QCD Lecture 1

October 14, 2025

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http://th-www.if.uj.edu.pl/~michal/QCD_2025/

Two years ago we celebrated 50 years of QCD

https://indico.cern.ch/event/1276932/

Volume 47B, number 4 PHYSICS LETTERS 26 November 1973

ADVANTAGES OF THE COLOR OCTET GLUON PICTURE☆

H. FRITZSCH*, M. GELL-MANN and H. LEUTWYLER**
California Institute of Technology, Pasadena, Calif. 91109, USA

Received 1 October 1973

It is pointed out that there are several advantages in abstracting properties of hadrons and their currents from a Yang–Mills gauge model based on colored quarks and color octet gluons.

VOLUME 30, NUMBER 26 PHYSICAL REVIEW LETTERS 25 JUNE 1973

Reliable Perturbative Results for Strong Interactions?*

H. David Politzer
Jefferson Physical Laboratories, Harvard University, Cambridge, Massachusetts 0
(Received 3 May 1973)

An explicit calculation shows perturbation theory to be arbitrarily good for the dee Euclidean Green's functions of any Yang-Mills theory and of many Yang-Mills theori with fermions. Under the hypothesis that spontaneous symmetry breakdown is of dyn cal origin, these symmetric Green's functions are the asymptotic forms of the physically significant spontaneously broken solution, whose coupling could be strong.

VOLUME 30, NUMBER 26 PHYSICAL REVIEW LETTERS 25 JUNE 1973

Ultraviolet Behavior of Non-Abelian Gauge Theories*

David J. Gross† and Frank Wilczek
Joseph Henry Laboratories, Princeton University, Princeton, New Jersey 08540
(Received 27 April 1973)

It is shown that a wide class of non-Abelian gauge theories have, up to calculable logarithmic corrections, free-field–theory asymptotic behavior. It is suggested that Bjorken scaling may be obtained from strong-interaction dynamics based on non-Abelian gauge symmetry.

“My own feeling is that we have learned a great deal from field theory... that I am quite happy to discard it as an old, but rather friendly, mistress who I would be willing to recognize on the street if I should encounter her again. From a philosophical point of view and certainly from a practical one the S-matrix approach at the moment seems to me by far the most attractive.”

Marvin Goldberger, konferencja Solvay, 1961

[D.J. Gross, Nucl. Phys. B (Proc. Suppl.) 135 (2004) 193-211]

Marvin Goldberger

Goldberger was a professor of physics at Princeton University 1957 - 1977.
He received the Dannie Heineman Prize for Mathematical Physics in 1961.
In 1963 was elected to the U.S. National Academy of Sciences.
In 1965 he was elected a Fellow of the American Academy of Arts and Sciences.
From 1978 through 1987 he served as president of Caltech.
He was the Director of the Institute for Advanced Study from 1987 to 1991.
From 1991 to 1993 he was a professor of physics at the University of California, Los Angeles.
From 1993 until his death in November, 2014, he served on the faculty of the University of California, San Diego,
Goldberger also served as Dean of Natural Sciences for UC San Diego from 1994 to 1999.

In physics mostly known from so called Goldberger-Treiman relation:

$$g_{\pi NN} F_\pi = G_A M_N$$

and it is obeyed to 10% accuracy.

A: Perturbative QCD

Introduction, why QCD.
Interactions in field theory, Feynman rules (reminder).
Nonabelian SU(N) field theory, free case, interaction, color factors.
Deep Inelastic ep scattering, Bjorken scaling.
Elastic and inelastic cross-section.
Parton model.
Renormalization, example: self-energy.
Running coupling constant.
Infrared singularities.
DGLAP evolution equations.
Perturbative calculation of axial anomaly.

B: Path integral formulation of QCD

Introduction, reminder on the Dirac notation, path integral in QM.
Path integral for a classical scalar field, 2-point Green function, propagator in momentum space.
Fermions, functional determinants, Grassmann variables, Berezin integral.
Chiral transformation and its Jacobian.
Computing anomaly with the Fujikawa method, Atiyah-Singer theorem.
Theta term in QCD, topological current K_{mu}, fermion masses and theta term.
Quantization of the non-Abelian gauge theories: QCD vs. QED, Feynman rules, gauge fixing, Jacobian in the path integral.
Faddeev-Popov ghosts, Feynman rules for ghosts, on-shell Ward identities for QCD.
Lattice QCD.

C: Low energy QCD

Effective QCD: chiral symmetry, conserved currents and charges, chiral algebra, inclusion of quark masses.
Chiral Ward identities, QCD spectrum and chiral symmetry, quark condensate and Goldstone bosons, PCAC.
Nonlinear realization of chiral symmetry and Goldstone bosons, chiral lagrangian.
Gell-Mann-Okubo mass relation, Goldberger-Treiman relation.
Heavy quarks and heavy quark spin symmetry.

Francois Gelis,
"A Stroll Through Quantum Fields" (internet)

Richard Feynman,
"Feynman Lectures on Strong Interactions" (internet)

W. Greiner, S. Schramm, E. Stein
"Quantum Choromodynamics" (library)

J. Collins,
"Foundations of Perturbative QCD" (library)

S. Scherer,
"Introduction to Chiral Perturbation Theory" (internet)

Assessment:

Active participation in tutorials (min. 3 full solutions)

Absence: max. 3 (no consequences),
up to 5 – written solutions of selected problems required,
over 5 – no assesment (in the case of illness individual assesment scheme might be possible)

Oral exam: list of ~ 10 broad topics (problems)
Each student can choose one problem she/he would like to discuss, and then she/he will be asked to discuss another problem chosen by the examiner.

Global symmetry multiplication by phase

Both scalar complex field theory

$$\mathcal{L} = \partial_\mu \Phi^\dagger \partial^\mu \Phi - m^2 \Phi^\dagger \Phi$$

and spinor field theory

$$\mathcal{L} = \overline{\psi}(x)(i\partial\!\!\!/ - m)\psi(x) \qquad \overline{\psi} = \psi^\dagger \gamma^0$$

are invariant under the global transformation:

$$\Phi(x) \to \Phi'(x) = e^{-i\theta}\Phi(x)$$
$$\psi(x) \to \psi'(x) = e^{-i\theta}\psi(x)$$

The invariance due to the *local* transformation breaks down due to the derivative.

$$\psi(x) \to \psi'(x) = e^{-i\theta(x)}\psi(x)$$

Local U(1) symmetry

We need to "pull" the phase through the derivative

$$\psi(x) \to \psi'(x) = e^{-i\theta(x)}\psi(x)$$

$$\partial_\mu \psi'(x) = e^{-i\theta(x)}\Big(\partial_\mu - i(\partial_\mu \theta(x))\Big)\,\psi(x)$$

A new term appears: the phase derivative. To make the Lagrange density invariant we introduce the covariant derivative, i.e., the four-potential $A^\mu(x)$

$$\partial_\mu \to D_\mu = \partial_\mu + iqA_\mu$$

where q is a constant (as it turns out a *charge* of field ψ). Then

$$\begin{aligned} D'_\mu \psi'(x) &= \left(\partial_\mu + iqA'_\mu\right) e^{-i\theta(x)}\psi(x) \\ &= e^{-i\theta(x)}\big(\partial_\mu \underbrace{- i\partial_\mu\theta(x) + iqA'_\mu}_{=iq\,A_\mu}\big)\psi(x) \\ &= e^{-i\theta(x)} D_\mu \psi(x). \end{aligned}$$

← makes Lagrangian invariant

gauge transformation:

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) + \frac{1}{q}\partial_\mu\theta(x)$$

Local U(1) symmetyry

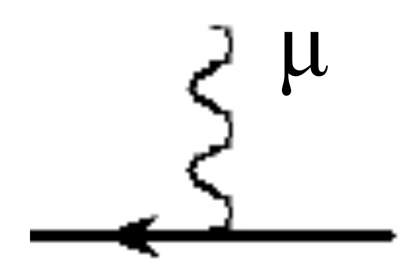
Rewrite: $$D'_\mu \psi'(x) = e^{-i\theta(x)} D_\mu \psi(x) = e^{-i\theta(x)} D_\mu e^{i\theta(x)} \psi'(x)$$

$$\psi'(x) = e^{-i\theta(x)} \psi(x)$$

Then,
the transformation law: $$D'_\mu = e^{-i\theta(x)} D_\mu e^{i\theta(x)}$$

Replacing the ordinary derivative with a covariant derivative generates a new term, which we call the *interaction lagrangian*:

$$\mathcal{L}_{\text{int}}(x) = -q\,\overline{\psi}(x)\gamma^\mu \psi(x)\, A_\mu(x)$$



we have explicit form of *current density*:

$$j^\mu = q\,\overline{\psi}(x)\gamma^\mu \psi(x)$$

q – coupling constant. Graphical representation
(Feynman diagram – time flows from right to left)

Quantum Chromo Dynamics

$$\Psi = \begin{bmatrix} \psi_1 \\ \vdots \\ \psi_N \end{bmatrix}$$

Gauge theory based on SU(N) group

$$\Psi(x) \to \Psi'(x) = U(x)\Psi(x) \qquad U(x) = e^{-i\theta_m(x)T^m} \qquad (m = 1, 2, \ldots N^2 - 1)$$

covariant derivative

$$D_\mu = \partial_\mu + igT^m A_\mu^m(x) = \partial_\mu + ig\boldsymbol{A}_\mu(x)$$

transforms as

$$D'_\mu = U(x) D_\mu U^\dagger(x) \longrightarrow$$

$$\longrightarrow \boldsymbol{A}'_\mu(x) = U(x)\boldsymbol{A}_\mu(x)U^\dagger(x) - \frac{i}{g}U(x)\partial_\mu U^\dagger(x)$$

SU(N) group

in fundamental representation generators are given as $N \times N$ hermitean matrices that satisfy commutation relations

$$[T_m, T_n] = i f_{mnl} T_l$$

f_{mnl} are totally antisymmetric tensors known as structure constants. To define the group we either give explicit form of the generators or a complete set of structure constants.

Examples:
SU(2)
Pauli matrices

$$T^i = \frac{1}{2}\tau^i$$

$$\tau^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \tau^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \tau^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Normalization:

$$\mathrm{Tr}(T_m T_n) = \frac{1}{2}\delta_{mn}$$

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Examples:
SU(3)
Gell-Mann
matrices

$$T^i = \frac{1}{2}\lambda^m$$

$$\lambda^1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \lambda^2 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \lambda^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\lambda^4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \lambda^5 = \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix},$$

$$\lambda^6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \lambda^7 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix},$$

$$\lambda^8 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Conjugated fundamental rep.

obviously, there are infintely many matrix representations related by the unitary transformation

$$T'_n = U^\dagger T_n U$$

let's complex conjugate the commutation relation

$$[T_m, T_n] = i f_{mnl} T_l$$

and multiply all generators by minus

$$[-T^*_m, -T^*_n] = i f_{mnl} \left(-T^*_l\right)$$

we have constructed conjugated representation $T'_n = -T^*_n$ satysfying commutation relation

is this representation unitary equivalent to the fundamental one?

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is this representation unitary equivalent to the fundamental one?

SU(2) – yes
SU(3) and higher – no

complication

$$\tau_i \tau_j = \delta_{ij} + i\varepsilon_{ijk}\tau_k,$$

$$\lambda_a \lambda_b = \frac{2}{3}\delta_{ab} + i f_{abc}\lambda_c + d_{abc}\lambda_c$$

therefore quarks and antiquarks are different objects

Adjoint representation

it follows from the Jacobi identity

$$[T_m, [T_n, T_l]] + [T_n, [T_l, T_m]] + [T_l, [T_m, T_n]] = 0$$

that

$$f_{nlk} f_{kmr} + f_{lmk} f_{knr} + f_{mnk} f_{klr} = 0$$

this relation can be writen in terms of $(N^2\text{-}1)$ x $(N^2\text{-}1)$ matrices defined as

$$\left(T_l^{\text{adj}}\right)_{mn} = -i f_{lmn}$$

in the following way

$$[T_m, T_n] = i f_{mnl} T_l$$

which means that T_l^{adj} are SU(3) generators, they form adjoint representation
note that

$$-\,T_l^{\text{adj}\,*} = T_l^{\text{adj}}$$

so adjoint representation is self-conjugated (real)

Adjoint representation

consider vector in the adjoint representation $A = (a^1, \ldots, a^{N^2-1})$

which transforms as $A' = U^{\mathrm{adj}} A \to a'^m = a^m - \theta^l f_{lmn} a^n + \ldots$.

because $U(x) = e^{-i\theta_m(x)T^m}$ and $\left(T_l^{\mathrm{adj}}\right)_{mn} = -i f_{lmn}$

one can write this transformation differently, defining $\boldsymbol{A} = \sum_{n=1}^{N^2-1} a^n T_n$

then $\boldsymbol{A}' = U \boldsymbol{A} U^\dagger$

leads to

$$\begin{aligned} a'^m T_m &= (1 - i\,\theta^n T_n + \ldots) a^m T_m (1 + i\,\theta^n T_n + \ldots) \\ &= a^m T_m - i\,\theta^n \left[T_n, T_m\right] a^m \\ &= a^m T_m + \theta^n\, f_{nmk} T_k\, a^m \\ &= \left(a^m - \theta^l\, f_{lmn}\, a^n\right)\, T_m , \end{aligned}$$

Adjoint representation

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gauge fields transform according to the adjoint representation of SU(N)

QED vs.QCD

field tensor in QED $F^{\mu\nu} = \partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu}$

can be expressed in terms of covariant derivatives, because the the field is Abelian:

$$F^{\mu\nu} = D^{\mu} A^{\nu} - D^{\nu} A^{\mu} = (\partial^{\mu} + iq\underline{A^{\mu}) A^{\nu}} - (\partial^{\nu} + iq\underline{A^{\nu}) A^{\mu}}$$

this can be generalized to the non Abelian case where the commutator does not vanish

$$\boldsymbol{F}_{\mu\nu} = D_{\mu}\boldsymbol{A}_{\nu} - D_{\nu}\boldsymbol{A}_{\mu} = \partial_{\mu}\boldsymbol{A}_{\nu} - \partial_{\nu}\boldsymbol{A}_{\mu} + ig\left[\boldsymbol{A}_{\mu}, \boldsymbol{A}_{\nu}\right]$$

in order to find transformaion law, we have first to prove that

$$\boldsymbol{F}_{\mu\nu} = \frac{1}{ig}[D_{\mu}, D_{\nu}]$$

commutator is in principle an operator and the field tensor is a function!

because

$$D'_{\mu} = U(x) D_{\mu} U^{\dagger}(x)$$

we have

$$\boldsymbol{F}'_{\mu\nu} = U(x)\boldsymbol{F}_{\mu\nu}U^{\dagger}(x)$$

QCD Lagrangian

gauge boson part (yang-Mills)

$$\mathcal{L}_{\mathrm{YM}} = -\frac{1}{2}\,\mathrm{Tr}(\boldsymbol{F}_{\mu\nu}\boldsymbol{F}^{\mu\nu}) = -\frac{1}{4}\sum_m F^m_{\mu\nu}F^{m\,\mu\nu}$$

in QED $(\partial_\mu A_\nu - \partial_\nu A_\mu)^2$

in QCD $(\partial_\mu \boldsymbol{A}_\nu - \partial_\nu \boldsymbol{A}_\mu + ig[\boldsymbol{A}_\mu, \boldsymbol{A}_\nu])^2$

QCD lagrangian contains interactions!
gluons interact with themselves, they carry adjoint color charge

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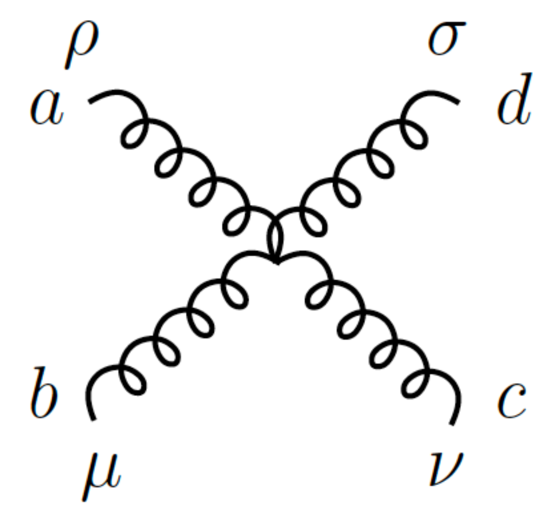

QCD lagrangian contains interactions!
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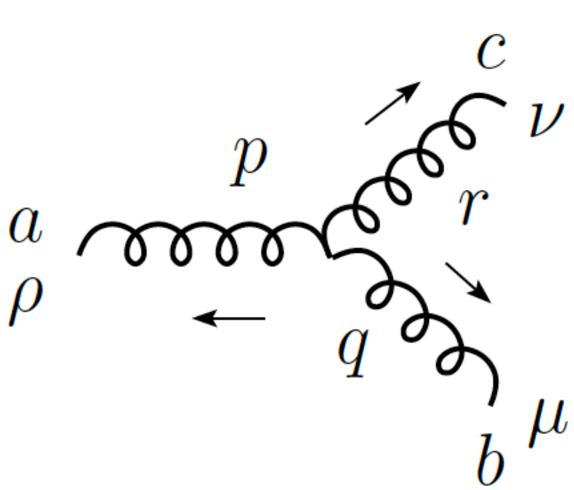
$$\begin{aligned}&-ig_s^2 f^{abe} f^{cde}\,(g_{\rho\nu}g_{\mu\sigma} - g_{\rho\sigma}g_{\mu\nu})\\&-ig_s^2 f^{ace} f^{bde}\,(g_{\rho\mu}g_{\nu\sigma} - g_{\rho\sigma}g_{\mu\nu})\\&-ig_s^2 f^{ade} f^{cbe}\,(g_{\rho\nu}g_{\mu\sigma} - g_{\rho\mu}g_{\sigma\nu})\end{aligned}$$



$$-g_s f^{abc}\left[(p-q)_\nu\, g_{\rho\mu} + (q-r)_\rho\, g_{\mu\nu} + (r-p)_\mu\, g_{\nu\rho}\right]$$

QCD Lagrangian

gauge boson part (yang-Mills)

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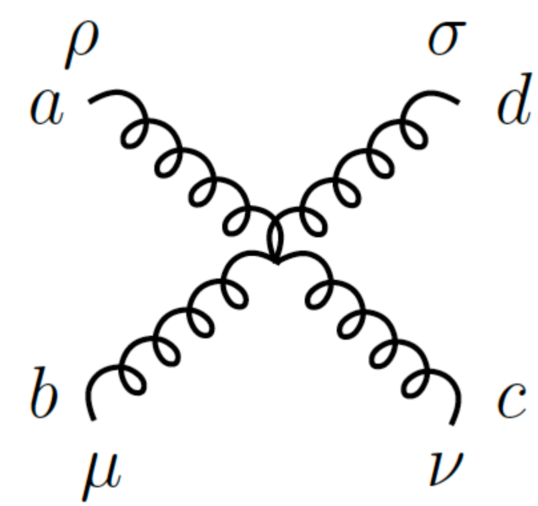

QCD lagrangian contains interactions!
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$$-ig_s^2 f^{abe} f^{cde}(g_{\rho\nu}g_{\mu\sigma} - g_{\rho\sigma}g_{\mu\nu})$$
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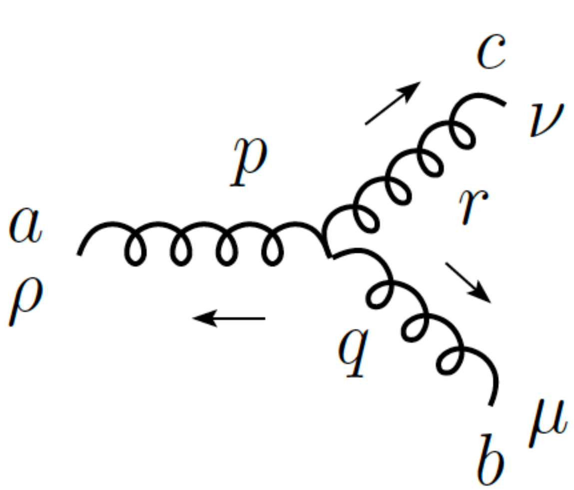
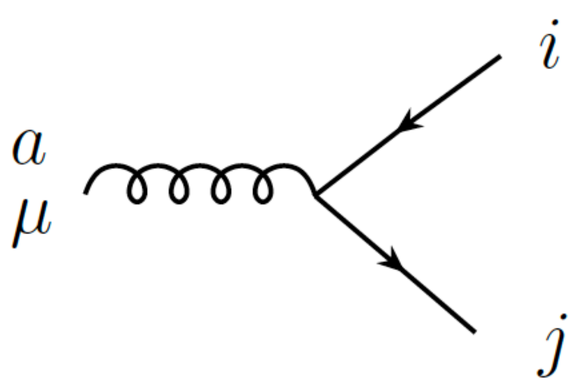


$$-g_s f^{abc}\left[(p-q)_\nu\, g_{\rho\mu} + (q-r)_\rho\, g_{\mu\nu} + (r-p)_\mu\, g_{\nu\rho}\right]$$

Full QCD lagrangian

$$\mathcal{L} = -\frac{1}{2}\,\mathrm{Tr}\,[\boldsymbol{F}_{\mu\nu}\boldsymbol{F}^{\mu\nu}] + \sum_{f=1}^{6}\left[\overline{q}_f\, i\gamma^{\mu} D_{\mu} q_f - m_f \overline{q}_f\, q_f\right]$$

quarks interact via covariant derivative



$$i g_s \gamma_\mu T^a_{ji}$$

propagators:

$$i S_F(p) = \cdot\; i\,\delta_{ij}\,\frac{(\not{k} + m)}{k^2 - m^2 + i\epsilon}$$

gauge choice!

$$i D_F(p)_{\mu\nu} = \frac{-i\,\delta_{ab}}{k^2 + i\epsilon}\left[g_{\mu\nu} - (1-\eta)\frac{k_\mu k_\nu}{k^2}\right]$$

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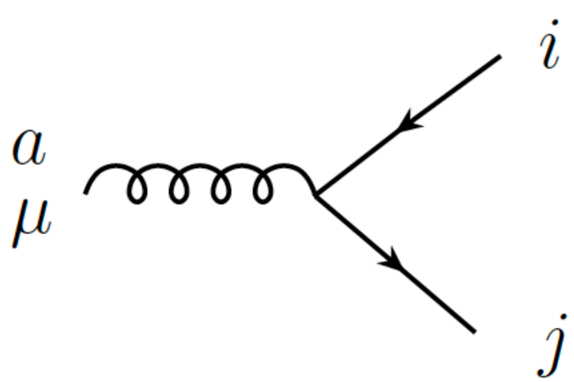

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