagger(\vec{x}, t) q_L(\vec{x}, t) + q_R^\dagger(\vec{x}, t) q_R(\vec{x}, t) \right].\end{aligned}$$

Recall anti-commutation relations for quark fields

$$\begin{aligned}\{q_{\alpha,r}(\vec{x}, t), q_{\beta,s}^\dagger(\vec{y}, t)\} &= \delta^3(\vec{x} - \vec{y}) \delta_{\alpha\beta} \delta_{rs} \\ \{q_{\alpha,r}(\vec{x}, t), q_{\beta,s}(\vec{y}, t)\} &= 0, \\ \{q_{\alpha,r}^\dagger(\vec{x}, t), q_{\beta,s}^\dagger(\vec{y}, t)\} &= 0,\end{aligned}$$

QCD commutation rules

QCD charges form Lie algebra (exercise)

$$\begin{aligned}[Q_L^a, Q_L^b] &= if_{abc}Q_L^c, \\ [Q_R^a, Q_R^b] &= if_{abc}Q_R^c, \\ [Q_L^a, Q_R^b] &= 0, \\ [Q_L^a, Q_V] &= [Q_R^a, Q_V] = 0\end{aligned}$$

of $SU(3)_L \times SU(3)_R \times U(1)_V$ group

For conserved charges:

$$[Q_L^a, H_{\text{QCD}}^0] = [Q_R^a, H_{\text{QCD}}^0] = [Q_V, H_{\text{QCD}}^0] = 0$$

Axial current is anomalous, but otherwise it would commute with the hamiltonian as well.

QCD commutation rules

QCD charges form Lie algebra (exercise)

$$[Q_L^a, Q_L^b] = if_{abc}Q_L^c,$$

$$[Q_R^a, Q_R^b] = if_{abc}Q_R^c,$$

$$[Q_L^a, Q_R^b] = 0,$$

$$[Q_L^a, Q_V] = [Q_R^a, Q_V] = 0$$

$$[Q_V^a, Q_V^b] = if_{abc}Q_V^c$$

$$[Q_A^a, Q_A^b] = if_{abc}Q_V^c$$

$$[Q_V^a, Q_A^b] = if_{abc}Q_A^c$$

of $SU(3)_L \times SU(3)_R \times U(1)_V$ group

For conserved charges:

$$[Q_L^a, H_{\text{QCD}}^0] = [Q_R^a, H_{\text{QCD}}^0] = [Q_V, H_{\text{QCD}}^0] = 0$$

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Quark masses – χ SB

(chiral symmetry breaking)

$$M = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix}$$
$$= \frac{m_u + m_d + m_s}{\sqrt{6}} \lambda_0 + \frac{(m_u + m_d)/2 - m_s}{\sqrt{3}} \lambda_8 + \frac{m_u - m_d}{2} \lambda_3.$$
$$\lambda_0 = \sqrt{\frac{2}{3}} \mathbf{1}$$

Symmetry breaking lagrangian:

$$\mathcal{L}_M = -\bar{q}Mq = -(\bar{q}_R M q_L + \bar{q}_L M q_R)$$

Now we calculate variation of \mathcal{L}_M under chiral transformations

$$\exp\left(-i \sum_{a=1}^8 \Theta_a^L \frac{\lambda_a}{2}\right) e^{-i\Theta^L} \quad \text{and} \quad \exp\left(-i \sum_{a=1}^8 \Theta_a^R \frac{\lambda_a}{2}\right) e^{-i\Theta^R}$$

Quark masses – χ SB

(chiral symmetry breaking)

$$\delta\mathcal{L}_M = -i \left[\sum_{a=1}^8 \Theta_a^R \left(\bar{q}_R \frac{\lambda_a}{2} M q_L - \bar{q}_L M \frac{\lambda_a}{2} q_R \right) + \Theta^R (\bar{q}_R M q_L - \bar{q}_L M q_R) \right. \\ \left. + \sum_{a=1}^8 \Theta_a^L \left(\bar{q}_L \frac{\lambda_a}{2} M q_R - \bar{q}_R M \frac{\lambda_a}{2} q_L \right) + \Theta^L (\bar{q}_L M q_R - \bar{q}_R M q_L) \right],$$

From this we can easily calculate currents and current derivatives (lecture 12):

$$\partial_\mu L^{\mu,a} = \frac{\partial \delta\mathcal{L}_M}{\partial \Theta_a^L} = -i \left(\bar{q}_L \frac{\lambda_a}{2} M q_R - \bar{q}_R M \frac{\lambda_a}{2} q_L \right),$$

$$\partial_\mu R^{\mu,a} = \frac{\partial \delta\mathcal{L}_M}{\partial \Theta_a^R} = -i \left(\bar{q}_R \frac{\lambda_a}{2} M q_L - \bar{q}_L M \frac{\lambda_a}{2} q_R \right),$$

$$\partial_\mu L^\mu = \frac{\partial \delta\mathcal{L}_M}{\partial \Theta^L} = -i (\bar{q}_L M q_R - \bar{q}_R M q_L),$$

$$\partial_\mu R^\mu = \frac{\partial \delta\mathcal{L}_M}{\partial \Theta^R} = -i (\bar{q}_R M q_L - \bar{q}_L M q_R).$$

Quark masses – χ SB

(chiral symmetry breaking)

$$\partial_\mu V^{\mu,a} = i\bar{q}\left[M, \frac{\lambda_a}{2}\right]q,$$

➔
$$\partial_\mu A^{\mu,a} = i\left(\bar{q}_L\left\{\frac{\lambda_a}{2}, M\right\}q_R - \bar{q}_R\left\{\frac{\lambda_a}{2}, M\right\}q_L\right) = i\bar{q}\left\{\frac{\lambda_a}{2}, M\right\}\gamma_5 q,$$

$$\partial_\mu V^\mu = 0,$$

$$\partial_\mu A^\mu = 2i\bar{q}M\gamma_5 q + \text{anomaly}$$

Here we included anomaly, but for most of the time we will ignore it.

- Individual vector currents $\bar{u}\gamma^\mu u$, $\bar{d}\gamma^\mu d$ and $\bar{s}\gamma^\mu s$ are always conserved
- Vector current is a sum of them and is also conserved
- Baryon number is conserved
- Axial current is not conserved due to the quark masses (and anomaly)
- For equal quark mass all vector currents $V^{\mu,a}$ are conserved
- Axial flavor currents $A^{\mu,a}$ are not conserved, but their divergences are proportional to pseudoscalar densities. This leads to the concept of partially conserved axial currents (PCAC).

Chiral Ward identities

Define densities:

$$S_a(x) = \bar{q}(x)\lambda_a q(x), \quad P_a(x) = i\bar{q}(x)\gamma_5\lambda_a q(x), \quad a = 0, \dots, 8$$

$$S(x) = \bar{q}(x)q(x), \quad P(x) = i\bar{q}(x)\gamma_5 q(x)$$

Ward identities relate divergences of Green functions containing at least one current $V^{\mu,a}$ or $A^{\mu,a}$ to some linear combinations of other Green functions.

Example:

$$\begin{aligned} G_{AP}^{\mu,ab}(x, y) &= \langle 0 | T[A_a^\mu(x) P_b(y)] | 0 \rangle \\ &= \Theta(x_0 - y_0) \langle 0 | A_a^\mu(x) P_b(y) | 0 \rangle + \Theta(y_0 - x_0) \langle 0 | P_b(y) A_a^\mu(x) | 0 \rangle \end{aligned}$$

We shall calculate: $\partial_\mu^x G_{AP}^{\mu,ab}(x, y)$ remembering that

$$\partial_\mu^x \Theta(x_0 - y_0) = \delta(x_0 - y_0) g_{0\mu} = -\partial_\mu^x \Theta(y_0 - x_0)$$

Chiral Ward identities

Differentiating

$$\begin{aligned}
 G_{AP}^{\mu,ab}(x, y) &= \langle 0|T[A_a^\mu(x)P_b(y)]|0\rangle \\
 &= \Theta(x_0 - y_0)\langle 0|A_a^\mu(x)P_b(y)|0\rangle + \Theta(y_0 - x_0)\langle 0|P_b(y)A_a^\mu(x)|0\rangle
 \end{aligned}$$

we get:

$$\begin{aligned}
 \partial_\mu^x G_{AP}^{\mu,ab}(x, y) &= \delta(x_0 - y_0)\langle 0|A_0^a(x)P_b(y)|0\rangle - \delta(x_0 - y_0)\langle 0|P_b(y)A_0^a(x)|0\rangle \\
 &\quad + \Theta(x_0 - y_0)\langle 0|\partial_\mu^x A_a^\mu(x)P_b(y)|0\rangle + \Theta(y_0 - x_0)\langle 0|P_b(y)\partial_\mu^x A_a^\mu(x)|0\rangle \\
 &= \delta(x_0 - y_0)\langle 0|[A_0^a(x), P_b(y)]|0\rangle + \langle 0|T[\partial_\mu^x A_a^\mu(x)P_b(y)]|0\rangle,
 \end{aligned}$$

equal time commutator time ordered product
can be calculated from for conserved current
chiral algebra this term is zero
(time component!)

Chiral Ward identities

Generalization

$$\begin{aligned} \partial_\mu^x \langle 0 | T \{ J^\mu(x) A_1(x_1) \cdots A_n(x_n) \} | 0 \rangle &= \\ &= \langle 0 | T \{ [\partial_\mu^x J^\mu(x)] A_1(x_1) \cdots A_n(x_n) \} | 0 \rangle \\ &\quad + \delta(x^0 - x_1^0) \langle 0 | T \{ [J_0(x), A_1(x_1)] A_2(x_2) \cdots A_n(x_n) \} | 0 \rangle \\ &\quad + \delta(x^0 - x_2^0) \langle 0 | T \{ A_1(x_1) [J_0(x), A_2(x_2)] \cdots A_n(x_n) \} | 0 \rangle \\ &\quad + \cdots + \delta(x^0 - x_n^0) \langle 0 | T \{ A_1(x_1) \cdots [J_0(x), A_n(x_n)] \} | 0 \rangle \end{aligned}$$

Schwinger terms*

Schwinger has shown that naive commutation rules involving charge densities have extra contributions:

$$[J_0^a(\vec{x}, 0), J_i^b(\vec{y}, 0)] = iC_{abc}J_i^c(\vec{x}, 0)\delta^3(\vec{x} - \vec{y}) + S_{ij}^{ab}(\vec{y}, 0)\partial^j\delta^3(\vec{x} - \vec{y}),$$

where the Schwinger term satisfies

$$S_{ij}^{ab}(\vec{y}, 0) = S_{ji}^{ba}(\vec{y}, 0)$$

One can get rid of the Schwinger terms by redefining the time ordered product. In what follows we shall ignore Schwinger terms.

S. Treiman, R. Jackiw, and D. J. Gross, *Lectures on Current Algebra and Its Applications* (Princeton University Press, Princeton, 1972).

Current commutators

Full list:

$$[V_0^a(\vec{x}, t), V_b^\mu(\vec{y}, t)] = \delta^3(\vec{x} - \vec{y}) i f_{abc} V_c^\mu(\vec{x}, t),$$

$$[V_0^a(\vec{x}, t), V^\mu(\vec{y}, t)] = 0,$$

$$[V_0^a(\vec{x}, t), A_b^\mu(\vec{y}, t)] = \delta^3(\vec{x} - \vec{y}) i f_{abc} A_c^\mu(\vec{x}, t),$$

$$[V_0^a(\vec{x}, t), S_b(\vec{y}, t)] = \delta^3(\vec{x} - \vec{y}) i f_{abc} S_c(\vec{x}, t),$$

$$[V_0^a(\vec{x}, t), S_0(\vec{y}, t)] = 0,$$

$$[V_0^a(\vec{x}, t), P_b(\vec{y}, t)] = \delta^3(\vec{x} - \vec{y}) i f_{abc} P_c(\vec{x}, t),$$

$$[V_0^a(\vec{x}, t), P_0(\vec{y}, t)] = 0,$$

$$[A_0^a(\vec{x}, t), V_b^\mu(\vec{y}, t)] = \delta^3(\vec{x} - \vec{y}) i f_{abc} A_c^\mu(\vec{x}, t),$$

$$[A_0^a(\vec{x}, t), V^\mu(\vec{y}, t)] = 0,$$

$$[A_0^a(\vec{x}, t), A_b^\mu(\vec{y}, t)] = \delta^3(\vec{x} - \vec{y}) i f_{abc} V_c^\mu(\vec{x}, t),$$

$$[A_0^a(\vec{x}, t), S_b(\vec{y}, t)] = \delta^3(\vec{x} - \vec{y}) i f_{abc} P_c(\vec{x}, t),$$

$$[A_0^a(\vec{x}, t), S_0(\vec{y}, t)] = 0,$$

$$[A_0^a(\vec{x}, t), P_b(\vec{y}, t)] = \delta^3(\vec{x} - \vec{y}) i f_{abc} S_c(\vec{x}, t),$$

$$[A_0^a(\vec{x}, t), P_0(\vec{y}, t)] = 0.$$



Chiral Ward identities

Example: $G_{AP}^{\mu,ab}(x, y) = \langle 0|T[A_a^\mu(x)P_b(y)]|0\rangle$

we have shown: $\partial_\mu^x G_{AP}^{\mu,ab}(x, y)$

$$= \delta(x_0 - y_0) \langle 0|[A_0^a(x), P_b(y)]|0\rangle + \langle 0|T[\partial_\mu^x A_a^\mu(x)P_b(y)]|0\rangle$$

$$= \delta^4(x - y) i f_{abc} \langle 0|S_c(x)|0\rangle \quad < \text{follows from symmetry}$$

symmetry breaking $> +i \langle 0|T[\bar{q}(x) \{ \frac{\lambda_a}{2}, M \} \gamma_5 q(x) P_b(y)]|0\rangle$

We can now calculate
the anti-commutator
(no summation over a)
[exercise]

$$i\bar{q}(x) \{ \frac{\lambda_a}{2}, M \} \gamma_5 q(x) =$$

$$\left[\frac{1}{3}(m_u + m_d + m_s) + \frac{1}{\sqrt{3}} \left(\frac{m_u + m_d}{2} - m_s \right) d_{aa8} \right] P_a(x)$$

$$+ \left[\sqrt{\frac{1}{6}}(m_u - m_d) \delta_{a3} + \frac{\sqrt{2}}{3} \left(\frac{m_u + m_d}{2} - m_s \right) \delta_{a8} \right] P_0(x)$$

$$+ \frac{m_u - m_d}{2} \sum_{c=1}^8 d_{a3c} P_c(x).$$

QCD spectrum

Both vector and axial charges commute with QCD (massless) hamiltonian H_{QCD}^0 therefore the eigenstates organize themselves into irreducible representations of the chiral group $SU(3)_L \times SU(3)_R \times U(1)_V$ (axial $U(1)$ is broken by anomaly). States within each multiplet are (nearly) degenerate in mass and have well defined baryon number ($U(1)_V$ ensures baryon number conservation). Since axial and vector charges have opposite parity, one would expect that multiplets of opposite parity are degenerate in mass.

For positive parity states:
(e.g. baryon or meson
ground states)

$$H_{\text{QCD}}^0|i, +\rangle = E_i|i, +\rangle$$

$$P|i, +\rangle = +|i, +\rangle$$

Define now* $|\phi\rangle = Q_A^a|i, +\rangle$ and calculate its mass. Because $[H_{\text{QCD}}^0, Q_A^a] = 0$

$$H_{\text{QCD}}^0|\phi\rangle = H_{\text{QCD}}^0 Q_A^a|i, +\rangle = Q_A^a H_{\text{QCD}}^0|i, +\rangle = E_i Q_A^a|i, +\rangle = E_i|\phi\rangle$$

so the new state has the same energy (mass) but opposite parity

$$P|\phi\rangle = P Q_A^a P^{-1} P|i, +\rangle = -Q_A^a(+|i, +\rangle) = -|\phi\rangle$$

*charges and generators transforming Hilbert space states are identical

QCD spectrum

State $|\phi\rangle$ can be expanded in the basis of negative parity multiplet (in fact generators are Clebsch-Gordan coefficients)

$$|\phi\rangle = Q_A^a |i, +\rangle = -t_{ij}^a |j, -\rangle$$

But such degeneracy of opposite parity states is not seen in Nature.

$n^{2s+1}\ell_J$	J^{PC}	$l = 1$ $u\bar{d}, \bar{u}d,$ $\frac{1}{\sqrt{2}}(d\bar{d} - u\bar{u})$	$l = \frac{1}{2}$ $u\bar{s}, d\bar{s};$ $\bar{d}s, \bar{u}s$	$l = 0$ f'	$l = 0$ f	θ_{quad} [$^\circ$]	θ_{lin} [$^\circ$]
1^1S_0	0^{-+}	$\pi(138)$	$K(494)$	$\eta(548)$	$\eta'(958)$	-11.3	-24.5
1^3S_1	1^{--}	$\rho(770)$	$K^*(892)$	$\phi(1020)$	$\omega(782)$	39.2	36.5
1^1P_1	1^{+-}	$b_1(1235)$	K_{1B}^\dagger	$h_1(1415)$	$h_1(1170)$		
1^3P_0	0^{++}	$a_0(1450)$	$K_0^*(1430)$	$f_0(1710)$	$f_0(1370)$		
1^3P_1	1^{++}	$a_1(1260)$	K_{1A}^\dagger	$f_1(1420)$	$f_1(1285)$		
1^3P_2	2^{++}	$a_2(1320)$	$K_2^*(1430)$	$f_2'(1525)$	$f_2(1270)$	29.6	28.0
1^1D_2	2^{-+}	$\pi_2(1670)$	$K_2(1770)^\dagger$	$\eta_2(1870)$	$\eta_2(1645)$		
1^3D_1	1^{--}	$\rho(1700)$	$K^*(1680)^\ddagger$		$\omega(1650)$		
1^3D_2	2^{--}		$K_2(1820)^\dagger$				
1^3D_3	3^{--}	$\rho_3(1690)$	$K_3^*(1780)$	$\phi_3(1850)$	$\omega_3(1670)$	31.8	30.8
1^3F_4	4^{++}	$a_4(1970)$	$K_4^*(2045)$	$f_4(2300)$	$f_4(2050)$		
1^3G_5	5^{--}	$\rho_5(2350)$	$K_5^*(2380)$				
2^1S_0	0^{-+}	$\pi(1300)$	$K(1460)$	$\eta(1475)$	$\eta(1295)$		
2^3S_1	1^{--}	$\rho(1450)$	$K^*(1410)^\ddagger$	$\phi(1680)$	$\omega(1420)$		
2^3P_1	1^{++}	$a_1(1640)$					
2^3P_2	2^{++}	$a_2(1700)$	$K_2^*(1980)$	$f_2(1950)$	$f_2(1640)$		

J^P	(D, L_N^P)	S	Octet members				Singlets
$1/2^+$	$(56, 0_0^+)$	$1/2$	$N(939)$	$\Lambda(1116)$	$\Sigma(1193)$	$\Xi(1318)$	
$1/2^+$	$(56, 0_2^+)$	$1/2$	$N(1440)$	$\Lambda(1600)$	$\Sigma(1660)$	$\Xi(1690)^\dagger$	
$1/2^-$	$(70, 1_1^-)$	$1/2$	$N(1535)$	$\Lambda(1670)$	$\Sigma(1620)$	$\Xi(?)$	$\Lambda(1405)$
					$\Sigma(1560)^\dagger$		
$3/2^-$	$(70, 1_1^-)$	$1/2$	$N(1520)$	$\Lambda(1690)$	$\Sigma(1670)$	$\Xi(1820)$	$\Lambda(1520)$
$1/2^-$	$(70, 1_1^-)$	$3/2$	$N(1650)$	$\Lambda(1800)$	$\Sigma(1750)$	$\Xi(?)$	
					$\Sigma(1620)^\dagger$		
$3/2^-$	$(70, 1_1^-)$	$3/2$	$N(1700)$	$\Lambda(?)$	$\Sigma(1940)^\dagger$	$\Xi(?)$	
$5/2^-$	$(70, 1_1^-)$	$3/2$	$N(1675)$	$\Lambda(1830)$	$\Sigma(1775)$	$\Xi(1950)^\dagger$	
$1/2^+$	$(70, 0_2^+)$	$1/2$	$N(1710)$	$\Lambda(1810)$	$\Sigma(1880)$	$\Xi(?)$	$\Lambda(1810)^\dagger$
$3/2^+$	$(56, 2_2^+)$	$1/2$	$N(1720)$	$\Lambda(1890)$	$\Sigma(?)$	$\Xi(?)$	
$5/2^+$	$(56, 2_2^+)$	$1/2$	$N(1680)$	$\Lambda(1820)$	$\Sigma(1915)$	$\Xi(2030)$	
$7/2^-$	$(70, 3_3^-)$	$1/2$	$N(2190)$	$\Lambda(?)$	$\Sigma(?)$	$\Xi(?)$	$\Lambda(2100)$
$9/2^-$	$(70, 3_3^-)$	$3/2$	$N(2250)$	$\Lambda(?)$	$\Sigma(?)$	$\Xi(?)$	
$9/2^+$	$(56, 4_4^+)$	$1/2$	$N(2220)$	$\Lambda(2350)$	$\Sigma(?)$	$\Xi(?)$	

Spontaneous χ SB

What was wrong with the previous argument?

We have tacitly assumed that the ground state of QCD (vacuum) is annihilated by Q_A^a

To show this, consider a creation operator associated with positive parity fields a_i^\dagger creating positive parity state $|i, +\rangle$ and b_j^\dagger creates quanta of opposite parity. States $|i, +\rangle$ and $|j, -\rangle$ are basis states of an irreducible representation of $SU(3)_L \times SU(3)_R$

In analogy with (L.12 p. 11) $[Q^a(t), \Phi_k(\vec{y}, t)] = -t_{kj}^a \Phi_j(\vec{y}, t)$

we have $[Q_A^a, a_i^\dagger] = -t_{ij}^a b_j^\dagger$

Then $Q_A^a |i, +\rangle = Q_A^a a_i^\dagger |0\rangle = \left([Q_A^a, a_i^\dagger] + \underbrace{a_i^\dagger Q_A^a}_{\hookrightarrow 0} \right) |0\rangle = -t_{ij}^a b_j^\dagger |0\rangle$

If axial charges annihilate vacuum then we arrive at

$$|\phi\rangle = Q_A^a |i, +\rangle = -t_{ij}^a |j, -\rangle$$

What happens when $Q_A^a |0\rangle \neq 0$?

Spontaneous χ SB

Goldstone theorem:

For each charge (generator) of some symmetry group that does not annihilate vacuum, there corresponds a massless particle (Goldstone boson) of parity equal to the parity of this charge. In QCD natural candidates for Goldstone bosons are: π , K and η .

In QCD $Q_V^a|0\rangle = Q_V|0\rangle = 0$ so the vacuum is invariant under $SU(3)_V \times U(1)_V$. It follows that H_{QCD}^0 is also invariant (but not vice versa) and that the physical states correspond to some irreducible representations of $SU(3)_V \times U(1)_V$.

To each $Q_A^a|0\rangle \neq 0$ there corresponds a massless Goldstone boson field $\phi^a(x)$ with zero spin and

$$\phi^a(\vec{x}, t) \xrightarrow{P} -\phi^a(-\vec{x}, t)$$

Moreover:

$$[Q_V^a, \phi^b(x)] = if_{abc}\phi^c(x)$$

Quark masses break axial symmetry explicitly, so Goldstone bosons are not exactly massless.

Current commutators

Full list:

$$[V_0^a(\vec{x}, t), V_b^\mu(\vec{y}, t)] = \delta^3(\vec{x} - \vec{y}) i f_{abc} V_c^\mu(\vec{x}, t),$$

$$[V_0^a(\vec{x}, t), V^\mu(\vec{y}, t)] = 0,$$

$$[V_0^a(\vec{x}, t), A_b^\mu(\vec{y}, t)] = \delta^3(\vec{x} - \vec{y}) i f_{abc} A_c^\mu(\vec{x}, t),$$

$$[V_0^a(\vec{x}, t), S_b(\vec{y}, t)] = \delta^3(\vec{x} - \vec{y}) i f_{abc} S_c(\vec{x}, t),$$

$$[V_0^a(\vec{x}, t), S_0(\vec{y}, t)] = 0,$$

$$[V_0^a(\vec{x}, t), P_b(\vec{y}, t)] = \delta^3(\vec{x} - \vec{y}) i f_{abc} P_c(\vec{x}, t),$$

$$[V_0^a(\vec{x}, t), P_0(\vec{y}, t)] = 0,$$

$$[A_0^a(\vec{x}, t), V_b^\mu(\vec{y}, t)] = \delta^3(\vec{x} - \vec{y}) i f_{abc} A_c^\mu(\vec{x}, t),$$

$$[A_0^a(\vec{x}, t), V^\mu(\vec{y}, t)] = 0,$$

$$[A_0^a(\vec{x}, t), A_b^\mu(\vec{y}, t)] = \delta^3(\vec{x} - \vec{y}) i f_{abc} V_c^\mu(\vec{x}, t),$$

$$[A_0^a(\vec{x}, t), S_b(\vec{y}, t)] = \delta^3(\vec{x} - \vec{y}) i f_{abc} P_c(\vec{x}, t),$$

$$[A_0^a(\vec{x}, t), S_0(\vec{y}, t)] = 0,$$

$$[A_0^a(\vec{x}, t), P_b(\vec{y}, t)] = \delta^3(\vec{x} - \vec{y}) i f_{abc} S_c(\vec{x}, t),$$

$$[A_0^a(\vec{x}, t), P_0(\vec{y}, t)] = 0.$$



Quark condensate

Recall definitions

$$\begin{aligned} S_a(y) &= \bar{q}(y)\lambda_a q(y), \quad a = 0, \dots, 8, \\ P_a(y) &= i\bar{q}(y)\gamma_5\lambda_a q(y), \quad a = 0, \dots, 8. \end{aligned}$$

Generic quark bilinears

$$A_i(x) = q^\dagger(x)\hat{A}_i q(x)$$

have the following commutation rules

$$[A_1(\vec{x}, t), A_2(\vec{y}, t)] = \delta^3(\vec{x} - \vec{y})q^\dagger(x)[\hat{A}_1, \hat{A}_2]q(x)$$

Calculate commutators of vector currents $Q_V^a(t) = \int d^3x q^\dagger(\vec{x}, t)\frac{\lambda^a}{2}q(\vec{x}, t)$ with S and P

we have $[\frac{\lambda_a}{2}, \gamma_0\lambda_0] = 0$ and $[\frac{\lambda_a}{2}, \gamma_0\lambda_b] = \gamma_0 i f_{abc}\lambda_c$

scalar quark densities

transform as a singlet and

an octet

(similarly pseudoscalars)

$$[Q_V^a(t), S_0(y)] = 0, \quad a = 1, \dots, 8,$$

$$[Q_V^a(t), S_b(y)] = i \sum_{c=1}^8 f_{abc} S_c(y), \quad a, b = 1, \dots, 8$$

Quark condensate

$$[Q_V^a(t), S_b(y)] = i \sum_{c=1}^8 f_{abc} S_c(y), \quad a, b = 1, \dots, 8,$$

One can invert this relation with the help of (recall computation of the Casimir)

$$\sum_{a,b=1}^8 f_{abc} f_{abd} = 3\delta_{cd}$$

$$S_a(y) = -\frac{i}{3} \sum_{b,c=1}^8 f_{abc} [Q_V^b(t), S_c(y)]$$

Because vector charges annihilate vacuum $Q_V^a|0\rangle = 0$ we have

$$\langle 0|S_a(y)|0\rangle = \langle 0|S_a(0)|0\rangle \equiv \langle S_a\rangle = 0, \quad a = 1, \dots, 8$$

where we have used translation invariance of the ground state:

$$S(y) = e^{ipy} S(0) e^{-ipy}$$

Quark Condensate

$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_8 = \sqrt{\frac{1}{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

From $\langle S_a \rangle = 0$ we have:

$$a=3 \quad \langle \bar{u}u \rangle - \langle \bar{d}d \rangle = 0$$

$$a=8 \quad \langle \bar{u}u \rangle + \langle \bar{d}d \rangle - 2\langle \bar{s}s \rangle = 0$$

From these eqs. we have

$$\langle \bar{u}u \rangle = \langle \bar{d}d \rangle = \langle \bar{s}s \rangle$$

Because $[Q_V^a(t), S_0(y)] = 0$, $a = 1, \dots, 8$ the same argument cannot be used for singlet condensate.

However, it is clear that for nonzero quark condensates

$$0 \neq \langle \bar{q}q \rangle = \langle \bar{u}u + \bar{d}d + \bar{s}s \rangle = 3\langle \bar{u}u \rangle = 3\langle \bar{d}d \rangle = 3\langle \bar{s}s \rangle$$

Quark condensate

Now we shall calculate commutator $i[Q_a^A(t), P_a(y)]$ for fixed a

This is related to

$$(i)^2 [\gamma_5 \frac{\lambda_a}{2}, \gamma_0 \gamma_5 \lambda_a] = \lambda_a^2 \gamma_0$$

We have

$$\lambda_1^2 = \lambda_2^2 = \lambda_3^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_4^2 = \lambda_5^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\lambda_6^2 = \lambda_7^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\lambda_8^2 = \frac{1}{3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

Quark condensate

Now we shall calculate commutator $i[Q_a^A(t), P_a(y)]$ for fixed a

This is related to

$$(i)^2 [\gamma_5 \frac{\lambda_a}{2}, \gamma_0 \gamma_5 \lambda_a] = \lambda_a^2 \gamma_0$$

We have (suppressing y dependence)

$$i[Q_a^A(t), P_a(y)] = \begin{cases} \bar{u}u + \bar{d}d, & a = 1, 2, 3 \\ \bar{u}u + \bar{s}s, & a = 4, 5 \\ \bar{d}d + \bar{s}s, & a = 6, 7 \\ \frac{1}{3}(\bar{u}u + \bar{d}d + 4\bar{s}s), & a = 8 \end{cases}$$

which gives vacuum expectation value

$$\langle 0 | i[Q_a^A(t), P_a(y)] | 0 \rangle = \frac{2}{3} \langle \bar{q}q \rangle, \quad a = 1, \dots, 8$$

Goldstone bosons

Expectation value is non-zero and time independent

$$\begin{aligned} \langle 0 | i[Q_a^A(t), P_a] | 0 \rangle &= i \int d^3x \langle 0 | [A_a^0(x), P_a] | 0 \rangle \\ &= i \int d^3x \sum_n \{ \langle 0 | A_a^0(x) | n \rangle \langle n | P_a | 0 \rangle - \langle 0 | P_a | n \rangle \langle n | A_a^0(x) | 0 \rangle \} \end{aligned}$$

where

$$\sum_n = \sum_n \int \frac{d^4p_n}{(2\pi)^3} \delta(p_n^2 - m_n^2) = \sum_n \int \frac{d^3p_n}{(2\pi)^3 2p_n^0}$$

$$= i \int d^3x \sum_n \{ e^{-ip_n x} \langle 0 | A_a^0(0) | n \rangle \langle n | P_a | 0 \rangle - e^{ip_n x} \langle 0 | P_a | n \rangle \langle n | A_a^0(0) | 0 \rangle \}$$

only states with zero energy contribute (time indep.)

$$e^{-ip_n x} = e^{-i(p_n^0 t - \mathbf{p}_n \cdot \mathbf{x})}$$

$$\langle 0 | i[Q_a^A(t), P_a] | 0 \rangle = \frac{i}{2} \lim_{p^0 \rightarrow 0} \sum_b \int \frac{d^3p}{(2\pi)^3} \int d^3x \left\{ e^{i\mathbf{p} \cdot \mathbf{x}} \frac{\langle 0 | A_a^0 | \phi^b \rangle}{p^0} \langle \phi^b | P_a | 0 \rangle - \text{h.c.} \right\}$$

Integral over d^3x gives Dirac delta, which eats up integration over d^3p

Goldstone bosons

$$\langle 0 | i[Q_a^A(t), P_a] | 0 \rangle = \frac{i}{2} \lim_{p^0 \rightarrow 0} \sum_b \left\{ \frac{\langle 0 | A_a^0 | \phi^b \rangle}{p^0} \langle \phi^b | P_a | 0 \rangle - \langle 0 | P_a | \phi^b \rangle \frac{\langle \phi^b | A_a^0 | 0 \rangle}{p^0} \right\}$$

From hermicity and Lorentz invariance $\langle 0 | A_a^\mu | \phi^b(p) \rangle = ip^\mu F_\phi \delta^{ab}$

and we get

$$\langle 0 | i[Q_a^A(t), P_a] | 0 \rangle = -F_\phi \langle \phi^a | P_a | 0 \rangle = \frac{2}{3} \langle \bar{q}q \rangle$$

Here F_ϕ is Goldstone boson (pion) decay constant. Its value is ~ 93 MeV (different normalizations).

- There must exist states for which $\langle 0 | A_a^0(0) | n \rangle$ and $\langle 0 | P_a | n \rangle$ are non-zero
- It is not vacuum, because $\langle 0 | P_a | 0 \rangle = 0$
- Energy of these states must vanish, because the quark condensate is time independent
- So we need $E_n = 0$
- Such states are massless Goldstone bosons $|\phi^b\rangle$
- GBs are (pseudo)scalars – still to be proven

Dimensions

Field dimensions:

$$\left[\int d^3x \mathcal{L} \right] = [\text{energy}] = 1 \quad [d^3x] = [\text{distance}^3] = -3 \rightarrow [\mathcal{L}] = 4$$

$$4 = [\mathcal{L}_D] = [\bar{q}\partial q] = [q]^2 + 1 \rightarrow [q] = \frac{3}{2} \rightarrow [\langle \bar{q}q \rangle] = 3$$

$$4 = [\mathcal{L}_{YM}] = [F_{\mu\nu}F^{\mu\nu}] = [F_{\mu\nu}]^2 \rightarrow [F_{\mu\nu}] = 2$$

$$4 = [\mathcal{L}_\phi] = [(\partial_\mu\phi)^2] \rightarrow [\phi] = 1$$

Phenomenological values of condensates:

$$\begin{aligned} \langle \bar{q}q \rangle &\simeq -(250 \text{ MeV})^3 \\ \left\langle \frac{\alpha_s}{\pi} F_{\mu\nu}^a F^{a\mu\nu} \right\rangle &\simeq (400 \text{ GeV})^4 \end{aligned}$$

Dimension of currents

$$[J_\mu] = [\bar{q}\Gamma_\mu q] = 3$$

Dimensions

In the case of quantum fields there are different conventions. Here we follow:
 T-P. Cheng and L-F. Li *Gauge theory of elementary particle physics*

$$\phi_a(\mathbf{x}, t) = \int \frac{d^3k}{(2\pi)^{3/2}} \frac{1}{\sqrt{2E_k}} [a_a(\mathbf{k})e^{-ik \cdot x} + a_a^\dagger(\mathbf{k})e^{+ik \cdot x}]$$

$$[\phi_a] = 1 \rightarrow [a_a(\mathbf{k})] = -\frac{3}{2}$$

Indeed $[a_a(\mathbf{k}), a_a^\dagger(\mathbf{k}')] = \delta^{(3)}(\mathbf{k} - \mathbf{k}')$

Fock state: $|\phi_a(k)\rangle = \sqrt{(2\pi)^3 2E_k} a_a^\dagger(\mathbf{k}) |0\rangle \rightarrow [|\phi_a(k)\rangle] = -1$

Matrix element of axial current:

$$\begin{aligned} [\langle 0 | J_A^{\mu, a}(0) | \phi^b(p) \rangle] &= 3 - 1 = 2 \\ [ip^\mu F_0 \delta^{ab}] &= 2 \end{aligned}$$