1 Introduction

The aim of this short course is to relate the problem of modular automorphisms of Poisson manifolds and of their quantizations.

The modular automorphism is maybe the easiest defined invariant on a Poisson manifold, and certainly one of the few that can be explicitly computed in most examples.

A Poisson manifold is said to be unimodular when its modular automorphism can be chosen to be trivial. This property can be seen as a type II property. When the symplectic foliation is regular, in fact, it implies the existence of a transverse measure. In the non-regular case its full meaning is subtler and, maybe, still not exploited at its fullness.

In which sense can we talk of a quantum analogue of this invariant? Roughly speaking quantizing the algebra of function on a Poisson manifold means replacing it by a non-commutative associative deformation. The Poisson bivector describes the direction of the infinitesimal deformation. For this reason the quantized algebra carries much information on the underlying Poisson geometry. As an example, it is usually still possible to find a shadow of modular properties at the quantum level; they survive in the form of a special algebra automorphism (often defined only up to inner ones). For general smooth algebras this is called van den Bergh automorphism, and plays a role in homological duality statements. For quantum groups (or, in general, for any Noetherian Hopf algebra), it is called Nakayama automorphism.

Quite recently Dolgushev showed how these modularity issues are not just re-
lated by analogy but indeed connected one to the other, at least in the context of formal deformation quantization. As we think that much consequences will follow from such results, and many more could be obtained in specific examples, our plan here is to clarify the general framework surrounding Dolgushev’s theorems (their statement only; we’ll refrain from dwelling the deeper math underground required for the proofs). We will comment on the role such statement should have in linking Poisson geometry with non commutative geometry à la Connes. We will also add some material for the specific case in which the Poisson manifolds we consider have additional symmetry properties.

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2 Poisson geometry

In this section we will review some basic results of the general theory of Poisson manifolds. Everything is quite standard and can be found in [11, 24] (and references therein).

2.1 General theory

Definition 2.1 Let $M$ be a manifold and let $\pi \in \Gamma(\wedge^2 TM)$ be a bivector. Then we will say that $(M, \pi)$ is a Poisson manifold if the bracket on $C^\infty(M)$ given by

$$\{f, g\} = \langle \pi, df \wedge dg \rangle$$

is a Lie bracket.

Let us remark that being $\pi$ a bivector the bracket above always satisfies the Leibniz identity

$$\{fg, h\} = f\{g, h\} + g\{f, h\}$$

The condition on $\pi$ being Poisson can also be expressed in terms of the Schouten-Nijenhuis bracket as

$$[\pi, \pi] = 0.$$  

For any $f \in C^\infty(M)$, the vector field

$$X_f(g) = \{f, g\} \quad \forall g \in C^\infty(M)$$

is called an hamiltonian vector field of Hamiltonian $f$. More generally one can define the so-called sharp map:

$$\sharp_{\pi} : \Omega^1 M \to \mathfrak{X}(M); \quad \alpha \mapsto i_\alpha \pi$$
so that $\mathcal{I}_\pi(df) = X_f$. Vector fields in the image of such map are called \textit{locally hamiltonian} vector fields.

Locally hamiltonian vector fields span an involutive non regular\footnote{By non regular we mean that its local dimension is non globally constant but is a lower semi-continuous function; thus integrability does not follow applying Frobenius theorem but requires the more subtle Stefan-Sussmann theorem (see \cite{4,11,24} for more details).} distribution which is integrable. Any Poisson manifold is therefore equipped with a canonical non regular foliation. Furthermore each leaf carries a well defined symplectic form induced by the Poisson tensor. This is what is called the symplectic foliation of $(M, \pi)$.

The sharp map allows also to consider an additional structure, namely a Lie bracket between 1–forms:

$$[\alpha, \beta] = L_{\mathcal{I}_\pi(\alpha)}\beta - L_{\mathcal{I}_\pi(\beta)}(\alpha) - d\pi(\alpha, \beta)$$

The cotangent bundle $T^*M$, with this bracket between its sections and the sharp map is a \textit{Lie algebroid}.

### 2.2 Cohomology and homology

Let us denote with $\mathfrak{X}^k(M)$ the space of all $k$–multivector fields on $M$. By mean of the Schouten bracket we can define the following degree 1 operator:

$$d_\pi : \mathfrak{X}^k(M) \rightarrow \mathfrak{X}^{k+1}(M); \quad d_\pi(P) = [\pi, P]$$

From the graded Jacobi identity for $[-, -]$ one easily gets that $d_\pi^2 = 0$. The cohomology of the complex $(\mathfrak{X}^k(M), d_\pi)$ is called the (Lichnerowicz)–Poisson cohomology of $M$ and will be denoted as $H^k_\pi(M)$.

This cohomology has a (well founded) reputation for being quite hard to compute. This is due to the fact that it depends both on the topology of the foliation and on the variation of the symplectic form from leaf to leaf. Some interesting examples on linear spaces were explicitly computed recently.

Let us try to understand it a little bit by having, as usual, a closer look at the meaning of low–dimensional cohomology groups. If we take a 0–cochain $f \in C^\infty(M)$, then $d_\pi(f) = X_f$. The 0-th cohomology group can be therefore described as the set of functions such that $X_f = 0$. Such functions are called \textit{Casimir functions} on the Poisson manifold and are constant along the leaves of the symplectic foliation.

**Examples:**

- Let $M = \mathbb{R}^{2n+1}$, with coordinates $p_i, q_i, t, i = 1, \ldots, n$ and with the canonical Poisson brackets having as only nontrivial commutators

$$\{p_i, q_j\} = \delta_{i,j}$$
Then $H^0(M)$ is isomorphic to $\mathcal{C}^\infty(\mathbb{R})$, identified with functions on $M$ depending only on $t$. In this case the symplectic leaves coincide with the level sets of the Casimir function $t$.

- Let $\pi = (x^2 + y^2)\partial_x \wedge \partial_y$ on $\mathbb{R}^2$. Then a non constant function $f \in H^0_\pi(M)$ if and only if $(x^2 + y^2)f \equiv 0$. This implies $f = 0$ on $\mathbb{R}^2 \setminus \{(0,0)\}$ and, by continuity, $f \equiv 0$. Thus $H^0_\pi(M) \cong \mathbb{R}$ and the Poisson manifold has no nonconstant Casimirs. Remark that this may happen even if the foliation is non trivial (hence leaves do no always coincide with level sets of Casimirs).

Let’s now take a 1–cochain $X \in \mathfrak{x}^1(M)$. Then $d_\pi X = 0$ if and only if $X \in \text{Der}(\mathcal{C}^\infty(M), \{\cdot,\cdot\})$ is a derivation of the Poisson bracket. Such a vector field is called a Poisson vector field. Therefore $H^1_\pi(M)$ is the space of Poisson vector fields modulo Hamiltonian vector fields. Our main invariant will sit inside this cohomology groups and we will not need the higher order ones.

In the special case of a symplectic manifold $M$ the sharp map can be extended to an isomorphism of cochain complexes and thus $H^k_\pi(M) \cong H^k_{\text{deR}}(M)$.

On Poisson manifolds it is possible also to give an homology theory, using forms, which is to a certain extent dual to the previous one. In fact it is this duality problem the core of these notes. Let $\Omega^k M$ denote the space of smooth $k$–forms on $M$. Let us define an homology operator $\partial_\pi$ as the graded commutator

$$\partial_\pi : \Omega^k M \to \Omega^{k-1} M; \quad \partial_\pi = [d, i_\pi]$$

It can be checked (the computation being not so trivially easy) that $\partial_\pi^2 = 0$. The corresponding homology (introduced by Brylinski) will be called the Poisson homology of $M$ and denoted $H^k_\pi(M)$.

Let us consider, as before, the simplest low-dimensional case. Let $\alpha = fdg \in \Omega^1 M$. Then

$$\partial_\pi (fdg) = di_\pi (fdg) - i_\pi (df \wedge dg) = -\{f, g\}$$

Therefore

$$H^0_\pi(M) = \mathcal{C}^\infty(M)/\{\mathcal{C}^\infty(M),\mathcal{C}^\infty(M)\}.$$ 

### 2.3 The modular class

Let, from now on, $M$ be orientable of dimension $n$ (this hypothesis is not, strictly speaking, necessary, but will simplify things a little bit). Let $\Omega$ be a volume form on $M$. Take any $f \in \mathcal{C}^\infty(M)(M)$. Then

$$L_{X_f} \Omega = \phi_\Omega(f) \Omega.$$ 

We have the following facts (which follows through a straightforward check from definitions):
1. the map $\phi_\Omega : f \mapsto \phi_\Omega(f)$ is a vector field;
2. the map $\phi_\Omega$ is a Poisson vector field;
3. let $\Omega' = a\Omega$ be another volume form; then
   \[ \phi_{\Omega'} = \phi_\Omega + X_{-log|a|}. \]
These three facts together imply that the vector field $\phi_\Omega$ defines a Poisson cohomology class $[\phi_\Omega] \in H^1_\pi(M)$ which does not depend on $M^2$. This class is called the Poisson modular class and, as everyone may have guess by now, is the main character entering our story.

A Poisson manifold $(M, \pi)$ such that $[\phi_\Omega] = 0$ will be called unimodular.

1. Let $M$ be symplectic; then $M$ is unimodular (use $\Omega = \omega^n/n!$).
2. Let $M = g^*$. Then the modular class is the Lie algebra cohomology class defined by the adjoint trace $X \mapsto tr(ad_X)$. Therefore $M$ is unimodular if and only if $g$ is.
3. Let $M$ be regularly foliated into symplectic manifolds. Then one can prove that there exists an injection $H^1(M) \hookrightarrow H^1_\pi(M)$ sending the Reeb class of the foliation to the Poisson modular class.

Let us consider fixed $(M, \pi)$ and the volume form $\Omega$. Then:
\[
\int_M \{f, g\} \Omega = \int_M \langle L_X f, g \rangle \Omega \\
= \int_M L_X (g \Omega) - g L_X \Omega \\
= \int_M (d\iota_X g \Omega) + \iota_X d(g \Omega) - g L_X \Omega \\
= -\int_M g \phi_\Omega(f) \Omega
\]
The equality between the first and last line of this chain is called Poisson KMS condition.

A couple of remarks can be made at this point:

1. Let $(M, \pi)$ be Poisson unimodular; let $\Omega$ be the volume form such that $\phi_\Omega = 0$. Then from the Poisson KMS condition we get:
\[
\int_M \{f, g\} \Omega = 0 \quad \forall f, g \in C^\infty(M)
\]

\[ ^{2}\text{In the algebraic setting which will be mentioned in the next section some additional care is needed; the appearance of a logarithm in the above formula forces to work with the subtler notion of log-hamiltonian vector fields.} \]
i.e. \( \int_M \) is a Poisson trace on the Poisson algebra \( \mathcal{C}^\infty(M) \) (remark that in general the space of Poisson traces is dual to \( H_0^\pi(M) \)).

2. Let us remark that the Poisson boundary of a volume form can be expressed through the simpler formula

\[
\partial_\pi \Omega = -i_{\varphi_\Omega} \Omega
\]

Therefore \((M, \pi)\) is unimodular if and only if there exists a volume form \(\Omega\) such that \(\partial_\pi \Omega = 0\). This means that such volume form is a non trivial cycle for the Poisson homology and therefore implies \( H_{\text{top}}^\pi(M) \neq 0 \).

### 2.4 Duality

To express correctly the duality between Poisson cohomology and homology we need their version with coefficients. Coefficients can be thought either, algebraically, as Poisson modules or, more geometrically, as being a vector bundle with a flat "connection". The right idea of connection in this context is that of contravariant connection

**Definition 2.2** Let \( E \to M \) be a vector bundle on the Poisson manifold \( M \). A flat contravariant connection on \( M \) is a linear map

\[
D : \Omega^1 M \otimes \Gamma(E) \to \Gamma(E)
\]

such that:

1. \( D_\alpha(fs) = fD_\alpha s + (\sharp_\pi(\alpha)f)s \);
2. \( D_fs = fD_s \);
3. \([D_\alpha, D_\beta] = D[\alpha, \beta]_\pi\)

As we were mentioning this notion admits other interpretations. We can say that the space of sections has a \( \mathcal{C}^\infty(M) \)-Poisson module structure\(^3\) given by \( f \cdot s = D_{df}s \), or that the Poisson Lie algebroid \((T^*M, [,]_\pi)\) has a Lie algebroid representation on \( E \).

We will just need the easiest possible example: let \( L \to M \) be a trivial line bundle on \( M \). Any Poisson vector field \( X \in \mathfrak{X}(M) \) defines a flat contravariant connection on \( L \) as:

\[
D_{f \varphi g}h = fD_{\varphi g}h = f([g, h] + (Xg)h]
\]

and, in fact, any flat contravariant connection on \( L \) arises in this way: \( Xg = D_{\varphi g}1 \).

**Examples:**

\(^3\)The defining properties of a Poisson module can then be derived just by rewriting 1.-2.-3. in this setting.
1. Let $M$ be symplectic, $E \to M$ a flat vector bundle over $M$ and $\nabla$ a flat connection on $E$. Then $D_{df} = \nabla_{\pi(df)} = \nabla_{X_f}$ defines a flat contravariant connection on $E$.

2. In case $M$ is Poisson and orientable we can consider the canonical line $\wedge^n T^* M$, together with the trivialization defined by the volume form $\Omega$. Then there is also a canonical Poisson vector field $\phi_\Omega$. The corresponding flat contravariant connection on $\wedge^n T^* M$ is called the canonical Poisson line bundle. In this language saying that a Poisson manifold is unimodular is tantamount to saying that its canonical Poisson line bundle is Poisson trivial.

The key point now is that both Poisson homology and Poisson cohomology can be defined with coefficients in any Poisson vector bundle\(^4\). We will not give the exact definition here, which will take us too far away from our purposes but rather clarify our easy case of trivial line bundles. If $L \to M$ is such a bundle carrying a flat contravariant connection defined by a vector field $X$, then the twisted Poisson cohomology with coefficients in $L$ is given by the coboundary operator $d_\pi + X \wedge -$.

Dually, the Poisson homology with coefficients is given by the boundary operator $\partial_\pi - \iota_X$.

In case $M$ is symplectic the flat contravariant connection is just an ordinary flat connection and the Poisson cohomology with coefficients turns out to be exactly the de Rham cohomology with coefficients in a flat bundle (see [2] for more on the subject).

We are now ready to state our duality result ([12] is the first paper in which a simpler version of this result was obtained; for more on the subject look at [11] and references therein; complete results expressed in the language of Lie algebroids can be found in [17]).

**Theorem 2.3** The following Poincaré duality between Poisson homology and cohomology holds true

\[
H^k_\pi(M; E \otimes \wedge^n T^* M) \simeq H^n_{n-k}(M; E)
\]

In particular if $(M, \pi)$ is unimodular:

\[
H^k_\pi(M) \simeq H^n_{n-k}(M)
\]

### 3 Quantum modular class

In this section we will explain results quite recently obtained by Dolgushev in [9], settling down the problem of quantization of the modular class in the formal setting.

\(^4\)In the Lie algebroid setting we are dealing with Lie algebroid (co)homology with coefficients in a representation, a natural generalization of what is usually done for Lie algebras.
3.1 Deformation quantization

Let \((M, \pi_0)\) be an orientable Poisson manifold. Let \(A_\hbar\) be a deformation quantization of this manifold. This basically means that \(A_\hbar\) is a topologically free \(C[[\hbar]]\)-associative algebra (topologically free means Hausdorff, complete in the \(\hbar\)-adic topology and torsion free with respect to \(\hbar\); this implies that as a vector space \(A_\hbar \cong A[[\hbar]]\)) and

\[
\frac{A_\hbar}{\hbar A_\hbar} \cong \mathcal{F}(M); \quad \frac{[f_1, f_2]}{\hbar} \mod \hbar = \{f_1, f_2\}
\]

Kontsevich formality theorem ([18]) implies that for any Poisson manifold such a deformation quantization exists. Furthermore it is uniquely determined by the so called Kontsevich class, which is a formal Poisson bivector, i.e. \(\pi \in \hbar \Gamma(\wedge^2 TX)[[\hbar]]\), such that \(\pi = \hbar \pi_0 + O(\hbar^2)\) and \([\pi, \pi] = 0\). Remark here that if the formal Poisson class is unimodular then also \(\pi_0\) is unimodular, and if \(X\) is a formal Poisson vector field for \(\pi\) then its lower order term \(X_0\) is a Poisson vector field for \(\pi_0\).

Our aim here is to give an answer to the following questions:

1. Does the quantization of unimodular Poisson manifolds carries special features?
2. Can the modular vector field be lifted in quantization?

To address such question we will first address a seemingly unrelated purely algebraic duality result.

3.2 van den Bergh duality

The van den Bergh duality theorem is a purely algebraic results which clarifies the extent to which Hochschild homology and cohomology \(^6\) are dual theories. Let \(B\) be any complex associative algebra; denote with \(B^e = B \otimes B^{op}\) its associative envelope, with its natural bimodule structure. Let us define \(\dim B\) as the projective dimension \(^7\) in the category of finitely generated \(B\)-bimodules. The algebra \(B\) is said to be smooth if \(\dim B < \infty\). With \(HH^k(B; M)\) (resp. \(H\Gamma_k(B; M)\)) we will denote the Hochschild cohomology (resp. homology) with coefficients in a \(B\)-bimodule \(M\) ([21] for general definitions).

**Theorem 3.1 (van den Bergh)** Let \(B\) be a smooth algebra. Suppose that there exist \(n \in \mathbb{N}\) and an invertible bimodule \(\omega_B\) such that:

\[
HH^k(B, B^e) = \begin{cases} 
\omega_B & k = n \\
0 & k \neq n 
\end{cases}
\]

\(^5\)The setting in which Dolgushev’s results are obtained is that of a smooth affine variety over \(\mathbb{C}\) with trivial canonical bundle.

\(^6\)We will not even attempt to provide full definitions here; the standard reference being [21]

\(^7\)this means the length of projective resolutions.
then

1. \( n = \dim B \);

2. \( HH_k(B, \omega_B \otimes_B M) \cong HH^{n-k}(B, M) \) for any \( B \)-bimodule \( M \).

In this case \( \omega_B \) is called the dualizing bimodule of \( B \).

The reason for our interest in this theorem is the following:

**Proposition 3.2** Let \( \mathcal{A}_\hbar \) be a deformation quantization of \((M, \pi_0)\). Then \( \mathcal{A}_\hbar \) satisfies the hypothesis of van den Bergh theorem, \( \dim \mathcal{A}_\hbar = \dim M \) and the dualizing bimodule is isomorphic to \( \mathcal{A}_\hbar \nu \), where \( \nu \in \text{Aut}(\mathcal{A}_\hbar) \), \( \nu = \text{Id} \mod \hbar \).

This proposition, as mentioned in the introduction, clarifies that any deformation quantization algebra comes equipped with a distinguished automorphism. To be precise, the automorphism \( \nu \) appearing in the proposition is not unique; it is uniquely defined only up to inner automorphisms of \( \mathcal{A}_\hbar \) (i.e. conjugation by elements of \( \mathcal{A}_\hbar \)). Any such automorphism will be called the modular automorphism of \( \mathcal{A}_\hbar \). It seems appropriate to remark here that the proof of this proposition does not allow to identify a modular automorphism in explicit form. It is also worth mentioning that the statement dose not apply only to Kontsevich’s deformation quantization but to any deformation quantization, in the loose sense specified at the beginning of this section.

**Theorem 3.3 (Dolgushev)** The v.d.Bergh dualizing bimodule of \( \mathcal{A}_\hbar \) is isomorphic to \( \mathcal{A}_\hbar \) (thus \( \nu \) is inner) iff \( \pi \) is unimodular.

**Proof**

Let us first suppose that \( \omega_B \cong \mathcal{A}_\hbar \) as a bimodule. Then van den Bergh theorem implies that there is an isomorphism:

\[
V : HH^0(\mathcal{A}_\hbar, \mathcal{A}_\hbar) \rightarrow HH_n(\mathcal{A}_\hbar, \mathcal{A}_\hbar).
\]

The formality theorems for chains and cochains ([10]) guarantee the existence of isomorphisms

\[
\mu_1 : HH^0(\mathcal{A}_\hbar, \mathcal{A}_\hbar) \rightarrow H^0_\pi(M); \quad \mu_2 : HH_n(\mathcal{A}_\hbar, \mathcal{A}_\hbar) \rightarrow H^n_\pi(M).
\]

(3.1)

Let us now consider \([1] \in H^0_\pi(M)\). Then \( \mu_2 \circ V \circ \mu_1^{-1}[1] = [\omega] \), where \( \omega \in \Omega^n M[[\hbar]] \) is such that \( \partial_\pi \omega = 0 \); if we prove that \( \omega \) is a volume form (i.e. nowhere vanishing) on \( M \) then we are done (see remark 2 after the definition of the modular class). Let \( \omega = \omega_0 + O(\hbar) \). Dolgushev shows that \( \omega \) is a volume form iff \( \omega_0 \) is. To prove that \( \omega_0 \) never vanishes it is enough to use the map \( V_0 \) (the \( \hbar = 0 \) term of the van den Bergh isomorphism) and Hochschild-Kostant-Rosenberg theorem.

9
Let us now start from \( \pi \) being unimodular. There exists then a volume form \( \omega \in \Omega^n M[[h]] \) such that \( \partial_h \omega = 0 \) and \( \omega_0 \) is a volume form. Now van den Bergh theorem establishes an isomorphism

\[
\tilde{V} : HH_n(\mathcal{A}_h, \mathcal{A}_h) \to HH^0(\mathcal{A}_h, \mathcal{A}_h)
\]

Let us consider \( \tilde{V}(\mu_2^{-1}[\omega]) \). This is represented by an invertible \( b \in \mathcal{A}_h \) which is a twisted 0–cocycle, i.e.:

\[
\nu^{-1}(a) \star b - b \star a = 0, \quad \forall a \in \mathcal{A}_h
\]

Thus \( \nu^{-1}(a) = b \star a \star b^{-1} \), which means that \( \nu \) is an inner automorphism and therefore \( \mathcal{A}_h \nu \simeq \mathcal{A}_h \) as \( \mathcal{A}_h \)–bimodules.

In the case in which this theorem holds \( \mathcal{A}_h \) is what is called a Calabi–Yau algebra([14]).

We would like to remark here that the proof of this theorem does not really need formality theorems in their full form. What is really needed is the existence of the isomorphisms given in (3.1). In specifica cases such isomorphisms can be obtained without relying on the \( \infty \)-algebra’s approach (e.g. through spectral sequences, as we will see for quantum groups).

### 3.3 Quantization of Poisson vector fields

Let \( X \in \mathfrak{X}(M)[[h]] \) be a formal vector field. How does this quantize? Certainly any derivation \( D \) of \( \mathcal{A}_h \) has a lower order term \( D_0^8 \) which is a derivation on functions, thus a vector field. We would like to build a section of this map, at least for Poisson vector fields (at the formal level).

**Proposition 3.4** There exists a \( \mathbb{C}[[h]] \)–linear map

\[
\mathfrak{X}[[h]] \cap \ker d_\pi \to \text{Der}(\mathcal{A}_h) ; X \mapsto D_X
\]

(3.2)

such that:

1. \( D_X = X \mod (h) \);
2. \( [D_X, D_Y] = D_{[X,Y]} + I \) where \( I \) is an inner derivation.

Let us remark that if \( X \in \mathfrak{X}[[h]] \cap \ker d_\pi \) then its lower order term \( X_0 \) is a \( \pi_0 \)–Poisson vector field. Let us also stress the point that this map is nothing but a lifting to the level of cycles of the isomorphism \( \mu : H^1_\pi(M) \simeq HH^1(\mathcal{A}_h, \mathcal{A}_h) \) provided by formality. The existence of such map can be expressed by saying that all Poisson vector fields can be quantized.

\footnote{Say \( D \in \mathfrak{X}^{n-1} \text{Der}(\mathcal{A}_h) \) but \( D \notin \mathfrak{X}^{n-1} \text{Der}(\mathcal{A}_h) \); then \( D_0 = D/\hbar^n \mod \hbar \).}
vector fields can be quantized as derivations of the quantum algebra. We do not know, apart from a bunch of examples, results of this kind which do not rely on formality techniques.

Take now $\phi_\Omega$ to be a modular vector field for $(M, \pi)$. Recall that the Kontsevich class of the quantization is $\pi = h\pi_0 + O(h^2)$, thus $\phi_\Omega \in h\Gamma(TM)[[h]] \cap \text{Ker} \, d_\pi$, i.e. $\phi_\Omega = 0 \mod h$. Then the quantization of the modular vector field via (3.2) verifies $D_{\phi_\Omega} = 0 \mod h$. This allows to define an automorphism $\exp(D_{\phi_\Omega})$ of $A_h$. It is this automorphism that we plan to study in the next section.

### 3.4 Main theorem

**Theorem 3.5 (Dolgushev)** Let $A_h$ be a deformation quantization of $(M, \pi_0)$ and let $\pi$ be a representative of the Kontsevich class, with modular vector field $\phi_\Omega$ w.r. to a formal volume form $\Omega \in \Gamma(\wedge^n T^* M)[[h]]$. Consider the corresponding derivation $D_{\phi_\Omega} \in \text{Der}(A_h)$. Then:

$$\nu = \exp(D_{\phi_\Omega})$$

modulo inner automorphisms.

In another language the same results can be expressed by saying that the semiclassical limit of the v.d.Bergh dualizing bimodule of $A_h$ is the canonical flat contravariant connection on $M$.

**Hints on the proof**

Let $\sigma = \exp(D_{\phi_\Omega})$. Define on $A_h[t, t^{-1}]$ the associative product

$$(at^n) \cdot (bt^m) = a \star \sigma^n(b)t^{n+m}$$

this product extend to a star product on $A[t, t^{-1}][[h]]$ which is a deformation quantization of

$$\pi_0 + t\partial_t \wedge \phi_\Omega.$$ 

This last bivector is always Poisson and unimodular\(^9\) on $M \times \mathbb{C}^*$. We can now apply theorem 3.3 to this deformation quantization and conclude that its v.d.Bergh dualizing bimodule is isomorphic to the algebra itself.

At this point the missing (more technical) step is relating the Hochschild (co)homology $A[t, t^{-1}][[h]]$ to the one of $A_h$. \(\blacksquare\)

**Remark:** What we did here was to consider the standard unimodular Poisson Ore extension of a non unimodular Poisson manifold. It would be interesting to check whether this can provide a new way to prove the duality result in Poisson

\(^{9}\)This kind of Poisson bivectors are called also Poisson-Ore extensions; in particular this is called the standard unimodular extension in [5].
(co)homologies. More precisely: start with \( M \). Relate its Poisson homology and cohomology to the one of the canonical unimodular extension (the Euler vector field will play a role here: basically one need to prove a Poincaré lemma kind of statement). Then use duality of unimodular Poisson manifolds (without coefficients) to prove duality for the non unimodular one (with coefficients).

**Remark** What Dolgushev proves, in passing, is that the Ore extension via an automorphism of the Kontsevich quantization of \( M \) gives a Kontsevich quantization of \( M \times \mathbb{C}^* \). It would be interesting to understand whether such result still holds true for general Ore extensions with twisted derivations.

### 3.5 Applications to NC geometry

The reference here is the work of Eli Hawkins ([16]); we will only scrap its surface, in this limited space. Let \( M \) be a manifold with a volume form. Obviously integration defines a linear map on \( C^\infty(M)_c \) which is trivially a trace. For this reason it is reasonable to expect that non commutative integration is a non commutative trace.

Say now we have a Poisson manifold \((M, \pi)\), with a fixed volume form \( \Omega \) and let \( A_\hbar \) be a deformation quantization.

**Proposition 3.6** If there exists a trace on \( A_\hbar \) such that

\[
\int_M f \Omega = \tau(\bar{f}) \mod \hbar,
\]

where \( f = \bar{f} \mod \hbar \), then \( \phi_\Omega = 0 \), i.e. \((M, \pi)\) is unimodular.

This can be seen as a simplified version of Dolgushev theorem (3.3). It tells us that if we insist on using traces as a non commutative version of integration then we are forced to restrict ourselves to the quantization of unimodular Poisson manifolds.

Now the problem is that if you wish to build up a NC spectral triple à la Connes basing on \( A_\hbar \) then indeed a trace is necessary. This simple proposition shows that it is unreasonable to expect a spectral triple when you start with a non unimodular Poisson manifolds. From another point of view the orientability and dimension axioms together requires that \( HH_n(A_\hbar, A_\hbar) \neq 0 \) and a non trivial cohomology class quantizes a volume form. But then formality for chains\(^{10}\) imply that \( H^n_\pi(M) \neq 0 \) with a volume form having non zero class. This is exactly Poisson unimodularity.

To summarize things non unimodularity in Poisson geometry is what explains the phenomenon known as *dimension drop* in Hochschild homology (see [15]). This phenomenon occurs, for example, in standard quantum groups. To recover the correct

\(^{10}\)The already invoked formality for Hochschild chains is a statement about the existence of a specific quasi isomorphism certain of \( L_\infty \) modules. Rather than trying to explain it, even vaguely, which would bring us quite far from this discussion we recall its main corollary: Hochschild (co)homology of Kontsevich’s deformation quantization \( A_\hbar \) is given by formal power series in the Poisson (co)homology of the Kontsevich’s representative \( \pi \) (a formal Poisson bivector).
dimension in those cases we now know that we have to look for twisted traces (see also [7] for its implications from the point of view of spectral triples). Twisted traces (in a sense which could be made precise) provide a quantization of the non trivial Poisson line bundle structure on the canonical bundle. The non triviality here, is of course contained in the modular flow.

4 Group manifolds

In this section we will consider some special cases of Poisson manifolds carrying a large set of symmetries. This will result in a wide class of examples of unimodular and non unimodular Poisson manifolds on which the modular flow is pretty well understood. It would be interesting to apply to this class of examples directly Dolgushev’s results. However it is not known whether the standard quantum groups are Kontsevich’s quantizations.

4.1 Poisson-Lie groups

The aim of this last lecture is to describe in more detail what happens when the manifold $M$ is, in fact, a Lie group and all the structures considered up to now are compatible with the multiplication.

**Definition 4.1** Let $G$ be a Lie group. A Poisson structure $\pi$ on $G$ is said to be multiplicative (and $(G, \pi)$ is then called a Poisson-Lie group) if

$$\pi(gh) = l_{g,\ast}\pi(h) + r_{h,\ast}\pi(g)$$

As a Poisson manifold a non trivial Poisson-Lie group is always non regular (can you guess why?). Let us now consider the following maps:

$$\eta : g \mapsto l_{g^{-1},\ast}\pi(g); \quad \eta : G \to \wedge^2g$$

$$\delta : X \mapsto \left. \frac{d}{dt}\eta(exp(tX)) \right|_{t=0}; \quad \delta : g \to \wedge^2g$$

Non the two properties of being multiplicative and of being Poisson for $\pi$ can be rewritten as properties of $\delta$. They respectively turn out to be the fact that $\delta$ is a 1–cocycle in Lie algebra cohomology with values in $\wedge^2g$, and that its transpose map $\delta^* : \wedge^2g^* \to g^*$ is a Lie bracket on $g^*$. A Lie algebra endowed with a map $\delta$ with such properties is called a Lie bialgebra.

The Lie algebra $g^*$ integrates to a unique connected simply connected Lie group $G^*$ which turns out to be Poisson–Lie as well (the notion is self-dual).

As an example we will consider the case in which $G$ is compact. Say that $G_\mathbb{C}$ is the connected, simply connected, semisimple Lie group that integrates the complexified Lie algebra $g_\mathbb{C}$, so that $G$ is the (unique up to inner automorphisms of $G_\mathbb{C}$) compact
Let \( h \) be a Cartan subalgebra of \( \mathfrak{g}_C \), such that \( h \cap \mathfrak{g} = \mathfrak{t} \) is the Lie algebra of the maximal torus of \( G \), and choose a basis \( \alpha_1, \ldots, \alpha_n \in h^* \) of simple roots. Let \( X_i^\pm \) and \( H_i \) \((i = 1, \ldots, n)\) be the Chevalley generators corresponding to the simple roots. Denote with \( X_\alpha \) and \( H_\alpha \) the Chevalley generators corresponding to generic roots.

The standard Poisson–Lie group structure on the compact group \( G \) is determined by the (coboundary) Lie bialgebra \( \delta(X) = \text{ad}_X r \) where

\[
r = \sum_{\alpha > 0} X_{-\alpha} \wedge X_\alpha \in \wedge^2 \mathfrak{g}
\]

### 4.2 Poisson homogeneous spaces and \( \theta \)-manifolds

Let \((G, \pi)\) be a Poisson–Lie group and let \( M \) be a \( G \)-homogeneous space, with a Poisson bivector \( \pi_M \). Denote with \( \phi : G \times M \to M \) the action and the orbit maps as

\[
\phi_g : x \mapsto \phi(g, x); \quad \phi_x : g \mapsto \phi(g, x)
\]

where \( g \in G \) and \( x \in M \). Then \((M, \pi_M)\) is called a Poisson homogeneous space if

\[
\pi_M(\phi(g, x)) = \phi_{g,*} \pi_M(x) + \phi_{x,*} \pi_G(g).
\]

Examples of Poisson homogeneous spaces are given by quotients by Poisson-Lie subgroups (i.e. subgroups such that the Poisson bivector is tangent to them). However such case is far from exhaustive.

Let \( \pi^r_G \) now be simply a right invariant (i.e. \( l_{g,*} \pi^r_G = \pi^r_G \)) Poisson structure on \( G \). The upper index \( r \) may be considered to describe right invariance or remembering the fact that such structures are determined by elements \( r \in \wedge^2 \mathfrak{g} \) satisfying the classical Yang–Baxter equation \([r, r] = 0\). Let \( P \) be a \( G \)-manifold, with a \( G \)-invariant Poisson structure \( \pi_P \) (i.e. \( l_{g,*} \pi_P = \pi_P \) for all \( g \in G \));\(^{11}\) and consider \( G \times P \) as a product Poisson manifold. The diagonal \( G \) action \( g \cdot (h, x) = (hg^{-1}, gx) \) preserves the Poisson structure and therefore it induces a Poisson structure on the quotient \((G \times P)/G \simeq P\). This structure is just the sum of \( \pi_P \) with the image of \( r \) via the (wedge square of) infinitesimal \( G \)-action.

In the case in which \( G \simeq \mathbb{T}^n \) and \( \pi^r_G \) has maximal rank, one has that \( \pi^r_G \) is determined by an antisymmetric matrix \( \theta \). The corresponding projected Poisson structure \( \pi_\theta \) on \( P \), where \( \pi_P = 0 \), is called a Poisson \( \theta \)-manifold.

Let us now consider the specific example in which \( P = G \), and the left action is simply product on the left. Fix \( r \in \wedge^2 \mathfrak{g} \) satisfying CYBE, let \( \pi^r_G(g) = r_{g,*}r \) and take as left invariant Poisson structure just \( \pi_P(g) = l_{g,*}r \). Then the projected Poisson structure is the sum of this two bivectors and turns out to be a Poisson–Lie structure. This also works if \( r \in \wedge^2 \mathfrak{h} \) where \( \mathfrak{h} \) is a Lie subalgebra of \( \mathfrak{g} \). In the

\(^{11}\)it is always possible to chose \( \pi_P = 0 \)
special case in which \( G \) is compact and \( \mathfrak{h} \) is the Lie algebra of a maximal torus, the CYBE equation turns out to be trivially satisfied. Therefore any antisymmetric matrix \( \theta \in \Lambda^2 \mathfrak{h} \) gives a Poisson–Lie structure \( \pi_\theta \) on \( G \). Together with the standard structure \( \pi_{\text{std}} \) it can be proved that ANY Poisson–Lie structure on a compact \( G \) is a linear combination \( \pi_{\text{std}} + \lambda \pi_\theta \), \( \lambda \in \mathbb{R} \) (the proof relies on the celebrated Belavin Drinfel’d theorem: see [19]).

4.3 Poisson modular class computations

Let \( G \) be a Poisson-Lie group. Let \( \Omega \) be a right invariant volume form on the group. Let \( x_0 \) be the modular character of \( \mathfrak{g} \), i.e. \( x_0(\xi) = \text{tr} \text{ad}_\xi^* \) and let \( \xi_0 \) be the modular character of the Lie algebra \( \mathfrak{g}^* \). Let us denote with \( x_0^L \) (resp. \( x_0^R \)) the left (resp. right) invariant vector field on \( G \) such that its value at the identity is \( x_0 \). Similarly for \( \xi_0^L \) and \( \xi_0^R \) (which are read as right invariant 1–forms on \( G \)). Then Evens-Lu-Weinstein ([12]) proved, with a rather direct computation, that

\[
\phi_\Omega = \frac{1}{2} (x_0^L + x_0^R - \sharp_\pi(\xi_0^R)) \tag{4.3}
\]

A similar formula can be proved for Poisson homogeneous spaces, in full generality ([22]); however, since explaining it here would require some (not so relevant) additional complications we will stick to the simpler case in which the homogeneous space is the quotient of a Poisson–Lie subgroup. In that case, and having chosen a \( G \)–invariant form on the quotient, the modular vector field on \( G/H \) is simply the projection of the one on \( G \).

Let us move to the standard compact case.

**Proposition 4.2** Let \((G, \pi_{\text{std}})\) be a standard compact Poisson Lie group and \( \Omega \) be an invariant volume form. Let us denote with \( H_\rho \) the Cartan element corresponding to the semisum \( \rho \) of all positive roots. Then

\[
\phi_\Omega = 2iH_\rho
\]

and \((G, \pi_{\text{std}})\) is non unimodular.

**Proof**

The computation of the modular vector field is simply an application of formula (4.3), once the explicit form of the Poisson–Lie bivector is given. Remark that being in the \( G \)–compact case, \( \mathfrak{g} \) is unimodular and therefore provides no contribution in formula (4.3), i.e.:

\[
\phi_\Omega = -\frac{1}{2} \sharp_\pi(\xi_0^L).
\]

Knowing that \( \phi_\Omega \neq 0 \) for a single volume form is, of course, not enough to check unimodularity. We need to know here whether its cohomology class is zero. The
point here is that even without computing the full Poisson cohomology of $G$ (which
is not known) we can conclude that $[\phi_{\Omega}] \neq 0$. In fact $\phi_{\Omega}$ is tangent to the maximal
torus $T$ having as Lie algebra $\mathfrak{t}$ the imaginary part of Cartan root vectors. The
restriction of its flow on this torus is not trivial. On the other hand $\pi_{\text{std}} = 0$ when
restricted to this torus. Hence the flow of $\phi_{\Omega}$ does not take place on symplectic leaves
but rather moves one 0-leaf to another. This is possible (basically by definition)
effectively if $\phi_{\Omega}$ is a Poisson derivation which is not Hamiltonian, therefore $[\phi_{\Omega}] \neq 0$ in $H^1_\pi(G)$.

Let us now consider the case of $\theta$–manifolds. Let $\rho : \mathfrak{t}^n \to M$ be the infinitesimal
action, and let $\theta \in \wedge^2 \mathfrak{t}$, so that $\pi_{\theta}(x) = \rho^{\wedge 2}(\theta)(x)$. Now $\rho^{\wedge 2}(\theta) = \sum_{i<j} \theta_{ij} \xi_i \wedge \xi_j$, with $\theta_{ij} \in \mathbb{R}$. Let $\Omega$ be a left invariant volume form on $M$. If we prove that
$\partial_{\pi_{\theta}} \Omega = \sum_{i<j} \theta_{ij} \partial i_{\xi_i} \wedge \frac{\partial}{\partial \xi_j} \Omega$ is zero then we’ve proven that $\theta$–manifolds are unimodular.
Now take into account that $\mathfrak{t}^n$ is commutative. Therefore $[\xi_i, \xi_j] = 0, \forall i, j$. Hence:

$$0 = i_{[\xi_i, \xi_j]} \Omega = (L_{\xi_i} i_{\xi_j} - i_{\xi_j} L_{\xi_i}) \Omega = -di_{\xi_i \wedge \xi_j} \Omega$$

In the case of $\theta$-groups one can perform a completely analogous computation\(^\text{13}\). Thus also $\theta$–Poisson groups are unimodular.

### 4.4 Quantization

Let’s come, now, to quantization. Of course you may say: so what? After Dolgushev’s results is there something interesting to be said about quantization? Yes there is. As already stressed Kontsevich deformation quantization allows to relate
properties of the noncommutative algebra $\mathcal{A}_\hbar$ with properties of the formal Poisson
bivector $\pi \in \hbar \Gamma(\wedge^2 TM)[[\hbar]]$ and not to properties of the Poisson bivector $\pi_0$ one
usually starts with.

Generally this means that while results in the direction $NC \Rightarrow \text{Poisson}$ remain
valid for $\pi_0$, results in the opposite direction are quite hard to prove. Symmetries
(together with the fact that they often provide an explicit - not formal - quantization)
can be a useful tool to circumvent this problem to obtain what one can call *exact
quantization properties*. We give a clearer explanation of what we have in mind
with the example of Hochschild homology computation for quantum groups due to
Feng-Tsygan ([13]).

There, using spectral sequences it was shown that if $\hat{\mathcal{A}}_h$ is the standard quanti-
zation of the standard compact Poisson–Lie group $K$ then the Hochschild homology
of, respectively, the quantized algebra of formal functions $\mathcal{C}^\hbar(G)$ and of quantized

\(^{12}\)But upcoming new results on it were recently announced in ([22])

\(^{13}\)the same proof holds true whenever $M$ carries an invariant $G$–form, regardless of abelianity
and compactness of the group $G$ in our special case. We stuck to the special case of torus actions
as it was the one raising some interest, in last years, in the context of quantization issues (e.g.
[6, 23]). On the other hand $\mathbb{R}^n$–actions may well provide nonunimodular Poisson $\theta$–manifolds. A
family of interesting such manifolds is currently under investigation in [1].
regular functions $\mathbb{C}_h[G]$ are respectively given by:

\[
HH_n(\mathbb{C}_h(G)) = \Omega^n_f(H) \otimes \mathbb{C}((h))
\]
\[
HH_n(\mathbb{C}_h[G]) = \Omega^n(N(H)) \otimes \mathbb{C}((h))
\]

where $\Omega^n_f(H)$ represents the space of formal differential forms of degree $n$ on the Cartan subgroup $H$ and $\Omega^n(N(H))$ the space of regular differential forms of degree $n$ on the normalizer of $H$. It would be interesting to understand whether an extension of these results to cohomology (eventually with coefficients) could allow to directly prove an analogue of Dolgushev’s results for standard quantizations. This, in principle should allow to recover from different means (and maybe to obtain explicit formulas for it) the Nakayama automorphism studied in [3] from a purely algebraic point of view.

Feng-Tsygan results were extended to a specific subclass of compact Poisson homogeneous spaces by [8]. Results on its quantization were recently given in [20]. Another class of examples on which it is reasonable to expect explicit results, as already mentioned, is the one of $\theta$-manifolds; this class is especially interesting since it was widely studied from the point of view of quantizations (non standard quantum groups, strict deformation quantization, geometric quantization with and without groupoid techniques, spectral triples).

References


