QCD 2025
problem set 7

1. Find classical action for the harmonic oscillator with external force $F(t)$. Take limit $\omega \to 0$ to obtain classical action for a particle moving in external force. Finally take limit $F \to 0$ to obtain action of a free particle.

   Hint

   The first step is to find the classical trajectory $\bar{x}(t_a) = a,\ \bar{x}(t_b) = b$. Note that for the harmonic oscillator e.o.m. with external force reads (prove it)

   $$\frac{d^2}{dt^2}x(t) + \omega^2 x(t) = j(t), \qquad j(t) = \frac{F(t)}{m}\,.$$

   This is an inhomogeneous differential equation, which one solves by the Green function method. The Green function satisfies

   $$\left[\frac{d^2}{dt^2} + \omega^2\right] G(t,s) = \delta(t-s)\,.$$

   Function G is symmetric in s and t, and satisfies boundary conditions $G(t_a, s) = G(t_b, s) = 0$. General solution for $\bar{x}$ reads

   $$\bar{x}(t) = x_0(t) + x_F(t)\,,$$

   where x_0 is the classical trajectory for $F = 0$ computed previously, and

   $$x_F(t) = \int_{t_a}^{t_b} ds\, G(t,s)\, j(s)\,.$$

   To compute the action use the same trick as previously, namely integrate by parts.

2. At the lecture we have expanded action S of a free harmonic oscillator around the classical trajectory: $x(t) = \bar{x}(t) + y(t)$ and showed that quantum contribution to K, denoted by F, where

   $$K = F(T)\, e^{\frac{i}{\hbar} S[\bar{x}(t)]}$$

   reads as follows

   $$F(T) = \int [\mathcal{D}y(t)]\, e^{\frac{i}{\hbar}\int_0^T \frac{1}{2} m(\dot{y}^2 - \omega^2 y^2)\, dt}\,.$$

   Compute F.

   From the equation for F above, one can easily find operator D defined at the lecture. Note that the system does not distinguish any specific time, hence the amplitude may depend only on the difference $T = t_b - t_a$.

One of the methods of calculating F is to expand

$$y(t) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi t}{T}\,, \qquad n > 0\,.$$

This representation of $y(t)$ satisfies the boundary conditions, $y(0) = y(T) = 0$. Note that

$$\int [\mathcal{D}y(t)] \sim \prod_n da_n$$

with all kinds of factors in front, *but we do not need to calculate them.* This is so because we know the normalization of F in the limit $\omega \to 0$, which is just the free particle propagator from problem 1. Using the fact that functions $\sin \frac{n\pi t}{T}$ form a complete set of orthogonal functions over the time interval $0 \le t \le T$ one can easily compute the argument of the exponent in F, and then perform the Gaussian integrals over da_n's. Final answer can be obtained by means of the following identity (prove it!):

$$\lim_{N\to\infty} \prod_{n=1}^{N} \left(1 - \frac{\omega^2 T^2}{n^2\pi^2}\right)^{-\frac{1}{2}} = \left(\frac{\sin \omega T}{\omega T}\right)^{-\frac{1}{2}}.$$

3. Lagrangian of the free scalar field reads as follows

$$\mathcal{L}_0 = \frac{1}{2}(1+0^+)\dot{\varphi}^2 - \frac{1}{2}(1-0^+)\left((\boldsymbol{\nabla}\varphi)\cdot(\boldsymbol{\nabla}\varphi) + m^2\varphi^2\right).$$

Generating functional reads

$$Z_0[j] = \int [D\varphi(x)] \exp\left\{ i \int d^4x \left(\mathcal{L}_0 + j(x)\varphi(x)\right) \right\}.$$

Show that

$$Z_0[j] = \exp\left\{ -\frac{1}{2} \int d^4x\, d^4y\; j(x) G_F^0(x,y) j(y) \right\},$$

where $G_F^0(x,y)$ is an inverse of

$$i\Big[(1+0^+)\partial_t^2 - (1-0^+)\left(\boldsymbol{\nabla}^2 - m^2\right)\Big].$$

Evaluate G_F^0 in momentum space. Show that it has the same pole structure as

$$\tilde{G}_F^0(k) = \frac{i}{k^2 - m^2 + i0^+}\,.$$

4. Consider Gaussian integral

$$J(\mathcal{M}) = \int d^N\xi\, d^N\psi\, \exp\left(\psi_i \mathcal{M}_{ij} \xi_j\right)$$

where ψ_i and ξ_i ($i = 1, 2, \ldots N$) are independent Grassmann variables. Expanding in a power series and commuting ξ's and ψ's show that

$$J(\mathcal{M}) = \det(\mathcal{M}).$$