Short resume: path integrals in quantum mechanics

1 From the Schrödinger equation to path integrals

From the Schrödinger equation we know the evolution in time of a state given at t_a, $\Psi(x,t_a)$. Writing

$$\Psi(x,t_b) = e^{-\frac{i}{\hbar}H(t_b-t_a)}\Psi(x,t_a)\,, \tag{1}$$

we see that differentiation with respect to t_b gives the Schrödinger equation for $\Psi(x,t_b)$

$$i\hbar\,\frac{\partial\Psi(x,t_b)}{\partial t_b} = H\Psi(x,t_b)\ . \tag{2}$$

So, indeed, the operator $\exp\left(-iH(t_b-t_a)/\hbar\right)$ performs the evolution in time of a state given at t_a.

We denote the vector representing particle at x by $|x\rangle$, and the amplitude we seek is

$$K(b,a) = \langle x_b|\, e^{-\frac{i}{\hbar}H(t_b-t_a)}\, |x_a\rangle\,. \tag{3}$$

We will call $K(b,a)$ a propagator.

Let us remind that the state $|p\rangle$ which is the eigenstate of momentum belonging to the eigenvalue p has the following wave function, in other words its representation in the position space is

$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}}e^{\frac{i}{\hbar}px}\,, \tag{4}$$

with $1/\sqrt{2\pi}$ being a normalization constant. The state $\langle p|$ is conjugate to $|p\rangle$, hence

$$\langle p|x\rangle = \frac{1}{\sqrt{2\pi\hbar}}e^{-\frac{i}{\hbar}px}\,. \tag{5}$$

Indeed, this is the correct normalization factor since

$$\langle p|p'\rangle = \int dy\langle p|y\rangle\langle y|p'\rangle = \frac{1}{2\pi\hbar}\int dy e^{-\frac{i}{\hbar}(p-p')y} = \delta(p-p')\ .$$

Let us construct an analog of the classical evolution in time (t_b-t_a) from a given position x_a to a given position x_b (see Fig.1.1). We compute (3) introducing a discretization of time: $t_b-t_a=N\epsilon$ where N is very large, hence ϵ very small. For the sake of simplicity we set $\hbar=m=1$. Since

$$e^{-i(t_b-t_b)H}\ =\ e^{-i\epsilon NH}\ =\ e^{-i\epsilon H}\,e^{-i\epsilon H}\ ...\ e^{-i\epsilon H} \tag{6}$$

and

$$1 = \int dx_j\,|x_j\rangle\langle x_j|\ , \tag{7}$$

we have

$$\langle x_b|e^{-i(t_b-t_a)H}|x_a\rangle \;=\; \int \langle x_b|e^{-i\epsilon H}|x_{N-1}\rangle dx_{N-1}\langle x_{N-1}|e^{-i\epsilon H}|x_{N-2}\rangle$$
$$\ldots \langle x_2|e^{-i\epsilon H}|x_1\rangle dx_1\langle x_1|e^{-i\epsilon H}|x_a\rangle\,. \tag{8}$$

Looking at Fig. 1.1 and formula (8) we see that the quantal analog of the classical evolution in time given by just one trajectory going from the initial spacetime point to the final spacetime point consist of *infinitely many* trajectories between the two spacetime points. These trajectories appear with different weighting factors which we will presently calculate.

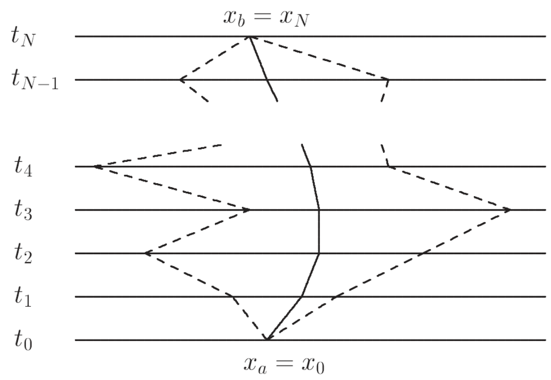

Figure 1: Classical path (solid) and two possible quantum paths (dashed) in a discretized time.

We do it for a simple hamiltonian

$$H = \frac{p^2}{2m} + V(x) = K + V\,. \tag{9}$$

Using the Campbell–Baker–Hausdorff formula for

$$e^A\, e^B = e^C \tag{10}$$

where

$$C = A + B + \frac{1}{2}[A,B] + \frac{1}{12}[A,[A,B]] + \frac{1}{12}[[A,B],B] + \;\ldots \tag{11}$$

we write

$$e^{-i\epsilon H} = e^{-i\epsilon(K+V)} = e^{-i\epsilon K}e^{-i\epsilon V} + O(\epsilon^2)\,, \tag{12}$$

where $O(\epsilon^2) = -\frac{1}{2}\epsilon^2[V,K]$, and $[V,K] = VK - KV \neq 0$ is the commutator.

In our amplitude (8) we can replace

$$\lim_{\substack{\epsilon\to 0,\, N\to\infty \\ N\epsilon=\text{const.}}} \left(e^{-i\epsilon(K+V)}\right)^N \;=\; \left(e^{-i\epsilon K}\, e^{-i\epsilon V}\right)^N\,. \tag{13}$$

Indeed, since the contents of the brackets in (13) differ by $O(\epsilon^2) \sim N^{-2}$, taking their N-th power makes that the r.h.s and the l.h.s of relation (13) differ by a term of $O(N^{-1})$. This relation is known as the Trotter product formula.

So, we write our propagator as follows

$$
\begin{aligned}
K(b,a) \;=&\; \langle x_b|e^{-i(t_b-t_a)H}|x_a\rangle \\
=&\; \int \langle x_b|e^{-i\epsilon K}|x_{N-1}\rangle e^{-i\epsilon V(x_{N-1})}\, dx_{N-1}\langle x_{N-1}|e^{-i\epsilon K}|x_{N-2}\rangle \\
&\times\, e^{-i\epsilon V(x_{N-2})} dx_{N-2} \;\;\ldots\;\; dx_1\langle x_1|e^{-i\epsilon K}|x_a\rangle e^{-i\epsilon V(x_a)}\,. \qquad (14)
\end{aligned}
$$

Now we calculate $\langle x|\exp\left(-i\epsilon K/\hbar\right)|y\rangle$ inserting back $\hbar$ and m. Since $|p\rangle$ is the eigenvector of the operator $K = p^2/2m$, we insert $1 = \int dp|p\rangle\langle p|$ and have

$$
\begin{aligned}
\langle x|e^{-\frac{i}{\hbar}\epsilon K}|y\rangle &= \int dp\langle x|e^{-\frac{i}{\hbar}\epsilon\frac{p^2}{2m}}|p\rangle\langle p|y\rangle \\
&= \int dp\langle x|p\rangle e^{\frac{-i\epsilon p^2}{\hbar 2m}}\langle p|y\rangle\,. \qquad (15)
\end{aligned}
$$

Using (4) and (5) we get

$$
\langle x|e^{-\frac{i}{\hbar}\epsilon K}|y\rangle = \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} dp\, e^{\frac{-i\epsilon p^2}{\hbar 2m}}\, e^{\frac{i}{\hbar}(y-x)p} = \sqrt{\frac{m}{2i\pi\hbar\epsilon}}\, e^{im\frac{(y-x)^2}{2\epsilon\hbar}}\,, \qquad (16)
$$

where we employed the formula

$$
\int_{-\infty}^{+\infty} dx\, e^{-ax^2+bx} = \sqrt{\frac{\pi}{a}} e^{\frac{b^2}{4a}}\,, \qquad \mathrm{Re}\, a \geq 0\,. \qquad (17)
$$

Note that for free particle (i.e. $V = 0$) Eq. (16) is the **exact** propagator $K(x, y)$ and $\varepsilon = T$ does not have to be infinitesimal.

We insert (16) into (14) and get our final expression for the propagator

$$
\begin{aligned}
K(b,a) &= \lim_{\epsilon\to 0}\sqrt{\frac{m}{2i\epsilon\hbar\pi}}\int\prod_{j=1}^{N-1} dx_j\sqrt{\frac{m}{2i\epsilon\hbar\pi}} e^{\frac{i}{\hbar}\epsilon L_j} \\
&\stackrel{\text{def}}{=} \int[\mathcal{D}x(t)]\, e^{\frac{i}{\hbar}\int_{t_a}^{t_b} dt L[x(t),\dot{x}(t)]} = \int[\mathcal{D}x(t)]\, e^{\frac{i}{\hbar}S[x(t)]} \qquad (18)
\end{aligned}
$$

where

$$
L_j = \tfrac{1}{2}m\left(\frac{x_{j+1}-x_j}{\epsilon}\right)^2 - V(x_j)\,.
$$

This is the key formula which gives the propagator in the form of a path integral. Its discretized form (the first part of (18)), which is a product of exponentials, can also be written as an exponential of the sum

$$
K(b,a) \;=\; \lim_{\epsilon\to 0}\int dx_1 \ldots dx_{N-1}\left(\frac{m}{2i\epsilon\hbar\pi}\right)^{\frac{1}{2}N} e^{\frac{i\epsilon}{\hbar}\sum_{j=0}^{N-1} L_j}\,. \qquad (19)
$$

Clearly

$$\lim_{\epsilon\to 0}\sum_{j=0}^{N-1}\epsilon L_j = \int_{t_a}^{t_b} dt\, L(x(t),\dot{x}(t)) = S[x(t)] \tag{20}$$

where $S[x(t)]$ is the functional of classical action. Hence from (18) and (19) we get an explicit expression for $\mathcal{D}x(t)$ for a discretized form of a "path integral":

$$[\mathcal{D}x(t)] = dx_1 \dots dx_{N-1}\left(\frac{m}{2i\epsilon\hbar\pi}\right)^{\frac{1}{2}N} . \tag{21}$$

2 Gaussian functional integrals

A very important class of Feynman propagators results from Lagrangians which are quadratic forms of $x(t)$ and $\dot{x}(t)$:

$$L(\dot{x},x,t) = a(t)\,\dot{x}^2(t) + b(t)\,\dot{x}x + c(t)\,x^2 + d(t)\dot{x} + e(t)\,x + f(t)\,. \tag{22}$$

In this case the propagator

$$K(x_b,x_a,t_b-t_a) = \int [\mathcal{D}x(t)]\, e^{\frac{i}{\hbar}\int_{t_a}^{t_b} dt L(\dot{x},x,t)} \;,$$

is evaluated as follows. We decompose the quantal trajectory into the classical trajectory, $\bar{x}(t)$,

$$\delta S[x(t)] = 0 \qquad \text{gives} \qquad \bar{x}(t)\;, \tag{23}$$

and a fluctuation, $y(t)$, around it

$$x(t) = \bar{x}(t) + y(t)\,, \qquad\qquad y(t_b) = y(t_a) = 0\,. \tag{24}$$

The action $S[x(t)]$ is stationary around $\bar{x}(t)$, hence terms linear in $y(t)$ vanish. Thus

$$S[\bar{x}(t)+y(t)] = S[\bar{x}(t)] + \tfrac{1}{2}\delta^2 S[y(t)] \tag{25}$$

where

$$\tfrac{1}{2}\delta^2 S[y(t)] = \frac{1}{2}\int_{t_a}^{t_b} dt\Big[\dot{y}\frac{\partial^2 L}{\partial\dot{x}^2}\dot{y} + 2y\frac{\partial^2 L}{\partial x\partial\dot{x}}\dot{y} + y\frac{\partial^2 L}{\partial x^2}y\Big]\,. \tag{26}$$

We shall integrate (26) by parts using

$$\begin{aligned} \dot{y}\frac{\partial^2 L}{\partial\dot{x}^2}\dot{y} &= \frac{d}{dt}\Big(y\frac{\partial^2 L}{\partial\dot{x}^2}\dot{y}\Big) - y\frac{d}{dt}\Big(\frac{\partial^2 L}{\partial\dot{x}^2}\dot{y}\Big)\,, \\ 2y\frac{\partial^2 L}{\partial x\partial\dot{x}}\dot{y} &= \frac{d}{dt}\Big(\frac{\partial^2 L}{\partial x\partial\dot{x}}y^2\Big) - y\frac{d}{dt}\Big(\frac{\partial^2 L}{\partial x\partial\dot{x}}\Big)y\,. \end{aligned} \tag{27}$$

This gives

$$\delta^2 S = -\int_{t_a}^{t_b} dt\, y\left[\frac{d}{dt}\left(\frac{\partial^2 L}{\partial\dot{x}^2}\frac{d}{dt}\right) + \left(\frac{d}{dt}\frac{\partial^2 L}{\partial x\partial\dot{x}}\right) - \frac{\partial^2 L}{\partial^2 x}\right] y = \int_{t_a}^{t_b} dt\, y D(t) y \tag{28}$$

where we have defined operator D as

$$D(t) = -\left[\frac{d}{dt}\left(\frac{\partial^2 L}{\partial \dot{x}^2}\frac{d}{dt}\right) + \left(\frac{d}{dt}\frac{\partial^2 L}{\partial x \partial \dot{x}}\right) - \frac{\partial^2 L}{\partial^2 x}\right]. \tag{29}$$

Since $\bar{x}(t)$ is fixed, the integration over paths reduces to integrating over all $y(t)$'s which vanish at the ends, thus $\mathcal{D}x(t) = \mathcal{D}y(t)$ and

$$K(x_b, x_a, t_b - t_a) = F(t_b - t_a)\, e^{\frac{i}{\hbar}S[\bar{x}(t)]}, \tag{30}$$

with

$$F(t_b - t_a) = \int [\mathcal{D}y(t)]\, e^{\frac{i}{\hbar}\frac{1}{2}\delta^2 S[y(t)]} \tag{31}$$

where $F(t_b - t_a)$ does not depend on the spatial positions because they are always equal zero.

Equation (30) tells us that when L is a quadratic form of $\dot{x}(t)$ and $x(t)$ the dependence of K on x_b and x_a is completely determined by the classical trajectory $\bar{x}(t)$ (more specifically: by the value of the functional of action, $S[\bar{x}(t)]$, calculated at $\bar{x}(t)$). One may interpret the prefactor $F(t_b - t_a)$ as the contribution of quantal fluctuations around the classical trajectory. Note, however, that although $\exp\{iS[\bar{x}(t)]/\hbar\}$ is uniquely determined by the classical trajectory, it is, nevertheless, a quantal object.

We shall now show how to compute F. For simplicity we assume $t_a = 0$ and $t_b = T$. We have

$$F(T) = \int [\mathcal{D}y(t)]\, \exp\left[\frac{i}{\hbar}\frac{1}{2}\int_0^T dt\, yD(t)y\right]. \tag{32}$$

The operator D can be diagonalized in terms of its eigenfunctions

$$D(t)y_n(t) = \lambda_n y_n(t) \quad \text{with} \quad y_n(0) = y_n(T) = 0\,. \tag{33}$$

One expands y as

$$y(t) = \sum_n a_n y_n(t) \tag{34}$$

since functions y_n form a complete set of normalized and orthogonal basis functions on $[0, T]$. Hence

$$\int_0^T dt\, y_m(t)y_n(t) = \delta_{mn} \tag{35}$$

and therefore

$$\int_0^T dt\, yD(t)y = \sum_n \lambda_n a_n^2\,. \tag{36}$$

The integration over $\mathcal{D}y(t)$ can be replaced by integrations over coefficients a_n up to a normalization constant $\mathcal{N}$

$$F(T) = \mathcal{N}\prod_n \int_0^T da_n \exp\left[\frac{i}{\hbar}\frac{1}{2}\lambda_n a_n^2\right] = \mathcal{N}'\prod_n \frac{1}{\sqrt{\lambda_n}} = \mathcal{N}'\,\frac{1}{\sqrt{\det D}}\,. \tag{37}$$

We are a bit carless as far as normalization constants are concerned, but we will shortly show how to fix the correct normalization. We see that the quantal part of the propagator, namely F, is inversely proportional to the square root of the determinant of D.

We are going to discuss now a few important examples of Gaussian propagators.

3 Free particle

We already have expression for it, Eq.(16),

$$K_0(b,a,T) = \int [\mathcal{D}x(t)]\, e^{\frac{i}{\hbar}\int_{t_a}^{t_b} \frac{1}{2} m\dot{x}^2\, dt} = \sqrt{\frac{m}{2i\pi\hbar T}}\, e^{i\frac{1}{2}m\frac{(x_b-x_a)^2}{\hbar T}} .$$

$S[\bar{x}(t)]$ is trivially simple to evaluate, however the prefactor is not that obvious. In fact this situation is typical: the prefactor is, as a rule, the main problem. Luckily in the free particle case it is fully determined from (17).

4 Harmonic oscillator

Now we have to evaluate

$$K(x_b,x_a,T) = \int [\mathcal{D}x(t)]\, e^{\frac{i}{\hbar}\int_{t_a}^{t_b} \frac{1}{2} m(\dot{x}^2-\omega^2x^2)\, dt} = F(T)\, e^{\frac{i}{\hbar}S[\bar{x}(t)]} \tag{38}$$

where $T = t_b - t_a$, and

$$\bar{x}(t) = \frac{1}{\sin\omega T}\, [x_b \sin\omega(t-t_a) + x_a \sin\omega(t_b - t)] . \tag{39}$$

Note that $\bar{x}(t)$ satisfies the correct boundary conditions

$$\bar{x}(t_a) = x_a , \qquad \bar{x}(t_b) = x_b . \tag{40}$$

With the help of (39) we get

$$S[\bar{x}(t)] = \int_{t_a}^{t_b} \tfrac{1}{2} m(\dot{\bar{x}}^2 - \omega^2\bar{x}^2)\, dt = \frac{m\omega}{2\sin\omega T}\, [(x_b^2 + x_a^2)\cos\omega T - 2x_b x_a] , \tag{41}$$

and it remains to evaluate the prefactor F. Note that the system does not distinguish any specific time, hence the amplitude may depend only on the difference $T = t_b - t_a$, so we assume $t_a = 0$.

For the harmonic oscillator

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2 \tag{42}$$

and therefore operator D reads

$$D = -m\left(\frac{d^2}{dt^2} + \omega^2\right). \tag{43}$$

We can safely skip m, as it only changes the normalization constant $\mathcal{N}'$ in (37).

Expanding y in a Fourier series

$$y(t) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi t}{T} \tag{44}$$

defines eigenfunctions od D. This representation of $y_n(t)$ satisfies the boundary conditions, $y_n(0) = y_n(T) = 0$. We have

$$Dy_n(t) = -\left(\frac{d^2}{dt^2} + \omega^2\right)\sin\frac{n\pi t}{T} = \left(\left(\frac{n\pi}{T}\right)^2 - \omega^2\right) y_n(t)\,. \tag{45}$$

Hence the eigenvalues λ_n read

$$\lambda_n = \left(\frac{n\pi}{T}\right)^2 - \omega^2 = \left(\frac{n\pi}{T}\right)^2\left(1 - \left(\frac{\omega T}{n\pi}\right)^2\right). \tag{46}$$

Now we can compute the determinant

$$\det D = \prod_m \left(\frac{m\pi}{T}\right)^2 \prod_n \left(1 - \left(\frac{\omega T}{n\pi}\right)^2\right) = \mathcal{C}\prod_n \left(1 - \left(\frac{\omega T}{n\pi}\right)^2\right), \tag{47}$$

where $\mathcal{C}$ is ω independent constant to be included in $\mathcal{N}'$.

Final answer can be obtained by means of the following identity (prove it!):

$$\lim_{N\to\infty}\prod_{n=1}^{N}\left(1 - \frac{\omega^2 T^2}{n^2\pi^2}\right)^{-\frac{1}{2}} = \left(\frac{\sin\omega T}{\omega T}\right)^{-\frac{1}{2}}.$$

Hence

$$K(b,a,T) = \mathcal{N}''\sqrt{\frac{\omega T}{\sin\omega T}}\exp\left[\frac{i}{\hbar}\frac{m\omega}{2\sin\omega T}\left[(x_b^2 + x_a^2)\cos\omega T - 2x_b x_a\right]\right]. \tag{48}$$

To fix $\mathcal{N}''$ we will take the limit $\omega \to 0$ and compare (48) with (16):

$$K(b,a,T) \to \mathcal{N}''\exp\left[\frac{i}{\hbar}\frac{m}{2T}(x_b - x_a)^2\right], \tag{49}$$

which implies

$$\mathcal{N}'' = \sqrt{\frac{m}{2i\pi\hbar T}}\,. \tag{50}$$

So we finally have

$$K(b,a,T) = \sqrt{\frac{m\omega}{2i\pi\hbar\sin\omega T}}\exp\left[\frac{i}{\hbar}\frac{m\omega}{2\sin\omega T}\left[(x_b^2 + x_a^2)\cos\omega T - 2x_b x_a\right]\right]. \tag{51}$$

5 Classical trajectories for the oscillator

First we consider a free harmonic oscillator. The equation of motion reads

$$\ddot{x} + \omega^2 x = 0. \tag{52}$$

Solutions are given in terms of trigonometric functions. Since we need to satisfy boundary conditions $\bar{x}(t_a) = x_a$ and $\bar{x}(t_b) = x_b$ the classical trajectory is easy to guess

$$\bar{x}(t) = \frac{1}{\sin \omega T} \left[x_a \sin \omega (t_b - t) + x_b \sin(t - t_a) \right] \tag{53}$$

where $T = t_b - t_a$.

To compute the action we shall integrate the kinetic term by parts

$$\begin{aligned} S &= \frac{m}{2} \int_{t_a}^{t_b} dt \, \left(\dot{\bar{x}}^2 - \omega^2 \bar{x}^2 \right) \\ &= \frac{m}{2} \left(\bar{x}(t_b)\dot{\bar{x}}(t_b) - \bar{x}(t_a)\dot{\bar{x}}(t_a) \right) - \frac{m}{2} \int_{t_a}^{t_b} dt \underbrace{\left(\ddot{\bar{x}} + \omega^2 \bar{x} \right)}_{=0} \bar{x} \, . \end{aligned} \tag{54}$$

Last term vanishes due to the equation of motion. We need therefore derivatives of the classical trajectory

$$\dot{\bar{x}}(t) = \frac{\omega}{\sin \omega T} \left[-x_a \cos \omega (t_b - t) + x_b \cos \omega (t - t_a) \right], \tag{55}$$

which gives

$$\begin{aligned} S &= \frac{m\omega}{2 \sin \omega T} \left(x_b \left[-x_a + x_b \cos \omega T \right] - x_a \left[-x_a \cos \omega T + x_b \right] \right) \\ &= \frac{m\omega}{2 \sin \omega T} \left[\left(x_a^2 + x_b^2 \right) \cos \omega T - 2 x_a x_b \right]. \end{aligned} \tag{56}$$

Now we shall consider a forced harmonic oscillator. The equation of motion reads

$$\ddot{x} + \omega^2 x = j, \;\; j(t) = \frac{f(t)}{m}, \tag{57}$$

where f is a force. This is an inhomogenous differential equation, which can be solved in terms of the Green function $G(t, s)$. We split the solution into two parts

$$\bar{x}(t) = \bar{x}_0(t) + \bar{x}_f(t), \tag{58}$$

where

$$\bar{x}_0(t) = \frac{1}{\sin \omega T} \left[x_a \sin \omega (t_b - t) + x_b \sin(t - t_a) \right] \tag{59}$$

is a solution of a free equation and $x_f(t_a) = x_f(t_b) = 0$. To compute the action we apply the same trick as previously

$$
\begin{aligned}
S &= \frac{m}{2}\int_{t_a}^{t_b} dt\, \left(\dot{\bar{x}}^2 - \omega^2 \bar{x}^2 + 2j\,\bar{x}\right) \\
&= \frac{m}{2}\left(\bar{x}(t_b)\dot{\bar{x}}(t_b) - \bar{x}(t_a)\dot{\bar{x}}(t_a)\right) - \frac{m}{2}\int_{t_a}^{t_b} dt \left[\underbrace{\left(\ddot{\bar{x}} + \omega^2\bar{x}\right)}_{=j} - 2j\right]\bar{x} \\
&= \frac{m}{2}\left(\bar{x}(t_b)\dot{\bar{x}}(t_b) - \bar{x}(t_a)\dot{\bar{x}}(t_a)\right) + \frac{1}{2}\int_{t_a}^{t_b} dt\, f(t)\bar{x}(t). \qquad (60)
\end{aligned}
$$

To find x_f we compute the Green function

$$
\left(\frac{d^2}{dt^2} + \omega^2\right) G(t,s) = \delta(t-s). \qquad (61)
$$

Then

$$
x_f(t) = \int_{t_a}^{t_b} ds\, G(t,s) j(s). \qquad (62)
$$

Indeed

$$
\left(\frac{d^2}{dt^2} + \omega^2\right) \bar{x}_0(t) = 0 \qquad (63)
$$

and

$$
\begin{aligned}
\left(\frac{d^2}{dt^2} + \omega^2\right) x_f(t) &= \int_{t_a}^{t_b} ds \left(\frac{d^2}{dt^2} + \omega^2\right) G(t,s)\, j(s) \\
&= \int_{t_a}^{t_b} ds\, \delta(t-s) j(s) \\
&= j(s). \qquad (64)
\end{aligned}
$$

Note that Green function $G(t,s)$ is symmetric and must satisfy boundary conditions $G(t_a,s) = G(t_b,s) = 0$. Apart from $t = s$ function $G(t,s)$ is a solution of a free oscillator (note that $t_a \le t, s \le t_b$)

$$
G(t,s) = \begin{cases} A\sin\omega(t-t_a) & \text{for} \quad t < s \\ B\sin\omega(t_b - t) & \text{for} \quad s < t \end{cases}. \qquad (65)
$$

Function G has to be continous at $t = s$ and must have a derivative jump

$$\int\limits_{s-\varepsilon}^{s+\varepsilon} dt \left(\frac{d^2}{dt^2} + \omega^2 \right) G(t,s) = \left. \frac{dG(t,s)}{dt} \right|_{t=s+\varepsilon} - \left. \frac{dG(t,s)}{dt} \right|_{t=s-\varepsilon} = 1. \tag{66}$$

Substituting (65) we obtain

$$-B\omega \cos \omega(t_b - s) - A\omega \cos \omega(s - t_a) = 1. \tag{67}$$

Continuity condition reads

$$A \sin \omega(s - t_a) = B \sin \omega(t_b - s). \tag{68}$$

This equation is easy to solve

$$A = C \sin \omega(t_b - s),\ B = C \sin \omega(s - t_a), \tag{69}$$

and constant C can be found from (67)

$$\begin{aligned} -C \left(\sin \omega(s - t_a) \cos \omega(t_b - s) + \sin \omega(t_b - s) \cos \omega(s - t_a) \right) &= \frac{1}{\omega} \\ &\Downarrow \\ -C \sin \omega T &= \frac{1}{\omega} \\ &\Downarrow \\ C &= -\frac{1}{\omega \sin \omega T}. \end{aligned} \tag{70}$$

Hence

$$G(t,s) = -\frac{1}{\omega \sin \omega T} \begin{cases} \sin \omega(t_b - s) \sin \omega(t - t_a) & \text{for} \quad t < s \\ \sin \omega(s - t_a) \sin \omega(t_b - t) & \text{for} \quad s < t \end{cases}. \tag{71}$$

We see that indeed, $G(t,s) = G(s,t)$.

The final result for x_f reads therefore:

$$\begin{aligned} \bar{x}_f(t) &= \frac{1}{m} \int\limits_{t_a}^{t_b} ds\, G(t,s) f(s) \\ &= -\frac{1}{m\omega \sin \omega T} \times \\ &\left[\int\limits_{t_a}^{t} ds\ \sin \omega(s - t_a) \sin \omega(t_b - t) f(s) + \int\limits_{t}^{t_b} ds\ \sin \omega(t_b - s) \sin \omega(t - t_a) f(s) \right]. \end{aligned} \tag{72}$$

We can convince ourselves that (72) satisfies boundary conditions $x_f(t_a) = x_f(t_b) = 0$.

Now, we shall compute the action splitting it into a several pieces

$$
\begin{aligned}
S &= \frac{m}{2}\left(\bar{x}(t_b)\dot{\bar{x}}(t_b) - \bar{x}(t_a)\dot{\bar{x}}(t_a)\right) + \frac{1}{2}\int_{t_a}^{t_b} dt\, f(t)\bar{x}(t) \\
&= \underbrace{\frac{m}{2}\left(\bar{x}_0(t_b)\dot{\bar{x}}_0(t_b) - \bar{x}_0(t_a)\dot{\bar{x}}_0(t_a)\right)}_{S_0} \\
&\quad + \underbrace{\frac{m}{2}\left(\bar{x}_0(t_b)\dot{\bar{x}}_f(t_b) - \bar{x}_0(t_a)\dot{\bar{x}}_f(t_a)\right)}_{S_1} \\
&\quad + \underbrace{\frac{1}{2}\int_{t_a}^{t_b} dt\, f(t)\bar{x}_0(t)}_{S_2} + \underbrace{\frac{1}{2}\int_{t_a}^{t_b} dt\, f(t)\bar{x}_f(t)}_{S_3}.
\end{aligned} \tag{73}
$$

Here S_0 is given by (56). To compute S_1 we need velocities of x_f

$$
\begin{aligned}
\frac{d}{dt}\bar{x}_f(t) &= -\frac{1}{m\omega\sin\omega T}\times \qquad (74) \\
&\quad \frac{d}{dt}\left[\int_{t_a}^{t} ds\, \sin\omega(s-t_a)\sin\omega(t_b-t)f(s) + \int_{t}^{t_b} ds\, \sin\omega(t_b-s)\sin\omega(t-t_a)f(s)\right] \\
&= -\frac{1}{m\omega\sin\omega T}\times \\
&\quad \left\{+\sin\omega(t-t_a)\sin\omega(t_b-t)f(t) - \omega\int_{t_a}^{t} ds\, \sin\omega(s-t_a)\cos\omega(t_b-t)f(s)\right. \\
&\quad \left. -\sin\omega(t_b-t)\sin\omega(t-t_a)f(t) + \omega\int_{t}^{t_b} ds\, \sin\omega(t_b-s)\cos\omega(t-t_a)f(s)\right\}
\end{aligned}
$$

and

$$
\begin{aligned}
\dot{\bar{x}}_f(t) &= \frac{1}{m\sin\omega T}\times \qquad (75) \\
&\quad \left\{\int_{t_a}^{t} ds\, \sin\omega(s-t_a)\cos\omega(t_b-t)f(s) - \int_{t}^{t_b} ds\, \sin\omega(t_b-s)\cos\omega(t-t_a)f(s)\right\}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\dot{x}_f(t_a) &= -\frac{1}{m\sin\omega T}\int\limits_{t_a}^{t_b} ds\, \sin\omega(t_b-s)f(s), \\
\dot{x}_f(t_b) &= +\frac{1}{m\sin\omega T}\int\limits_{t_a}^{t_b} ds\, \sin\omega(s-t_a)f(s).
\end{aligned} \tag{76}
$$

Now we can compute S_1

$$
\begin{aligned}
S_1 &= \frac{1}{2\sin\omega T}\left(\int\limits_{t_a}^{t_b} ds\, x_b\sin\omega(s-t_a)f(s) + x_a\int\limits_{t_a}^{t_b} ds\, \sin\omega(t_b-s)f(s)\right) \\
&= \frac{1}{2\sin\omega T}\left(\int\limits_{t_a}^{t_b} ds\, \left[x_b\sin\omega(s-t_a) + x_a\, \sin\omega(t_b-s)\right]f(s)\right)
\end{aligned}
$$

Computing S_2 is rather straightforward

$$
S_2 = \frac{1}{2\sin\omega T}\int\limits_{t_a}^{t_b} dt\, f(t)\left[x_a\sin\omega(t_b-t) + x_b\sin(t-t_a)\right]. \tag{77}
$$

Adding $S_1 + S_2$ we obtain

$$
S_{12} = S_1 + S_2 = \frac{x_a}{\sin\omega T}\int\limits_{t_a}^{t_b} dt\, f(t)\sin\omega(t_b-t) + \frac{x_b}{\sin\omega T}\int\limits_{t_a}^{t_b} dt\, f(t)\sin(t-t_a). \tag{78}
$$

Finally the last piece

$$
\begin{aligned}
S_3 = &-\frac{1}{2m\omega\sin\omega T} \\
&\left\{\int\limits_{t_a}^{t_b} dt\, f(t)\int\limits_{t_a}^{t} ds\, \sin\omega(s-t_a)\sin\omega(t_b-t)f(s)\right. \\
&\left. +\int\limits_{t_a}^{t_b} dt\, f(t)\int\limits_{t}^{t_b} ds\, \sin\omega(t_b-s)\sin\omega(t-t_a)f(s)\right\}.
\end{aligned} \tag{79}
$$

In the last integral we can change order of integrations

$$\begin{aligned}
& \int_{t_a}^{t_b} dt \int_{t}^{t_b} ds \, \sin\omega(t_b - s)\sin\omega(t - t_a) f(t) f(s) \\
= & \int_{t_a}^{t_b} ds \int_{t_a}^{s} dt \sin\omega(t_b - s)\sin\omega(t - t_a) f(t) f(s) \\
= & \int_{t_a}^{t_b} dt \int_{t_a}^{t} ds \sin\omega(t_b - t)\sin\omega(s - t_a) f(t) f(s). \qquad (80)
\end{aligned}$$

In the last line we have renamed variables $s \longleftrightarrow t$. We see that the last integral in (80) is equal to the first integral in (79). Therefore

$$S_3 = -\frac{1}{m\omega \sin\omega T} \int_{t_a}^{t_b} dt \, f(t) \int_{t_a}^{t} ds \, \sin\omega(s - t_a)\sin\omega(t_b - t) f(s). \qquad (81)$$

We can now add all terms together

$$\begin{aligned}
S &= \frac{m\omega}{2\sin\omega T}\left[\left(x_a^2 + x_b^2\right)\cos\omega T - 2x_a x_b\right] \\
&+ \frac{x_a}{\sin\omega T}\int_{t_a}^{t_b} dt \, f(t)\sin\omega(t_b - t) + \frac{x_b}{\sin\omega T}\int_{t_a}^{t_b} dt \, f(t)\sin(t - t_a) \\
&- \frac{1}{m\omega\sin\omega T}\int_{t_a}^{t_b} dt \, f(t)\int_{t_a}^{t} ds \, \sin\omega(s - t_a)\sin\omega(t_b - t) f(s). \qquad (82)
\end{aligned}$$

Or alternatively

$$\begin{aligned}
S &= \frac{m\omega}{2\sin\omega T}\Big\{ \left[\left(x_a^2 + x_b^2\right)\cos\omega T - 2x_a x_b\right] \\
&+ \frac{2x_a}{m\omega}\int_{t_a}^{t_b} dt \, f(t)\sin\omega(t_b - t) + \frac{2x_b}{m\omega}\int_{t_a}^{t_b} dt \, f(t)\sin(t - t_a) \\
&- \frac{2}{m^2\omega^2}\int_{t_a}^{t_b} dt \, f(t)\int_{t_a}^{t} ds \, \sin\omega(s - t_a)\sin\omega(t_b - t) f(s)\Big\}. \qquad (83)
\end{aligned}$$