QCD 2025
problem set 4

1. In this problem we shall calculate the d-dimensional angular integral given in problem 2. To this end we shall introduce spherical coordinates in d dimensions. First we chose arbitrarily a d-th axis (equivalent of the z axis in three dimensions) and project on it $\vec{k}$ vector with $\cos\theta_{d-1}$. Therefore a projection on the $d-1$ dimensional subspace orthogonal do the d-th axis is $k\sin\theta_{d-1}$. Now we choose an axis in this $d-1$ dimensional subspace, the $d-1$ axis, and project on this axis this projection (*i.e.* $k\sin\theta_{d-1}$) with $\cos\theta_{d-2}$. Next, a projection on the the $d-2$ dimensional subspace orthogonal do the d-th and $d-1$ axes involves $\sin\theta_{d-2}$. We continue this procedure until we "run out of dimensions" with the result:

$$\begin{aligned} k_d &= k\cos\theta_{d-1}, \\ k_{d-1} &= k\sin\theta_{d-1}\cos\theta_{d-2}, \\ &\ldots \\ k_2 &= k\sin\theta_{d-1}\sin\theta_{d-2}\ldots\cos\theta_1, \\ k_1 &= k\sin\theta_{d-1}\sin\theta_{d-2}\ldots\sin\theta_1, \end{aligned} \tag{1}$$

where $\theta_1 \in (0, 2\pi), \quad \theta_{i>1} \in (0,\pi)$. Compute

$$\int d\Omega_d = \int \prod_{i=1}^{d-1} \left(\sin^{i-1}\theta_i d\theta_i\right) = 2\prod_{i=1}^{d-1}\left(\int_0^\pi \sin^{i-1}\theta_i d\theta_i\right) \tag{2}$$

using

$$\int_0^\pi \sin^n\theta\, d\theta = B\left(\frac{1+n}{2}, \frac{1}{2}\right). \tag{3}$$

2. For the Altarelli-Parisi probabilities defined below

$$\begin{aligned} P_{q\leftarrow q}(z) &= C_F\left(\frac{1+z^2}{1-z}\right)_+, \\ P_{q\leftarrow G}(z) &= \frac{1}{2}\Big[z^2+(1-z)^2)\Big], \\ P_{G\leftarrow q}(z) &= C_F\frac{1+(1-z)^2}{z}, \\ P_{G\leftarrow G}(z) &= 2C_A\Big[z\left(\frac{1}{1-z}\right)_+ + \frac{1-z}{z} + z(1-z)\Big] + \frac{1}{2}\left(\frac{11}{3}C_A - \frac{2}{3}n_f\right)\delta(1-z). \end{aligned}$$

calculate Mellin moments:

$$\int_0^1 dz\, z^{n-1} P_{a\leftarrow b}(z) = \gamma_{ab}^{(n)}.$$

The calculation of $\gamma_{qq}^{(n)}$ and $\gamma_{GG}^{(n)}$ requires a certain trick. We have

$$\gamma_{qq}^{(n)} = C_F \int_0^1 dz\, z^{n-1} \left(\frac{1+z^2}{1-z}\right)_+ = C_F \int_0^1 dz\, (z^{n-1} - 1) \left(\frac{1+z^2}{1-z}\right).$$

To this end use (and prove) the following identity

$$\frac{z^{n-1}}{1-z} = -\sum_{k=0}^{n-2} z^k + \frac{1}{1-z}.$$

Compute explicitly values of these moments for $n = 1, 2$.

3. For $n = 1$ write explicitly the DGLAP equation for $q_1^{NS}(t)$ with $\gamma_{ab}^{(1)}$ from the previous problem. What is the interpretation of this result?

4. For $n = 2$ write explicitly the DGLAP equations for

$$\frac{d}{dt} q_2^S(t) \quad \text{and} \quad \frac{d}{dt} G_2(t)$$

in Mellin representation. Add them together and interpret the result.

5. Consider a combination for $n = 2$

$$f(t) = \frac{4C_F}{3} \frac{d}{dt} q_2^S(t) - \frac{n_f}{3} \frac{d}{dt} G_2(t)$$

and write the corresponding DGLAP equation for df/dt. Solve this equation using

$$\frac{\alpha_s}{4\pi} = \frac{2}{\beta_0 t}.$$

6. Consider an integral that naively is equal to zero

$$\int_{-\infty}^{\infty} dx\, [f(x+a) - f(x)] \tag{4}$$

where f is a function that does not vanish at infinity:

$$f(\pm\infty) \neq 0. \tag{5}$$

Calculate (4) expanding in a up to a^2. What happens when $f'(\pm\infty) = 0$. Generalize this result to the n-dimensional Euclidean integral

$$\Delta(\vec{a}) = \int d^n\vec{r}\, [f(\vec{r} + \vec{a}) - f(\vec{r})].$$