

# QCD lecture 12

January 14

# Chiral symmetry

Quark masses (from sum rules at  $\mu = 1$  GeV)

$$\begin{pmatrix} m_u = 0.005 \text{ GeV} \\ m_d = 0.009 \text{ GeV} \\ m_s = 0.175 \text{ GeV} \end{pmatrix} \ll 1 \text{ GeV} \leq \begin{pmatrix} m_c = (1.15 - 1.35) \text{ GeV} \\ m_b = (4.0 - 4.4) \text{ GeV} \\ m_t = 174 \text{ GeV} \end{pmatrix}$$

Approximate symmetry: up, down, strange are massless.

QCD lagrangian (  $G_a^{\mu\nu}$  - field tensor)

$$\mathcal{L}_{\text{QCD}}^0 = \sum_{l=u,d,s} \bar{q}_l i \not{D} q_l - \frac{1}{4} G_{\mu\nu,a} G_a^{\mu\nu}$$

We know, that right-handed and left-handed fermions transform independently:

$$P_R = \frac{1}{2}(1 + \gamma_5) = P_R^\dagger, \quad P_L = \frac{1}{2}(1 - \gamma_5) = P_L^\dagger, \quad P_R + P_L = 1,$$

# Chiral symmetry

Define

$$q_R = P_R q, \quad q_L = P_L q$$

and rewrite the lagrangian

$$\mathcal{L}_{\text{QCD}}^0 = \sum_{l=u,d,s} (\bar{q}_{R,l} i \not{D} q_{R,l} + \bar{q}_{L,l} i \not{D} q_{L,l}) - \frac{1}{4} \mathcal{G}_{\mu\nu,a} \mathcal{G}_a^{\mu\nu}$$

Chiral symmetry (**global**  $U(3)_L \times U(3)_R$ )

$$\begin{pmatrix} u_L \\ d_L \\ s_L \end{pmatrix} \mapsto U_L \begin{pmatrix} u_L \\ d_L \\ s_L \end{pmatrix} = \exp \left( -i \sum_{a=1}^8 \Theta_a^L \frac{\lambda_a}{2} \right) e^{-i\Theta^L} \begin{pmatrix} u_L \\ d_L \\ s_L \end{pmatrix}$$
$$\begin{pmatrix} u_R \\ d_R \\ s_R \end{pmatrix} \mapsto U_R \begin{pmatrix} u_R \\ d_R \\ s_R \end{pmatrix} = \exp \left( -i \sum_{a=1}^8 \Theta_a^R \frac{\lambda_a}{2} \right) e^{-i\Theta^R} \begin{pmatrix} u_R \\ d_R \\ s_R \end{pmatrix}$$

# Parity

Consider Dirac equation

$$\{i\gamma^0\partial_t + i\boldsymbol{\gamma}\cdot\partial_{\mathbf{x}} - m\} \psi(t, \mathbf{x}) = 0$$

and space reflection  $\mathbf{x} \rightarrow -\mathbf{x}$

then 
$$\{i\gamma^0\partial_t - i\boldsymbol{\gamma}\cdot\partial_{\mathbf{x}} - m\} \psi(t, -\mathbf{x}) = 0$$

What is the wave function transformation generated by space reflection?

We have to change sign of space gamma matrices and leave unchanged time gamma matrix:

$$\gamma^0 = P^{-1}\gamma^0P \qquad -\boldsymbol{\gamma} = P^{-1}\boldsymbol{\gamma}P$$

Then 
$$\{i\gamma^0\partial_t + i\boldsymbol{\gamma}\cdot\partial_{\mathbf{x}} - m\} \psi^P(t, \mathbf{x}) = 0$$

where 
$$\psi^P(t, \mathbf{x}) = P \psi(t, -\mathbf{x})$$

# Parity

We need to solve  $\gamma^0 = P^{-1}\gamma^0 P$   $-\gamma = P^{-1}\gamma P$

and the solution reads (exercise):  $P = P^{-1} = \gamma^0$

Parity transformation  $P : q(\vec{x}, t) \mapsto \gamma_0 q(-\vec{x}, t)$ .

changes chirality (because  $\gamma^0$  anticommutes with  $\gamma^5$  )

$$q_R(\vec{x}, t) = P_R q(\vec{x}, t) \mapsto P_R \gamma_0 q(-\vec{x}, t) = \gamma_0 P_L q(-\vec{x}, t) \neq \pm q_R(-\vec{x}, t)$$

Parity transforms left and right fermions into each other.

QCD physical states (mesons) should be grouped in multiplets of some representations of  $U(3)_L \times U(3)_R$  and, because of the fact that parity transformation changes chirality, multiplets with positive and negative parity should be degenerate (in mass). This is not observed experimentally. We will make this statement more precise later.

# Conserved currents

Recall Noether theorem:

in order to find conserved currents of some global symmetry transformation, we have to promote this symmetry to a local one and calculate the currents.

Consider

$$\mathcal{L} = \mathcal{L}(\Phi_i, \partial_\mu \Phi_i)$$

which leads to the equations of motion:

$$\frac{\partial \mathcal{L}}{\partial \Phi_i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi_i} = 0, \quad i = 1, \dots, n$$

Suppose fields  $\Phi_i(x)$  transform according to some symmetry group (local). Consider infinitesimal transformation

$$\Phi_i(x) \mapsto \Phi'_i(x) = \Phi_i(x) + \delta \Phi_i(x) = \Phi_i(x) - i\epsilon_a(x) F_i^a[\Phi_j(x)]$$

which is not necessarily linear

$$\Phi_i(x) \mapsto \Phi'_i(x) = \Phi_i(x) - i\epsilon_a(x) t_{ij}^a \Phi_j(x)$$

# Conserved currents

Field transformation

$$\Phi_i(x) \mapsto \Phi'_i(x) = \Phi_i(x) + \delta\Phi_i(x) = \Phi_i(x) - i\epsilon_a(x)F_i^a[\Phi_j(x)]$$

Variation of the lagrangian

$$\begin{aligned}\delta\mathcal{L} &= \mathcal{L}(\Phi'_i, \partial_\mu\Phi'_i) - \mathcal{L}(\Phi_i, \partial_\mu\Phi_i) \\ &= \frac{\partial\mathcal{L}}{\partial\Phi_i}\delta\Phi_i + \frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi_i}\partial_\mu\delta\Phi_i \\ &= \epsilon_a(x) \left( -i\frac{\partial\mathcal{L}}{\partial\Phi_i}F_i^a - i\frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi_i}\partial_\mu F_i^a \right) + \partial_\mu\epsilon_a(x) \left( -i\frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi_i}F_i^a \right)\end{aligned}$$

Define current

$$J^{\mu,a} = -i\frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi_i}F_i^a$$

and calculate  
its divergence  
and use EoM

$$\begin{aligned}\partial_\mu J^{\mu,a} &= -i \left( \partial_\mu \frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi_i} \right) F_i^a - i \frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi_i} \partial_\mu F_i^a \\ &= -i \frac{\partial\mathcal{L}}{\partial\Phi_i} F_i^a - i \frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi_i} \partial_\mu F_i^a,\end{aligned}$$

$$\frac{\partial\mathcal{L}}{\partial\Phi_i} - \partial_\mu \frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi_i} = 0.$$

# Conserved currents

Field transformation

$$\Phi_i(x) \mapsto \Phi'_i(x) = \Phi_i(x) + \delta\Phi_i(x) = \Phi_i(x) - i\epsilon_a(x)F_i^a[\Phi_j(x)]$$

Variation of the lagrangian

$$\begin{aligned} \delta\mathcal{L} &= \mathcal{L}(\Phi'_i, \partial_\mu\Phi'_i) - \mathcal{L}(\Phi_i, \partial_\mu\Phi_i) \\ &= \frac{\partial\mathcal{L}}{\partial\Phi_i}\delta\Phi_i + \frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi_i}\partial_\mu\delta\Phi_i \\ &= \epsilon_a(x) \left( -i\frac{\partial\mathcal{L}}{\partial\Phi_i}F_i^a - i\frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi_i}\partial_\mu F_i^a \right) + \partial_\mu\epsilon_a(x) \left( -i\frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi_i}F_i^a \right) \end{aligned}$$

Define current

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$$\begin{aligned} \partial_\mu J^{\mu,a} &= -i \left( \partial_\mu \frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi_i} \right) F_i^a - i \frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi_i} \partial_\mu F_i^a \\ &= -i \frac{\partial\mathcal{L}}{\partial\Phi_i} F_i^a - i \frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi_i} \partial_\mu F_i^a, \end{aligned}$$

$$\frac{\partial\mathcal{L}}{\partial\Phi_i} - \partial_\mu \frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi_i} = 0.$$

# Conserved currents

We arrive at  $\delta\mathcal{L} = \epsilon_a(x)\partial_\mu J^{\mu,a} + \partial_\mu\epsilon_a(x)J^{\mu,a}$ .

This allows to define currents and current derivatives as

$$J^{\mu,a} = \frac{\partial\delta\mathcal{L}}{\partial\partial_\mu\epsilon_a},$$
$$\partial_\mu J^{\mu,a} = \frac{\partial\delta\mathcal{L}}{\partial\epsilon_a}.$$

If we demand the action to be invariant under global transformation, we conclude that the current is conserved:

$$\partial_\mu J^{\mu,a} = 0$$

It follows that there exists a conserved charge (exercise)

$$Q^a(t) = \int d^3x J_0^a(\vec{x}, t)$$

# Currents in QFT

Canonical quantization

define generalized momenta  $\Pi_i = \partial\mathcal{L}/\partial(\partial_0\Phi_i)$

and impose commutation rules:

$$\begin{aligned}[\Phi_i(\vec{x}, t), \Pi_j(\vec{y}, t)] &= i\delta^3(\vec{x} - \vec{y})\delta_{ij}, \\[\Phi_i(\vec{x}, t), \Phi_j(\vec{y}, t)] &= 0, \\[\Pi_i(\vec{x}, t), \Pi_j(\vec{y}, t)] &= 0.\end{aligned}$$

Suppose now that the symmetry transformation is linear

$$\Phi_i(x) \mapsto \Phi'_i(x) = \Phi_i(x) - i\epsilon_a(x)t_{ij}^a\Phi_j(x)$$

then (current and charge are operators now, normal ordering suppressed)

$$\begin{aligned}J^{\mu,a}(x) &= -it_{ij}^a\frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi_i}\Phi_j, \\Q^a(t) &= -i\int d^3x\Pi_i(x)t_{ij}^a\Phi_j(x)\end{aligned}$$

# Commutation rules

It is easy to show (exercise)

$$\begin{aligned} [Q^a(t), \Phi_k(\vec{y}, t)] &= -it_{ij}^a \int d^3x [\Pi_i(\vec{x}, t) \Phi_j(\vec{x}, t), \Phi_k(\vec{y}, t)] \\ &= -t_{kj}^a \Phi_j(\vec{y}, t). \end{aligned}$$

Field (operator) transformations induce transformations of the Hilbert space

$$|\alpha'\rangle = [1 + i\epsilon_a G^a(t)]|\alpha\rangle$$

where  $G^a$  are hermitian operators (they in principle could depend on time).  
We demand

$$\langle\beta|A|\alpha\rangle = \langle\beta'|A'|\alpha'\rangle$$

# Commutation rules

For a matrix element of a field we have

$$\begin{aligned} \langle \beta | \Phi_i(x) | \alpha \rangle &= \langle \beta' | \Phi'_i(x) | \alpha' \rangle \\ &= \langle \beta | [1 - i\epsilon_a G^a(t)] [\Phi_i(x) - i\epsilon_b t_{ij}^b \Phi_j(x)] [1 + i\epsilon_c G^c(t)] | \alpha \rangle. \end{aligned}$$

terms linear in  $\epsilon$  should vanish

$$0 = -i\epsilon_a [G^a(t), \Phi_i(x)] - \underbrace{i\epsilon_a t_{ij}^a \Phi_j(x)}_{i\epsilon_a [Q^a(t), \Phi_i(x)]},$$

From this we conclude that  $G^a(t) = Q^a(t)$

# Commutation rules

Finally

$$[Q^a(t), Q^b(t)] = -i(t_{ij}^a t_{jk}^b - t_{ij}^b t_{jk}^a) \int d^3x \Pi_i(\vec{x}, t) \Phi_k(\vec{x}, t)$$

Recalling that

$$t_{ij}^a t_{jk}^b - t_{ij}^b t_{jk}^a = iC_{abc} t_{ik}^c$$

We have

$$[Q^a(t), Q^b(t)] = iC_{abc} Q^c(t)$$

# QCD currents

$$\begin{pmatrix} u_L \\ d_L \\ s_L \end{pmatrix} \mapsto U_L \begin{pmatrix} u_L \\ d_L \\ s_L \end{pmatrix} = \exp\left(-i \sum_{a=1}^8 \Theta_a^L \frac{\lambda_a}{2}\right) e^{-i\Theta^L} \begin{pmatrix} u_L \\ d_L \\ s_L \end{pmatrix}$$

$$\begin{pmatrix} u_R \\ d_R \\ s_R \end{pmatrix} \mapsto U_R \begin{pmatrix} u_R \\ d_R \\ s_R \end{pmatrix} = \exp\left(-i \sum_{a=1}^8 \Theta_a^R \frac{\lambda_a}{2}\right) e^{-i\Theta^R} \begin{pmatrix} u_R \\ d_R \\ s_R \end{pmatrix}$$

Repeating the same steps we arrive at (for massless fermions)

$$\delta\mathcal{L}_{\text{QCD}}^0 = \bar{q}_R \left( \sum_{a=1}^8 \partial_\mu \Theta_a^R \frac{\lambda_a}{2} + \partial_\mu \Theta^R \right) \gamma^\mu q_R + \bar{q}_L \left( \sum_{a=1}^8 \partial_\mu \Theta_a^L \frac{\lambda_a}{2} + \partial_\mu \Theta^L \right) \gamma^\mu q_L$$

and (quark fields are now operators) we have 18 conserved currents:

$$L^{\mu,a} = \bar{q}_L \gamma^\mu \frac{\lambda_a}{2} q_L, \quad \partial_\mu L^{\mu,a} = 0,$$

$$R^{\mu,a} = \bar{q}_R \gamma^\mu \frac{\lambda_a}{2} q_R, \quad \partial_\mu R^{\mu,a} = 0.$$

# QCD currents

Define vector and axial currents

octet vector  $V^{\mu,a} = R^{\mu,a} + L^{\mu,a} = \bar{q}\gamma^\mu \frac{\lambda^a}{2} q,$

axial (exercise)  $A^{\mu,a} = R^{\mu,a} - L^{\mu,a} = \bar{q}\gamma^\mu \gamma_5 \frac{\lambda^a}{2} q,$

singlet vector  $V^\mu = \bar{q}_R \gamma^\mu q_R + \bar{q}_L \gamma^\mu q_L = \bar{q} \gamma^\mu q,$

axial (exercise)  $A^\mu = \bar{q}_R \gamma^\mu q_R - \bar{q}_L \gamma^\mu q_L = \bar{q} \gamma^\mu \gamma_5 q$

All these currents are conserved (modulo anomaly)

# Parity of currents

Parity operator:  $\gamma^0$

Transformation properties of gamma matrices

$\Gamma$	1	$\gamma^\mu$	$\sigma^{\mu\nu}$	$\gamma_5$	$\gamma^\mu\gamma_5$
$\gamma_0\Gamma\gamma_0$	1	$\gamma_\mu$	$\sigma_{\mu\nu}$	$-\gamma_5$	$-\gamma_\mu\gamma_5$

imply the following properties of currents

$$P : V^{\mu,a}(\vec{x}, t) \mapsto V_\mu^a(-\vec{x}, t),$$
$$P : A^{\mu,a}(\vec{x}, t) \mapsto -A_\mu^a(-\vec{x}, t).$$

# QCD charges

$$Q_L^a(t) = \int d^3x q_L^\dagger(\vec{x}, t) \frac{\lambda^a}{2} q_L(\vec{x}, t), \quad a = 1, \dots, 8,$$

$$Q_R^a(t) = \int d^3x q_R^\dagger(\vec{x}, t) \frac{\lambda^a}{2} q_R(\vec{x}, t), \quad a = 1, \dots, 8,$$

$$Q_V(t) = \int d^3x \left[ q_L^\dagger(\vec{x}, t) q_L(\vec{x}, t) + q_R^\dagger(\vec{x}, t) q_R(\vec{x}, t) \right].$$

Recall anti-commutation relations for quark fields

$$\{q_{\alpha,r}(\vec{x}, t), q_{\beta,s}^\dagger(\vec{y}, t)\} = \delta^3(\vec{x} - \vec{y}) \delta_{\alpha\beta} \delta_{rs}$$

$$\{q_{\alpha,r}(\vec{x}, t), q_{\beta,s}(\vec{y}, t)\} = 0,$$

$$\{q_{\alpha,r}^\dagger(\vec{x}, t), q_{\beta,s}^\dagger(\vec{y}, t)\} = 0,$$

# Commutators

To compute current commutators that are bilinears in quark fields, we will use

$$\begin{aligned} & \left[ q^\dagger(\mathbf{x}, t) \Gamma^{(1)} T^{(1)} q(\mathbf{x}, t), q^\dagger(\mathbf{y}, t) \Gamma^{(2)} T^{(2)} q(\mathbf{y}, t) \right] \\ &= \Gamma_{\alpha\beta}^{(1)} \Gamma_{\sigma\tau}^{(2)} T_{pq}^{(1)} T_{rs}^{(2)} \left[ q_{\alpha p}^\dagger(\mathbf{x}, t) q_{\beta q}(\mathbf{x}, t), q_{\sigma r}^\dagger(\mathbf{y}, t) q_{\tau s}(\mathbf{y}, t) \right] \end{aligned}$$

the identity

$$[ab, cd] = a\{b, c\}d - ac\{b, d\} + \{a, c\}db - c\{a, d\}b,$$

and canonical anti-commutation rules

$$\begin{aligned} \{q_{\alpha,r}(\vec{x}, t), q_{\beta,s}^\dagger(\vec{y}, t)\} &= \delta^3(\vec{x} - \vec{y}) \delta_{\alpha\beta} \delta_{rs}, \\ \{q_{\alpha,r}(\vec{x}, t), q_{\beta,s}(\vec{y}, t)\} &= 0, \\ \{q_{\alpha,r}^\dagger(\vec{x}, t), q_{\beta,s}^\dagger(\vec{y}, t)\} &= 0, \end{aligned}$$

# QCD commutation rules

QCD charges form Lie algebra (exercise)

$$\begin{aligned}[Q_L^a, Q_L^b] &= if_{abc}Q_L^c, \\ [Q_R^a, Q_R^b] &= if_{abc}Q_R^c, \\ [Q_L^a, Q_R^b] &= 0, \\ [Q_L^a, Q_V] &= [Q_R^a, Q_V] = 0\end{aligned}$$

of  $SU(3)_L \times SU(3)_R \times U(1)_V$  group

For conserved charges:

$$[Q_L^a, H_{\text{QCD}}^0] = [Q_R^a, H_{\text{QCD}}^0] = [Q_V, H_{\text{QCD}}^0] = 0$$

Axial current is anomalous, but otherwise it would commute with the hamiltonian as well.

# QCD commutation rules

QCD charges form Lie algebra (exercise)

$$[Q_L^a, Q_L^b] = if_{abc}Q_L^c,$$

$$[Q_R^a, Q_R^b] = if_{abc}Q_R^c,$$

$$[Q_L^a, Q_R^b] = 0,$$

$$[Q_L^a, Q_V] = [Q_R^a, Q_V] = 0$$

$$[Q_V^a, Q_V^b] = if_{abc}Q_V^c$$

$$[Q_A^a, Q_A^b] = if_{abc}Q_V^c$$

$$[Q_V^a, Q_A^b] = if_{abc}Q_A^c$$

of  $SU(3)_L \times SU(3)_R \times U(1)_V$  group

For conserved charges:

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