STATISTICAL HADRONIZATION and MICROCANONICAL ENSEMBLE

MOTIVATION: a Monte-Carlo code for the statistical model

OUTLINE

- Introduction
- Microcanonical hadron-resonance gas ensemble
- Monte-Carlo sampling algorithm
- Preliminary results and conclusions
Clusters: four-momentum $P$, charges $Q$ and volume $V$

Statistical hadronization: any multi-hadronic state compatible with cluster quantum numbers is equally likely

MICROCANONICAL ENSEMBLE
AIM: MONTE CARLO SAMPLING OF THE HADRON GAS MICROCANONICAL PHASE SPACE

Sample the single hadronic channel first, then assign momenta

\[ N_j : \text{number of particles for the } j^{th} \text{ hadronic species} \]

Statistical weight of a single channel

\[ \Omega_{[N_1, \ldots, N_K]} \equiv \sum_{\text{states}} \delta^4(P - P_{\text{state}}) \mid_{[N_1, \ldots, N_K]} \]

Phase space volume or microcanonical partition function

\[ \Omega = \sum_{[N_1, \ldots, N_K]} \Omega_{[N_1, \ldots, N_K]} \equiv \sum_{\text{states}} \delta^4(P - P_{\text{state}}) \delta Q, Q_{\text{state}} \]
General expression for a cluster with volume $V$ and four-momentum $P$ as a cluster expansion

$$\Omega_{[N_1, ..., N_K]} = \lim_{\eta \to 0} \frac{1}{(2\pi)^3} \int_{-\infty}^{+\infty} dx^0 \int d^3 x \exp i P \cdot x \prod_{j=1}^{K} (\mp 1)^{N_j} \sum_{n_j=1}^{N_j} \prod_{h_n} (\mp 1)^{h_n} \frac{z_{j(n)}(x)}{n_j h_n}$$

with $z_{j(n)}(x) = \frac{(2J_j + 1) V}{(2\pi)^3} \int d^3 p \exp [-i n x^0 \epsilon + i n x \cdot p]$ \quad $\sum_{n_j=1}^{N_j} n_j h_n = N_j$

In the limit of Boltzmann statistics, with $\sum_{j=1}^{K} N_j \equiv N$

$$\Omega_{[N_1, ..., N_K]} = \left[ \frac{V}{(2\pi)^3} \right]^N \prod_{i=1}^{N} \frac{(2J_i + 1)}{N_i!} \int \prod_{i=1}^{N} d^3 p_i \delta^4 (P - \sum_{i} p_i)$$

⚠️ Calculable analytically only in the NR and UR regimes
The problem of calculating $\Omega_{[N_1, \ldots, N_x]}$ for relativistic massive particles with suitable numerical methods tackled by several authors in the '50s. None of several devised approximations satisfactory for all the allowed channels. Cerulus and Hagedorn (Suppl. N. Cim. IX (1958) 646) successfully implemented a Monte-Carlo method.

**$\Omega$ distribution with 1000 MC steps**

- With QS
- No QS

**$N\pi^0$ channel - 2 GHz clock rate**

- Quantum statistics
- Classical statistics
MAIN DIFFICULTY: SIZE

Number of light-flavoured hadronic species (PDG 2002): 271

\[ N_{\text{max}} = \frac{M}{\langle m \rangle} \]

If \( N_{\text{max}} \ll 271 \Rightarrow N_{\text{channels}} \approx \frac{271^{N_{\text{max}}}}{N_{\text{max}}!} \)

Time needed to calculate all channels diverges exponentially with \( M \)

Need a non-deterministic sampling algorithm
METROPOLIS ALGORITHM


Random walk in the channel
(or multihadronic configuration) space

Master equation

\[ P_m(t + 1) - P_m(t) = \sum_n P_n(t) w(n \rightarrow m) - P_m(t) w(m \rightarrow n) \]

if

\[ \frac{w(n \rightarrow m)}{w(m \rightarrow n)} = \frac{\Omega_m}{\Omega_n} \]

at equilibrium \( P_m \propto \Omega_m \)

and a channel can be extracted

Design goal: minimize the number of steps needed to reach equilibrium
Start from fast-sampling distribution as close as possible to the final one.

Clever choice of acceptance and proposal matrix so as to maximize the transition probability

\[ \omega(n \rightarrow m) = T(n \rightarrow m) A(n \rightarrow m) \]

OPTIMAL CHOICE for \( A \) once \( T \) is known

\[ A(n \rightarrow m) = \min \left\{ 1, \frac{\Omega_m T(m \rightarrow n)}{\Omega_n T(n \rightarrow m)} \right\} \]

OPTIMAL CHOICE for \( T \)

\[ T(n \rightarrow m) = \Omega_m \quad A(n \rightarrow m) = 1 \]

set \( T \) as the grand-canonical limit of the multihadron multiplicity distribution, the multi-Poisson

\[ \Omega_m \rightarrow \prod_j \exp(-\langle n_j \rangle) \frac{\langle n_j \rangle^{N_j}}{N_j!} = T(n \rightarrow m) \]

\[ \langle n_j \rangle = \text{canonical averages} \]
FAST CONVERGENCE TO EQUILIBRIUM!

\( M = 2 \text{ GeV}, \) free charges, \( M/V = 0.4 \text{ GeV/fm}^3 \)

Total multiplicity distribution

'Simple' updating rule

Poissonian updating rule
Further advantages of the poissonian updating rule

- Efficient sampling
- Convergence faster for larger clusters (approach grand-canonical limit)
- Low auto-correlation length
COMPARISON BETWEEN MICRO AND CANONICAL AVERAGES

Statistical model calculations so far have been done within the canonical ensemble assuming the equivalence between the set of clusters and one global cluster.

**TEMPERATURE** introduced through a saddle-point approximation for large $M$ and $V$

$$\Omega = \sum_{\text{states}} \delta^4(P - P_{\text{state}}) \delta_{Q, Q_{\text{state}}} \sim \exp[M/T] Z_{\text{can}}(Q) \quad P = (M, 0)$$

canonical average multiplicity

$$\langle n_j \rangle = \frac{(2J_j + 1)V}{(2\pi)^3} \int d^3 p \exp(-\sqrt{p^2 + m^2_j} / T) \frac{Z_{\text{can}}(Q - q_j)}{Z_{\text{can}}(Q)}$$

HOW LARGE $M$ AND $V$ SHOULD BE?
Study of $\frac{\langle n\rangle_{\text{micro}} - \langle n\rangle_{\text{can}}}{\langle n\rangle_{\text{can}}}$ with $Q = 0$ and $M/V = 0.4$ GeV/fm$^3$

see also F. Liu et al. hep-ph 0304174

<table>
<thead>
<tr>
<th>Mass (GeV)</th>
<th>Temperature (MeV)</th>
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<tbody>
<tr>
<td>4</td>
<td>169.3</td>
</tr>
<tr>
<td>8</td>
<td>164.6</td>
</tr>
<tr>
<td>12</td>
<td>162.7</td>
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</table>
Study of $\frac{\langle n \rangle_{\text{micro}} - \langle n \rangle_{\text{can}}}{\langle n \rangle_{\text{can}}}$ with $Q = 0$ and $M/V = 0.4$ GeV/fm$^3$

Increasing the number of degrees of freedom gets microcanonical and canonical averages closer

![Graph showing the percentage change in $\pi^0$, all hadrons, and only $\pi^0$ as a function of $M$ (GeV).]
Free charges,

\[ M/V = 0.4 \text{ GeV/fm}^3 \]
Preliminary simplified version of an $e^+e^-$ event generator

- Run parton shower program
- Assign final (virtual) partons a volume $V = M/0.35 \text{ GeV/fm}^3$ (and $\gamma_s = 0.65$)
- Hadronize the clusters with Metropolis algorithm
- Check overall conservation of charge (accept or reject)

Charged track multiplicity distribution at

$\sqrt{s} = 91.2 \text{ GeV}$

10K events with parton shower in Jetset7.4
ISMD 2003

Average multiplicities

<table>
<thead>
<tr>
<th>Meson</th>
<th>Misura</th>
<th>GSPS</th>
<th>LPS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi^0$</td>
<td>9.61 ± 0.29</td>
<td>9.738 ± 0.031</td>
<td>9.783 ± 0.031</td>
</tr>
<tr>
<td>$\pi^\pm$</td>
<td>8.50 ± 0.10</td>
<td>8.644 ± 0.021</td>
<td>8.621 ± 0.020</td>
</tr>
<tr>
<td>$K^{\pm}$</td>
<td>1.127 ± 0.026</td>
<td>1.0955 ± 0.0074</td>
<td>1.0621 ± 0.0073</td>
</tr>
<tr>
<td>$K_S^{0}$</td>
<td>1.0376 ± 0.0096</td>
<td>1.062 ± 0.010</td>
<td>1.035 ± 0.010</td>
</tr>
<tr>
<td>$\eta$</td>
<td>1.059 ± 0.086</td>
<td>0.888 ± 0.0094</td>
<td>0.8858 ± 0.0094</td>
</tr>
<tr>
<td>$\rho^+$</td>
<td>1.94 ± 0.13</td>
<td>1.062 ± 0.010</td>
<td>1.050 ± 0.010</td>
</tr>
<tr>
<td>$\rho^{\pm}$</td>
<td>1.20 ± 0.22</td>
<td>0.9963 ± 0.0071</td>
<td>1.0003 ± 0.0071</td>
</tr>
<tr>
<td>$\omega$</td>
<td>1.024 ± 0.059</td>
<td>0.9468 ± 0.0097</td>
<td>0.9646 ± 0.0098</td>
</tr>
</tbody>
</table>

| $K^{*0}$ | 0.357 ± 0.022 | 0.3570 ± 0.0042 | 0.3448 ± 0.0041 |
| $K^{*\pm}$ | 0.370 ± 0.012 | 0.3544 ± 0.0042 | 0.3412 ± 0.0041 |
| $\eta'$ | 0.166 ± 0.047 | 0.0866 ± 0.0029 | 0.0890 ± 0.0030 |
| $\phi$ | 0.0977 ± 0.0058 | 0.1081 ± 0.0033 | 0.1141 ± 0.0034 |
| $f_2(1270)$ | 0.188 ± 0.020 | 0.1006 ± 0.0032 | 0.108 ± 0.0033 |
| $K_2^{*}$ | 0.036 ± 0.011 | 0.02275 ± 0.0011 | 0.0225 ± 0.0011 |
| $a_0(980)$ | 0.135 ± 0.054 | 0.0875 ± 0.0021 | 0.0823 ± 0.0020 |
| $f_0$ | 0.1555 ± 0.0085 | 0.0674 ± 0.0026 | 0.0700 ± 0.0026 |
| $f'_0(1525)$ | 0.0120 ± 0.0058 | 0.0087 ± 0.0009 | 0.0107 ± 0.0010 |

**GSPS:** Parton shower developed by R. Ugoccioni, A. Giovannini, S. Lupia

**LPS:** Parton shower in JETSET 7.4

<table>
<thead>
<tr>
<th>Baryon</th>
<th>Misura</th>
<th>GSPS</th>
<th>LPS</th>
</tr>
</thead>
<tbody>
<tr>
<td>p</td>
<td>0.519 ± 0.018</td>
<td>0.5963 ± 0.0077</td>
<td>0.58585 ± 0.0054</td>
</tr>
<tr>
<td>$\Lambda$</td>
<td>0.1943 ± 0.0038</td>
<td>0.2127 ± 0.0033</td>
<td>0.20015 ± 0.0032</td>
</tr>
<tr>
<td>$\Sigma^+$</td>
<td>0.0535 ± 0.0052</td>
<td>0.0512 ± 0.0023</td>
<td>0.0487 ± 0.0016</td>
</tr>
<tr>
<td>$\Sigma^0$</td>
<td>0.0389 ± 0.0041</td>
<td>0.05245 ± 0.0016</td>
<td>0.0479 ± 0.0015</td>
</tr>
<tr>
<td>$\Sigma^-$</td>
<td>0.0410 ± 0.0037</td>
<td>0.0451 ± 0.0015</td>
<td>0.0428 ± 0.0015</td>
</tr>
<tr>
<td>$\Sigma^\pm$</td>
<td>0.0868 ± 0.0087</td>
<td>0.0963 ± 0.0031</td>
<td>0.0915 ± 0.0030</td>
</tr>
<tr>
<td>$\Delta^{++}$</td>
<td>0.044 ± 0.016</td>
<td>0.0906 ± 0.0021</td>
<td>0.0870 ± 0.0021</td>
</tr>
<tr>
<td>$\Xi$</td>
<td>0.01319 ± 0.00052</td>
<td>0.01305 ± 0.0008</td>
<td>0.01315 ± 0.00081</td>
</tr>
<tr>
<td>$\Sigma^*$</td>
<td>0.0118 ± 0.0011</td>
<td>0.02225 ± 0.0010</td>
<td>0.02165 ± 0.0010</td>
</tr>
<tr>
<td>$\Lambda(1520)$</td>
<td>0.0112 ± 0.0014</td>
<td>0.01205 ± 0.0008</td>
<td>0.0114 ± 0.0007</td>
</tr>
<tr>
<td>$\Xi^*$</td>
<td>0.00289 ± 0.00050</td>
<td>0.0047 ± 0.0005</td>
<td>0.0046 ± 0.0005</td>
</tr>
<tr>
<td>$\Omega$</td>
<td>0.00062 ± 0.00012</td>
<td>0.0016 ± 0.0003</td>
<td>0.0005 ± 0.0002</td>
</tr>
</tbody>
</table>
Conclusions and Outlook

- Towards a Monte-Carlo code for the statistical model of hadronization. Next steps: inclusion of particle momenta and BEC (L. Ferroni, T. Gabbriellini, A. Keranen, collaboration with Nantes)

- Easily matcheable to parton shower cascade programs with pre-confinement cluster formation (HERWIG)

- Comparison between micro and canonical ensemble: for $M \geq 8$ GeV average multiplicities sufficiently close

- Many more tests possible