QCD 2025/26

problem set 12

Tuesday, January 27, 2026

1. For a system of SU(3) scalar fields $\hat{\phi}_i(x)$ with $i = 1, 2, 3$ that satisfy the following commutation rules
$$\left[\hat{\phi}_i(t, \vec{x}), \hat{\pi}_j(t, \vec{x}\,')\right] = i\delta^{(3)}(\vec{x} - \vec{x}\,')\delta_{ij}$$
one defines charge operators
$$\hat{Q}^a(t) = -i \int d^3\vec{x}\, \hat{\pi}_i(t, \vec{x}) T^a_{ij} \hat{\phi}_j(t, \vec{x})$$
where matrices T^a satisfy SU(3) commutation relations:
$$\left[T^a, T^b\right] = if^{abc}T^c.$$
Prove that
$$\left[\hat{Q}^a(t), \hat{Q}^b(t)\right] = if^{abc}\hat{Q}^c(t).$$

2. In the case of fermion fields, commutation relations of scalar fields are replaced by anticommutation relations:
$$\left\{q_{\alpha,k}(t, \vec{x}), q^\dagger_{\beta,l}(t, \vec{x}\,')\right\} = \delta^{(3)}(\vec{x} - \vec{x}\,')\delta_{\alpha\beta}\delta_{kl}$$
where α, β stand for Dirac indices and k, l denote SU(3) indices. Relevant charges are defined as
$$\begin{aligned} \hat{Q}^a_{L,R}(t) &= \int d^3\vec{x}\, q^\dagger_{L,R}(t, \vec{x}) T^a q_{L,R}(t, \vec{x}), \\ \hat{Q}_V(t) &= \int d^3\vec{x}\, \left[q^\dagger_L(t, \vec{x}) q_L(t, \vec{x}) + q^\dagger_R(t, \vec{x}) q_R(t, \vec{x})\right] \end{aligned}$$
where $T^a = \lambda^a/a$ are SU(3) generators (Gell-Mann matrices). Making use of the identity (prove it!)
$$[ab, cd] = a\{b, c\}d - ac\{b, d\} + \{a, c\}db - c\{a, d\}b$$
show that
$$\begin{aligned} \left[\hat{Q}^a_L, \hat{Q}^b_L\right] &= if^{abc}\hat{Q}^c_L, \\ \left[\hat{Q}^a_R, \hat{Q}^b_R\right] &= if^{abc}\hat{Q}^c_R, \\ \left[\hat{Q}^a_L, \hat{Q}^b_R\right] &= 0, \\ \left[\hat{Q}^a_{L,R}, \hat{Q}_V\right] &= 0. \end{aligned}$$

3. Effective Lagrangian describing Goldstone boson interactions in the chiral limit reads

$$\mathcal{L}_{\text{eff}} = \frac{F^2}{4}\,\mathrm{Tr}\left(\partial_\mu U \partial^\mu U^\dagger\right)$$

where $U = \exp(i\phi/F)$ can be expressed in terms of the meson fields ϕ^a:

$$\phi(x) = \sum_a \lambda_a \phi^a(x)$$

where $\lambda_a = \tau_a$ are Pauli matrices for SU(2) and Gell-Mann matrices for SU(3).

Expand $\mathcal{L}_{\text{eff}}$ up to 4-fields. You should get one term with two fields and two terms with four fields (still expressed in terms of flavor traces). Calculate flavor traces for the case of SU(2) and express them in terms of the physical pion fileds:

$$\phi(x) = \sum_a \lambda_a \phi^a(x) = \begin{bmatrix} \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -\pi^0 \end{bmatrix}$$

4. For the SU(2) case the mass term corresponding to the non-zero quark masses reads:

$$\mathcal{L}_{\text{eff}}^{(m)} = -\frac{F^2 m_\pi^2}{4}\,\mathrm{Tr}(U + U^\dagger - 2)$$

Expand $\mathcal{L}_{\text{eff}}^{(m)}$ up to 2-fields and calculate flavor trace.

5. There exists another possible parametrization of U in SU(2)

$$U = \frac{1}{F}\left[\sigma(x) + i\,\vec{\tau}\cdot\vec{\pi}(x)\right] \text{ where } \sigma(x) = \sqrt{F^2 - \vec{\pi}^{\,2}(x)}.$$

Calculate the effective lagrangian up to 4 fields in this case.