QCD
problem set 12/2025

1. Effective Lagrangian describing Goldstone boson interactions reads

$$\mathcal{L}_{\text{eff}} = \frac{F^2}{4}\,\text{Tr}\left(\partial_\mu U \partial^\mu U^\dagger\right)$$

where $U = \exp(i\phi/F)$ can be expressed in terms of the physical meson fields:

$$\phi(x) = \sum_a \tau_a \phi^a(x) = \begin{bmatrix} \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -\pi^0 \end{bmatrix}$$

Expand $\mathcal{L}_{\text{eff}}$ up to 4-field interactions and calculate flavor trace for the case of SU(2).

SOLUTION:

First expand U

$$U = 1 + \frac{i}{F}\left(\boldsymbol{\tau}\cdot\boldsymbol{\phi}\right) - \frac{1}{2F^2}\left(\boldsymbol{\tau}\cdot\boldsymbol{\phi}\right)^2 - \frac{i}{6F^3}\left(\boldsymbol{\tau}\cdot\boldsymbol{\phi}\right)^3 + \ldots$$

and $\partial_\mu U$

$$\begin{aligned}\partial_\mu U &= \frac{i}{F}\left(\boldsymbol{\tau}\cdot\partial_\mu\boldsymbol{\phi}\right) - \frac{1}{2F^2}\left\{\left(\boldsymbol{\tau}\cdot\partial_\mu\boldsymbol{\phi}\right)\left(\boldsymbol{\tau}\cdot\boldsymbol{\phi}\right) + \left(\boldsymbol{\tau}\cdot\boldsymbol{\phi}\right)\left(\boldsymbol{\tau}\cdot\partial_\mu\boldsymbol{\phi}\right)\right\} \\ &\quad - \frac{i}{6F^3}\left\{\left(\boldsymbol{\tau}\cdot\partial_\mu\boldsymbol{\phi}\right)\left(\boldsymbol{\tau}\cdot\boldsymbol{\phi}\right)\left(\boldsymbol{\tau}\cdot\boldsymbol{\phi}\right) + \left(\boldsymbol{\tau}\cdot\boldsymbol{\phi}\right)\left(\boldsymbol{\tau}\cdot\partial_\mu\boldsymbol{\phi}\right)\left(\boldsymbol{\tau}\cdot\boldsymbol{\phi}\right) + \left(\boldsymbol{\tau}\cdot\boldsymbol{\phi}\right)\left(\boldsymbol{\tau}\cdot\boldsymbol{\phi}\right)\left(\boldsymbol{\tau}\cdot\partial_\mu\boldsymbol{\phi}\right)\right\} + \ldots\end{aligned}$$

and the same for $\partial^\mu U^\dagger$

$$\begin{aligned}\partial^\mu U^\dagger &= -\frac{i}{F}\left(\boldsymbol{\tau}\cdot\partial^\mu\boldsymbol{\phi}\right) - \frac{1}{2F^2}\left\{\left(\boldsymbol{\tau}\cdot\partial^\mu\boldsymbol{\phi}\right)\left(\boldsymbol{\tau}\cdot\boldsymbol{\phi}\right) + \left(\boldsymbol{\tau}\cdot\boldsymbol{\phi}\right)\left(\boldsymbol{\tau}\cdot\partial^\mu\boldsymbol{\phi}\right)\right\} \\ &\quad + \frac{i}{6F^3}\left\{\left(\boldsymbol{\tau}\cdot\partial^\mu\boldsymbol{\phi}\right)\left(\boldsymbol{\tau}\cdot\boldsymbol{\phi}\right)\left(\boldsymbol{\tau}\cdot\boldsymbol{\phi}\right) + \left(\boldsymbol{\tau}\cdot\boldsymbol{\phi}\right)\left(\boldsymbol{\tau}\cdot\partial^\mu\boldsymbol{\phi}\right)\left(\boldsymbol{\tau}\cdot\boldsymbol{\phi}\right) + \left(\boldsymbol{\tau}\cdot\boldsymbol{\phi}\right)\left(\boldsymbol{\tau}\cdot\boldsymbol{\phi}\right)\left(\boldsymbol{\tau}\cdot\partial^\mu\boldsymbol{\phi}\right)\right\} + \ldots\end{aligned}$$

Next calculate the trace

$$\begin{aligned}\text{Tr}\left(\partial_\mu U \partial^\mu U^\dagger\right) &= \frac{1}{F^2}\,\text{Tr}\left[\left(\boldsymbol{\tau}\cdot\partial_\mu\boldsymbol{\phi}\right)\left(\boldsymbol{\tau}\cdot\partial^\mu\boldsymbol{\phi}\right)\right] \\ &\quad - \frac{i}{2F^3}\,\text{Tr}\left[\left(\boldsymbol{\tau}\cdot\partial_\mu\boldsymbol{\phi}\right)\left\{\left(\boldsymbol{\tau}\cdot\partial^\mu\boldsymbol{\phi}\right)\left(\boldsymbol{\tau}\cdot\boldsymbol{\phi}\right) + \left(\boldsymbol{\tau}\cdot\boldsymbol{\phi}\right)\left(\boldsymbol{\tau}\cdot\partial^\mu\boldsymbol{\phi}\right)\right\}\right] \\ &\quad + \frac{i}{2F^3}\,\text{Tr}\left[\left(\boldsymbol{\tau}\cdot\partial_\mu\boldsymbol{\phi}\right)\left\{\left(\boldsymbol{\tau}\cdot\partial^\mu\boldsymbol{\phi}\right)\left(\boldsymbol{\tau}\cdot\boldsymbol{\phi}\right) + \left(\boldsymbol{\tau}\cdot\boldsymbol{\phi}\right)\left(\boldsymbol{\tau}\cdot\partial^\mu\boldsymbol{\phi}\right)\right\}\right] \\ &\quad + \text{four field interaction}\end{aligned}$$

We see that three field interaction is zero. Leading term gives the properly normalized kinetic term

$$\mathcal{L}^{(2)}_{\text{eff}} = \frac{F^2}{4}\frac{1}{F^2}\,\text{Tr}\left[\left(\boldsymbol{\tau}\cdot\partial_\mu\boldsymbol{\phi}\right)\left(\boldsymbol{\tau}\cdot\partial^\mu\boldsymbol{\phi}\right)\right] = \frac{1}{2}\partial_\mu\phi^a\partial^\mu\phi^a.$$

We are now left with four field interaction

$$
\begin{aligned}
\mathcal{L}_{\mathrm{eff}}^{(4)} &= \frac{F^2}{4}\left(-\frac{2}{6F^4}\right) \\
&\quad \times \mathrm{Tr}\left[(\boldsymbol{\tau}\cdot\partial_\mu\boldsymbol{\phi})\left\{(\boldsymbol{\tau}\cdot\partial^\mu\boldsymbol{\phi})(\boldsymbol{\tau}\cdot\boldsymbol{\phi})(\boldsymbol{\tau}\cdot\boldsymbol{\phi})+(\boldsymbol{\tau}\cdot\boldsymbol{\phi})(\boldsymbol{\tau}\cdot\partial^\mu\boldsymbol{\phi})(\boldsymbol{\tau}\cdot\boldsymbol{\phi})+(\boldsymbol{\tau}\cdot\boldsymbol{\phi})(\boldsymbol{\tau}\cdot\boldsymbol{\phi})(\boldsymbol{\tau}\cdot\partial^\mu\boldsymbol{\phi})\right\}\right] \\
&\quad +\frac{F^2}{4}\left(\frac{1}{4F^4}\right) \\
&\quad \times \mathrm{Tr}\left[\left\{(\boldsymbol{\tau}\cdot\partial_\mu\boldsymbol{\phi})(\boldsymbol{\tau}\cdot\boldsymbol{\phi})+(\boldsymbol{\tau}\cdot\boldsymbol{\phi})(\boldsymbol{\tau}\cdot\partial_\mu\boldsymbol{\phi})\right\}\left\{(\boldsymbol{\tau}\cdot\partial^\mu\boldsymbol{\phi})(\boldsymbol{\tau}\cdot\boldsymbol{\phi})+(\boldsymbol{\tau}\cdot\boldsymbol{\phi})(\boldsymbol{\tau}\cdot\partial^\mu\boldsymbol{\phi})\right\}\right].
\end{aligned}
$$

We can now use periodicity of the trace to simplify this expression

$$
\begin{aligned}
\mathcal{L}_{\mathrm{eff}}^{(4)} &= -\frac{1}{12F^2}\left\{2\,\mathrm{Tr}\left[(\boldsymbol{\tau}\cdot\partial_\mu\boldsymbol{\phi})(\boldsymbol{\tau}\cdot\partial^\mu\boldsymbol{\phi})(\boldsymbol{\tau}\cdot\boldsymbol{\phi})(\boldsymbol{\tau}\cdot\boldsymbol{\phi})\right]+\mathrm{Tr}\left[(\boldsymbol{\tau}\cdot\partial_\mu\boldsymbol{\phi})(\boldsymbol{\tau}\cdot\boldsymbol{\phi})(\boldsymbol{\tau}\cdot\partial^\mu\boldsymbol{\phi})(\boldsymbol{\tau}\cdot\boldsymbol{\phi})\right]\right\} \\
&\quad +\frac{1}{8F^2}\left\{\mathrm{Tr}\left[(\boldsymbol{\tau}\cdot\partial_\mu\boldsymbol{\phi})(\boldsymbol{\tau}\cdot\partial^\mu\boldsymbol{\phi})(\boldsymbol{\tau}\cdot\boldsymbol{\phi})(\boldsymbol{\tau}\cdot\boldsymbol{\phi})\right]+\mathrm{Tr}\left[(\boldsymbol{\tau}\cdot\partial_\mu\boldsymbol{\phi})(\boldsymbol{\tau}\cdot\boldsymbol{\phi})(\boldsymbol{\tau}\cdot\partial^\mu\boldsymbol{\phi})(\boldsymbol{\tau}\cdot\boldsymbol{\phi})\right]\right\} \\
&= \frac{1}{4F^2}\left(-\frac{2}{3}+\frac{1}{2}\right)\mathrm{Tr}\left[(\boldsymbol{\tau}\cdot\partial_\mu\boldsymbol{\phi})(\boldsymbol{\tau}\cdot\partial^\mu\boldsymbol{\phi})(\boldsymbol{\tau}\cdot\boldsymbol{\phi})(\boldsymbol{\tau}\cdot\boldsymbol{\phi})\right] \\
&\quad +\frac{1}{4F^2}\left(-\frac{1}{3}+\frac{1}{2}\right)\mathrm{Tr}\left[(\boldsymbol{\tau}\cdot\partial_\mu\boldsymbol{\phi})(\boldsymbol{\tau}\cdot\boldsymbol{\phi})(\boldsymbol{\tau}\cdot\partial^\mu\boldsymbol{\phi})(\boldsymbol{\tau}\cdot\boldsymbol{\phi})\right],
\end{aligned}
$$

so we finally get

$$
\mathcal{L}_{\mathrm{eff}}^{(4)} = \frac{1}{24F^2}\left\{\mathrm{Tr}\left[(\boldsymbol{\tau}\cdot\partial_\mu\boldsymbol{\phi})(\boldsymbol{\tau}\cdot\boldsymbol{\phi})(\boldsymbol{\tau}\cdot\partial^\mu\boldsymbol{\phi})(\boldsymbol{\tau}\cdot\boldsymbol{\phi})\right]-\mathrm{Tr}\left[(\boldsymbol{\tau}\cdot\partial_\mu\boldsymbol{\phi})(\boldsymbol{\tau}\cdot\partial^\mu\boldsymbol{\phi})(\boldsymbol{\tau}\cdot\boldsymbol{\phi})(\boldsymbol{\tau}\cdot\boldsymbol{\phi})\right]\right\}.
$$

Calculating the trace for SU(2) is rather easy:

$$
\mathrm{Tr}\left[\tau^i\tau^k\tau^l\tau^m\right] = 2\left(\delta^{ik}\delta^{lm}-\delta^{il}\delta^{km}+\delta^{im}\delta^{kl}\right)
$$

and we get

$$
\begin{aligned}
\mathrm{Tr}\left[(\boldsymbol{\tau}\cdot\partial_\mu\boldsymbol{\phi})(\boldsymbol{\tau}\cdot\boldsymbol{\phi})(\boldsymbol{\tau}\cdot\partial^\mu\boldsymbol{\phi})(\boldsymbol{\tau}\cdot\boldsymbol{\phi})\right] &= 4\left(\partial_\mu\boldsymbol{\phi}\cdot\boldsymbol{\phi}\right)\left(\partial^\mu\boldsymbol{\phi}\cdot\boldsymbol{\phi}\right)-2\left(\partial_\mu\boldsymbol{\phi}\cdot\partial^\mu\boldsymbol{\phi}\right)\left(\boldsymbol{\phi}\cdot\boldsymbol{\phi}\right), \\
\mathrm{Tr}\left[(\boldsymbol{\tau}\cdot\partial_\mu\boldsymbol{\phi})(\boldsymbol{\tau}\cdot\partial^\mu\boldsymbol{\phi})(\boldsymbol{\tau}\cdot\boldsymbol{\phi})(\boldsymbol{\tau}\cdot\boldsymbol{\phi})\right] &= 2\left(\partial_\mu\boldsymbol{\phi}\cdot\partial^\mu\boldsymbol{\phi}\right)\left(\boldsymbol{\phi}\cdot\boldsymbol{\phi}\right),
\end{aligned}
$$

which gives

$$
\mathcal{L}_{\mathrm{eff}}^{(4)} = \frac{1}{6F^2}\left\{\left(\partial_\mu\boldsymbol{\phi}\cdot\boldsymbol{\phi}\right)\left(\partial^\mu\boldsymbol{\phi}\cdot\boldsymbol{\phi}\right)-\left(\partial_\mu\boldsymbol{\phi}\cdot\partial^\mu\boldsymbol{\phi}\right)\left(\boldsymbol{\phi}\cdot\boldsymbol{\phi}\right)\right\}.
$$

In order to express these lagragians in terms of the physical pion fields, we observe that

$$
\phi_a = \frac{1}{2}\,\mathrm{Tr}\left(\tau_a\phi\right),
$$

which gives

$$
\phi_1 = \frac{1}{2}\,\mathrm{Tr}\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -\pi^0 \end{bmatrix}\right) = \frac{1}{2}\,\mathrm{Tr}\left(\begin{bmatrix} \sqrt{2}\pi^- & -\pi^0 \\ \pi^0 & \sqrt{2}\pi^+ \end{bmatrix}\right) = \frac{1}{\sqrt{2}}\left(\pi^+ + \pi^-\right),
$$

$$
\phi_2 = \frac{1}{2}\,\mathrm{Tr}\left(\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -\pi^0 \end{bmatrix}\right) = \frac{1}{2}\,\mathrm{Tr}\left(\begin{bmatrix} -i\sqrt{2}\pi^- & i\pi^0 \\ i\pi^0 & i\sqrt{2}\pi^+ \end{bmatrix}\right) = \frac{i}{\sqrt{2}}\left(\pi^+ - \pi^-\right),
$$

$$\phi_3 = \frac{1}{2}\,\mathrm{Tr}\left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -\pi^0 \end{bmatrix}\right) = \frac{1}{2}\,\mathrm{Tr}\left(\begin{bmatrix} \pi^0 & \sqrt{2}\pi^+ \\ -\sqrt{2}\pi & \pi^0 \end{bmatrix}\right) = \pi^0.$$

From this we get

$$\begin{aligned} \partial_\mu\phi^a\partial^\mu\phi^a &= \frac{1}{2}\partial_\mu\left(\pi^+ + \pi^-\right)\partial^\mu\left(\pi^+ + \pi^-\right) - \frac{1}{2}\partial_\mu\left(\pi^+ - \pi^-\right)\partial^\mu\left(\pi^+ - \pi^-\right) + \partial_\mu\pi^0\partial^\mu\pi^0 \\ &= 2\partial_\mu\pi^+\partial^\mu\pi^- + \partial_\mu\pi^0\partial^\mu\pi^0. \end{aligned}$$

Other terms can be computed in a similar way.

2. Mass term

$$\mathcal{L}_{\text{eff}}^{(m)} = -\frac{F^2 m_\pi^2}{4}\,\mathrm{Tr}(U + U^\dagger - 2).$$

We have

$$U + U^\dagger = 2 - \frac{1}{F^2}\left(\boldsymbol{\tau}\cdot\boldsymbol{\phi}\right)^2.$$

Hence

$$\mathrm{Tr}(U + U^\dagger - 2) = -\frac{2}{F^2}\boldsymbol{\phi}\cdot\boldsymbol{\phi}$$

and

$$\mathcal{L}_{\text{eff}}^{(m)} = \frac{m_\pi^2}{2}\boldsymbol{\phi}\cdot\boldsymbol{\phi}.$$

3. There exists another possible parametrization of U in SU(2)

$$U = \frac{1}{F}\left[\sigma(x) + i\,\vec{\tau}\cdot\vec{\pi}(x)\right] \text{ where } \sigma(x) = \sqrt{F^2 - \vec{\pi}^{\,2}(x)}.$$

Calculate the effective lagrangian up to 4 fields in this case.

SOLUTION:

First, let's observe that

$$U = \exp\left(i\frac{\boldsymbol{\tau}\cdot\boldsymbol{\phi}}{F}\right) = 1 + \frac{i}{F}\left(\boldsymbol{\tau}\cdot\boldsymbol{\phi}\right) - \frac{1}{2F^2}\left(\boldsymbol{\tau}\cdot\boldsymbol{\phi}\right)^2 - \frac{i}{6F^3}\left(\boldsymbol{\tau}\cdot\boldsymbol{\phi}\right)^3 + \dots$$

But

$$\left(\boldsymbol{\tau}\cdot\boldsymbol{\phi}\right)^2 = \tau^a\tau^b\phi_a\phi_b = \boldsymbol{\phi}^2,\ \left(\boldsymbol{\tau}\cdot\boldsymbol{\phi}\right)^3 = \left(\boldsymbol{\tau}\cdot\boldsymbol{\phi}\right)\,\boldsymbol{\phi}^2$$

and so on. Denoting

$$\phi = \sqrt{\boldsymbol{\phi}^2}$$

we obtain

$$U = \cos\frac{\phi}{F} + i\frac{\left(\boldsymbol{\tau}\cdot\boldsymbol{\phi}\right)}{F}\sin\frac{\phi}{F}.$$

Comparing with the definition above we find

$$\begin{aligned} \frac{\sigma}{F} &= \cos\frac{\phi}{F} = \left(1 - \frac{1}{2!}\phi^3 + \frac{1}{4!}\phi^4 + \dots\right), \\ \frac{\boldsymbol{\pi}}{F} &= \frac{\boldsymbol{\phi}}{F}\sin\frac{\phi}{F}. = \frac{\boldsymbol{\phi}}{F}\left(\phi - \frac{1}{3!}\phi^3 + \dots\right) \end{aligned}$$

Now, let's compute the lagrangian. We have

$$\partial_\mu \frac{\sigma}{F} = \partial_\mu \sqrt{1 - \frac{\boldsymbol{\pi}^2(x)}{F^2}} = -\frac{1}{2\sqrt{1-\frac{\boldsymbol{\pi}^2(x)}{F^2}}}\frac{1}{F^2}\partial_\mu \boldsymbol{\pi}^2(x).$$

We have therefore

$$\begin{aligned}
\mathrm{Tr}\left(\partial_\mu U \partial^\mu U^\dagger\right) &= \frac{1}{F^2}\,\mathrm{Tr}\left[\left(\partial_\mu \sigma(x) + i\boldsymbol{\tau}\cdot\partial_\mu \boldsymbol{\pi}(x)\right)\left(\partial^\mu \sigma(x) - i\boldsymbol{\tau}\cdot\partial^\mu \boldsymbol{\pi}(x)\right)\right] \\
&= \frac{1}{F^2}\,\mathrm{Tr}\left[\partial_\mu \sigma(x)\partial^\mu \sigma(x) + \left(\boldsymbol{\tau}\cdot\partial_\mu \boldsymbol{\pi}(x)\right)\left(\boldsymbol{\tau}\cdot\partial^\mu \boldsymbol{\pi}(x)\right)\right] \\
&= \frac{2}{F^2}\left[\partial_\mu \sigma(x)\partial^\mu \sigma(x) + \partial_\mu \boldsymbol{\pi}(x)\cdot\partial^\mu \boldsymbol{\pi}(x)\right] \\
&= \frac{2}{F^2}\partial_\mu \boldsymbol{\pi}(x)\cdot\partial^\mu \boldsymbol{\pi}(x) + \frac{2}{F^2 - \boldsymbol{\pi}^2(x)}\frac{1}{F^2}\left(\partial_\mu \boldsymbol{\pi}(x)\cdot\boldsymbol{\pi}(x)\right)\left(\partial^\mu \boldsymbol{\pi}(x)\cdot\boldsymbol{\pi}(x)\right)
\end{aligned}$$

Up to four fields we have

$$\mathcal{L}_{\text{eff}} = \frac{1}{2}\partial_\mu \boldsymbol{\pi}(x)\cdot\partial^\mu \boldsymbol{\pi}(x) + \frac{1}{2F^2}\left(\partial_\mu \boldsymbol{\pi}(x)\cdot\boldsymbol{\pi}(x)\right)\left(\partial^\mu \boldsymbol{\pi}(x)\cdot\boldsymbol{\pi}(x)\right).$$

Now, we shall compute the mass term

$$U + U^\dagger - 2 = \frac{2}{F}\sigma - 2 = 2\left(\sqrt{1-\frac{\boldsymbol{\pi}^2(x)}{F^2}} - 1\right) = -\frac{\boldsymbol{\pi}^2(x)}{F^2} + \ldots$$

Therefore

$$\mathcal{L}_{\text{eff}}^{(m)} = -\frac{F^2 m_\pi^2}{4}\,\mathrm{Tr}(U + U^\dagger - 2) = \frac{m_\pi^2}{4}\,\mathrm{Tr}(\boldsymbol{\pi}^2(x)) = \frac{m_\pi^2}{2}\boldsymbol{\pi}(x)\cdot\boldsymbol{\pi}(x)$$