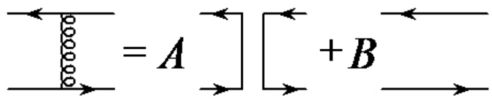
# QCD 2025
## problem set 1

1. Prove the Jacobi identity:
$$[T_m,[T_n,T_l]] + [T_n,[T_l,T_m]] + [T_l,[T_m,T_n]] = 0\,,$$
apply to it the SU($N$) algebra
$$[T_m,T_n] = i f_{mnl} T_l \tag{1}$$
and show that adjoint generators defined as
$$(T_l^{\mathrm{adj}})_{mn} = -i f_{lmn}$$
satisfy (1).

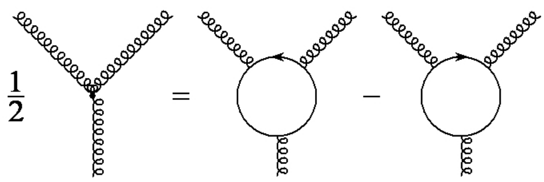
2. Find the transformation matrix $U$
$$-\tau_i^* = U^\dagger \tau_i U\,,$$
where $\tau_i$ are Pauli matrices.

3. Color factors.

   - With the help of the graphical methods for the SU($N$) group described at the lecture, find coefficients $A$ and $B$ for the following decomposition of the one gluon exchange:

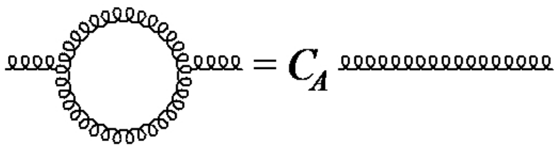
     
     

     and interpret the result for $N \to \infty$.
     HINT: contract fermion lines in two possible ways (remember that contractions must preserve the direction of the arrow).
   - Prove the following identity:

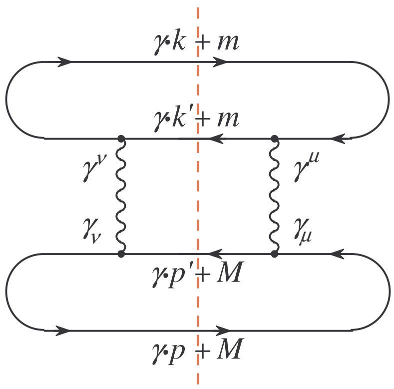
     HINT: start from the commutation relation for the generators in the fundamental representation and contract it with an extra generator keeping the arrows direction. Then use the normalization condition for the generators. In these color diagrams what really matters is topology, so we can turn them around or reflect without changing the result (except for possible change of sign if we flip legs of an antisymmetric tensor).

- Using relations from the previous problems calculate $C_A$, the Casimir operator for the adjoint representation:




4. During the lecture we have encountered the integral over $d\omega' d\cos\theta$, which can be expressed in terms of $dQ^2 d\nu$, where
$$Q^2 = 2\omega\omega'(1 - \cos\theta), \;\; \nu = \omega - \omega'.$$
Calculate the Jacobian for this change of variables.

5. Averaged scattering amplitude is shown in figure below and is given by the formula




$$\frac{1}{4}\sum_{\text{pol}} |\mathcal{M}_{fi}|^2 = \frac{e_1^2 e_2^2}{(q^2)^2} L^{\mu\nu}(k,k') L_{\mu\nu}(p,p').$$
Here
$$L^{\mu\nu}(p,p') = \frac{1}{2}\,\mathrm{Tr}\,(\gamma^\mu(\gamma\cdot p + M)\gamma^\nu(\gamma\cdot p' + M)) \tag{2}$$
where $\gamma$ denote Dirac matrices. Compute Dirac trace (2) using techniques described in the textbooks, you can also calculate them using Wolfram *Mathematica*. Express each of them in terms of $p^\mu$ and $q^\mu$. Check that they are gauge invariant, *i.e.*
$$q_\nu L^{\nu\mu} = q_\mu L^{\nu\mu} = 0.$$

6. Show that
$$\begin{aligned} p_\nu p_\mu L^{\nu\mu}(k,k') &= 4M^2\omega\omega' \cos^2\frac{\theta}{2}, \\ g_{\nu\mu} L^{\nu\mu}(k,k') &= -8\omega\omega' \sin^2\frac{\theta}{2}. \end{aligned}$$