QCD 2025
problem set 9

1. $f(\boldsymbol{\psi})$ is a function of N independent Grassmann variables ψ_i. Prove the property used during the lecture that if $\psi_i = J_{ij}\theta_j$ then

$$\int d^N\boldsymbol{\psi} f(\boldsymbol{\psi}) = \det{}^{-1}(J)\int d^N\boldsymbol{\theta} f(\boldsymbol{\theta}). \tag{1}$$

2. Consider Gaussian integral

$$J(\mathcal{M}) = \int d^N\xi\, d^N\psi\, \exp\left(\psi_i \mathcal{M}_{ij}\xi_j\right)$$

where ψ_i and ξ_i $(i = 1, 2, \ldots N)$ are independent Grassmann variables. Expanding in a power series and commuting ξ's and ψ's show that

$$J(\mathcal{M}) = \det(\mathcal{M}).$$

3. Anomaly is proportional to the integral

$$\int d^4\boldsymbol{k}\,\mathrm{Tr}\left\{\gamma^5 t\mathcal{F}\left(-\left[i\not{k} + \frac{\not{D}_x}{M}\right]^2\right)\right\}.$$

Expand $\mathcal{F}$ for large M and show that the only term contributing to the above integral is the term proportional to $\mathcal{F}''(k^2)$.

4. Winding number of the SU(2) gauge transformation U is defined as

$$N_{\mathrm{w}} = \frac{1}{24\pi^2}\varepsilon^{ijk}\int d^3r\ \mathrm{Tr}\left[\left(U^\dagger\partial_i U\right)\left(U^\dagger\partial_j U\right)\left(U^\dagger\partial_k U\right)\right]. \tag{2}$$

Calculate (2) for $U = \exp\left(i\,\vec{n}\cdot\vec{\tau}\,P(r)\right)$ where $\vec{n} = \vec{r}/r$. What are the boundary conditions for $P(r)$ that ensure that N_{w} is an integer?

HINT:

First decompose

$$U^\dagger\partial_i U = \frac{i}{2}\sum_{a=1}^{3}\xi_i^a\tau_a\,,$$

where τ_a are Pauli matrices. You should obtain that

$$\varepsilon^{ijk}\,\mathrm{Tr}\left[\left(U^\dagger\partial_i U\right)\left(U^\dagger\partial_j U\right)\left(U^\dagger\partial_k U\right)\right] \sim \det(\xi).$$

Due to the symmetry of U, elements of matrix ξ can be decomposed in the following way:

$$\xi_i^a = A\delta_{ia} + Bn_in_a + C\varepsilon_{iak}n_k.$$

Express $\det(\xi)$ in terms of A, B and C. You should get an answer, which is proportional to $(A^2 + C^2)(A + B)$.

In the last step calculate A, B and C. To this end expand U using de'Moivre (or Euler) formula for the exponent. For this you have to prove that $(\vec{n} \cdot \vec{\tau})^2 = 1$.

In order to differentiate U it is useful to use the following identities (prove them!)

$$\begin{aligned} \partial_i r &= n_i, \\ \partial_i n_k &= \frac{1}{r}(\delta_{ik} - n_i n_k). \end{aligned}$$