QCD 2025
problem set 2

1. Using the form of $L_{\mu\nu}$ obtained at the problem classes last time

$$L_{\mu\nu}(k,k') = 2\left(k_\mu k'_\nu + k'_\mu k_\nu + \frac{q^2}{2} g_{\mu\nu}\right)$$

show that

$$\begin{aligned} p_\nu p_\mu L^{\nu\mu}(k,k') &= 4M^2\omega\omega' \cos^2\frac{\theta}{2}, \\ g_{\nu\mu}L^{\nu\mu}(k,k') &= -8\omega\omega' \sin^2\frac{\theta}{2}. \end{aligned}$$

Assuming

$$L_{\mu,\nu}(p,p') = Ap_\mu p_\nu + Bg_{\mu\nu}$$

compute $L_{\mu\nu}(p,p')L^{\mu\nu}(k,k')$. For elastic cross-section and elementary fermions $A = 4$ and $B = -Q^2$.

2. In this problem we will discuss parton properties assuming very simple models for quark and gluon distributions. My advice is to use *Mathematica* for these calculations.

   (a) Using properties of parton distributions given at the last lecture in section *Quarks as Partons* calculate normalization constant A assuming

   $$u_v(x) = \frac{2A}{\sqrt{x}}(1-x)^3,\ d_v(x) = \frac{A}{\sqrt{x}}(1-x)^3$$

   where index v stands for *valence*. Recall that total u or d quark distribution is given as a sum of valence quarks and sea quarks u_s or d_s respectively. We assume that sea quark distributions are equal to antiquark distributions

   $$u_s(x) = \bar{u}(x),\ d_s(x) = \bar{d}(x).$$

   For this problem we assume that there are no strange quarks in the nucleon and we assume isospin symmetry, which says that u and d distributions in neutron, are equal to d and u distributions in proton. Calculate A. Check the value of charge of the proton and neutron. Calculate total momentum carried by the valence quarks. At this point we do not need any information on the sea quarks.

   (b) Gottfried sum rule. Calculate the difference of the structure functions of the proton and neutron:

   $$S_G = \int_0^1 \frac{dx}{x}\left(F_2^p(x) - F_2^n(x)\right).$$

Experimental value reads $S_G \simeq 0.24$. Compute S_G for valence quarks only. As you will see S_G will depend on the integral over the distributions of the sea quarks. Assume the sea quark distribution of the following form

$$u(x) = \bar{u}(x) = \frac{B}{x}(1-x)^8, \; d(x) = \bar{d}(x) = \frac{B}{x}(1-x)^\beta.$$

Note that constant B must be the same in both cases to assure that S_G is finite. From the experimental value of S_G calculate B for several choices of power β taking a few values around 8. Note that the antiquark and sea distributions must be positive. For these choices calculate total momentum carried by quarks. Is it possible to get the value of 100%?

(c) Choose gluon distribution of the form

$$g(x) = C\frac{1}{x}(1-x)^4.$$

For given β from the previous problem calculate C from the condition that the total momentum of the proton is 1.

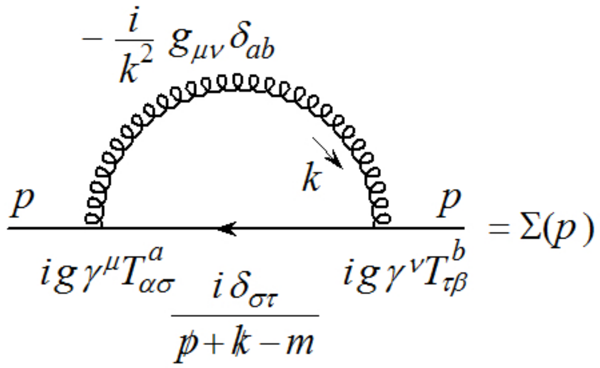

Figure 1: Feynman diagram corresponding to the quark self-energy. Time flows right to left.

In this problem set we shall perform detailed calculation of the fermion self-energy in QCD depicted in Fig. 1. Assume $p^2 \neq 0$. Note that the diagram (1) is almost the same as in QED, except for the color T generators, that enter in the quark-gluon vertices. Gluon propagator is in the Feynman gauge. The problem is divided into a few steps.

3. (a) Write mathematical expression $\Sigma(p)$ corresponding to the diagram (1). Show that the only effect of the fact that we calculate this diagram in QCD is a *color factor* in front.

(b) In the loop diagram (1) change the 4-dimensional integral over the gluon momentum k to a d dimensional one according to the following prescription:

$$\frac{d^4k}{(2\pi)^d} \to \mu^{4-d} \frac{d^dk}{(2\pi)^d}$$

Convince yourself that if we assume that

$$4 \to d = 4 - 2\varepsilon$$

where $\varepsilon \to 0_+$ then the integral over k is finite. The method of changing dimensionality of space-time, known as *dimensional regularization* proposed by Veltman and 't Hooft, has great advantage over some other regularization methods, namely it preserves gauge invariance. Note that in order to preserve dimensionality of $\Sigma(p)$ we had to introduce an arbitrary mass parameter μ.

(c) In the following we shall put $m = 0$. Using

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}.$$

and

$$g_{\mu\nu} g^{\mu\nu} = d.$$

calculate the numerator of $\Sigma(p)$. You should obtain that

$$\Sigma(p) \quad \sim \quad \int \frac{d^dk}{(2\pi)^d} \frac{\not{p} + \not{k}}{(p+k)^2 \, k^2} = \not{p} I + \gamma_\mu I^\mu$$