QCD 2025
problem set 3

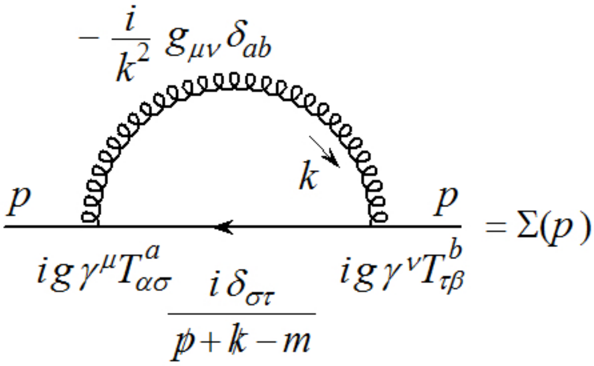

Figure 1: Feynman diagram corresponding to the quark self-energy. Time flow right to left.

In this problem set we shall perform detailed calculation of the fermion self-energy in QCD depicted in Fig. 1. Assume $p^2 \neq 0$. Note that the diagram (1) is almost the same as in QED, except for the color T generators, that enter in the quark-gluon vertices. Gluon propagator is in the Feynman gauge.

The problem is divided into a few steps.

1. Last time we have shown that mathematical expression $\Sigma(p)$ corresponding to the diagram (1) reads

$$\Sigma(p) = -g^2\mu^{4-d} C_F \delta_{\alpha\beta} \int \frac{d^d k}{(2\pi)^d} \frac{\gamma^\mu(\not p + \not k)\gamma_\mu}{(p+k)^2\, k^2}\,, \tag{1}$$

where $d = 4 - 2\varepsilon$ is the dimensionality of space-time. Using

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}.$$

and

$$g_{\mu\nu}g^{\mu\nu} = d$$

we have shown that

$$\begin{aligned}\Sigma(p) &= 2(1-\varepsilon)g^2\mu^{2\varepsilon} C_F \delta_{\alpha\beta} \int \frac{d^d k}{(2\pi)^d} \frac{\not p + \not k}{(p+k)^2\, k^2} \\ &= 2(1-\varepsilon)g^2\mu^{2\varepsilon} C_F \delta_{\alpha\beta} \left[\not p I + \gamma_\mu I^\mu\right]\,,\end{aligned} \tag{2}$$

where

$$\{I, I^\mu\} = \int \frac{d^d k}{(2\pi)^d} \frac{1}{(p+k)^2\, k^2} \{1, k^\mu\}\,. \tag{3}$$

2. Next, introducing Feynman parametrization of the propagators

$$\frac{1}{(p+k)^2\,k^2} = \int_0^1 dx \frac{1}{\left(k^2 + 2x\,pk + xp^2\right)^2} \tag{4}$$

show that

$$\{I, I^\mu\} = \int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \frac{1}{\left(\,k^2 - M^2\right)^2} \{1, k^\mu - xp^\mu\} \tag{5}$$

where

$$M^2 = -x(1-x)\,p^2\,. \tag{6}$$

HINT: employ change of variables

$$k^\mu \to k^\mu + xp^\mu. \tag{7}$$

3. In order to calculate the integral over d^dk, which is the integral in Minkowski space, we observe that (see Fig. 2):

$$\left\{ \int_{-\infty}^{\infty} + \int_{C_R} + \int_{+i\infty}^{-i\infty} \right\} dk^0 = 0. \tag{8}$$

To see this reinstall Feynman $+i\epsilon$ prescription.

$$k^2 - M^2 \to k^2 - M^2 + i\epsilon$$

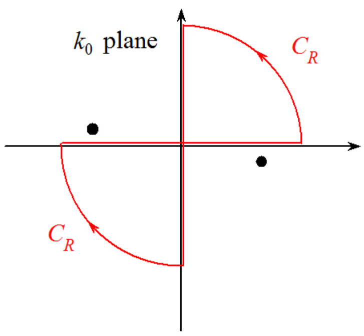

Figure 2: Integration contour over k_0. Black dots denote poles of Feynman propagators.

Since the integral over C_R vanishes

$$\int_{-\infty}^{\infty} dk^0 = -\int_{+i\infty}^{-i\infty} dk^0 = i\int_{-\infty}^{+\infty} dE \tag{9}$$

where $k^0 = iE$. Therefore the integral over $d^d k$ in Minkowski space transforms into the Euclidean integral

$$\{I, I^m\} = i\int_0^1 dx \int \frac{d^d\vec{k}}{(2\pi)^d} \frac{1}{\left(-\vec{k}^2 - M^2\right)^2} \{1, k^m - xp^m\} \tag{10}$$

where

$$k^m = (E, k^1, k^2, \ldots, k^{d-1}), \text{ with } m = 1, 2, \ldots d. \tag{11}$$

4. Since nothing depends on angles, except of k^m, which is nullified by the angular integration, we can use (we shall prove this later, but because full angular integral corresponds to the surface of a sphere of radius $r = 1$ in d dimensions, you can check that the formula below is right for $d = 2$ or 3):

$$\int d\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}. \tag{12}$$

After the angular integration we arrive at (using $d = 4 - 2\epsilon$):

$$\{I, I^\mu\} = \frac{i}{\Gamma(2-\varepsilon)} \frac{2\pi^{2-\varepsilon}}{(2\pi)^{4-2\varepsilon}} \int_0^1 dx \, \{1, -xp^\mu\} \int_0^\infty dk \frac{k^{d-1}}{(k^2 + M^2)^2}.$$

Changing variables to $r = k/M$, and then $t = r^2$, you should get two integrals that are representations of the Euler beta functions:

$$\begin{aligned} \int_0^\infty dt \, \frac{t^{x-1}}{(1+t)^{x+y}} &= B(x, y), \\ \int_0^1 dx \, x^{\alpha-1}(1-x)^{\beta-1} &= B(\alpha, \beta). \end{aligned} \tag{13}$$

Identify values of x, y, β and α and then write the final expression for I and I^μ in terms of Euler Γ functions only (use the well known expression for beta functions).

5. Using

$$z\Gamma(z) = \Gamma(z+1), \tag{14}$$

$$\Gamma(1/2) = \sqrt{\pi}. \tag{15}$$

and

$$\Gamma(1-\varepsilon) = \exp\left(\gamma\varepsilon + \frac{\pi^2}{12}\varepsilon^2 + \ldots\right) \tag{16}$$

where γ is Euler constant, you should arrive at the final answer for $\Sigma(p)$:

$$\Sigma(p) = i\not{p}\, C_F \frac{\alpha_s}{4\pi} \left(\frac{\mu^2 4\pi e^{-\gamma}}{-p^2}\right)^{\varepsilon} \left(\frac{1}{\varepsilon} + 1\right)$$

where

$$\alpha_s = \frac{g^2}{4\pi}. \tag{17}$$

6. In this problem we shall calculate the d-dimensional angular integral given in problem 2. To this end we shall introduce spherical coordinates in d dimensions. First we chose arbitrarily a d-th axis (equivalent of the z axis in three dimensions) and project on it $\vec{k}$ vector with $\cos\theta_{d-1}$. Therefore a projection on the $d-1$ dimensional subspace orthogonal do the d-th axis is $k \sin\theta_{d-1}$. Now we choose an axis in this $d-1$ dimensional subspace, the $d-1$ axis, and project on this axis this projection (*i.e.* $k\sin\theta_{d-1}$) with $\cos\theta_{d-2}$. Next, a projection on the the $d-2$ dimensional subspace orthogonal do the d-th and $d-1$ axes involves $\sin\theta_{d-2}$. We continue this procedure until we "run out of dimensions" with the result:

$$\begin{aligned} k_d &= k\cos\theta_{d-1}, \\ k_{d-1} &= k\sin\theta_{d-1}\cos\theta_{d-2}, \\ &\ldots \\ k_2 &= k\sin\theta_{d-1}\sin\theta_{d-2}\ldots\cos\theta_1, \\ k_1 &= k\sin\theta_{d-1}\sin\theta_{d-2}\ldots\sin\theta_1, \end{aligned} \tag{18}$$

where $\theta_1 \in (0, 2\pi), \quad \theta_{i>1} \in (0, \pi)$. Compute

$$\int d\Omega_d = \int \prod_{i=1}^{d-1} \left(\sin^{i-1}\theta_i d\theta_i\right) = 2\prod_{i=1}^{d-1}\left(\int_0^{\pi} \sin^{i-1}\theta_i d\theta_i\right) \tag{19}$$

using

$$\int_0^{\pi} \sin^n\theta\, d\theta = B\left(\frac{1+n}{2}, \frac{1}{2}\right) . \tag{20}$$