# QUANTIZATION OF INFINITESIMAL BRAIDINGS

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based on

- [A.Ardizzoni, L.Bottegoni, A.Sciandra, TW, **Commun. Contemp. Math.** 27, 5 (2025) 2450029]
- [C.Esposito, A.Rivezzi, J.Schnitzer, TW, arXiv:2505.17729]

# Motivation: Quantization of the Casimir

$\mathfrak{g}$ complex semi-simple Lie algebra, $C := \sum_i x_i x^i \in U\mathfrak{g}$ Casimir element

$$\chi := \frac{\Delta(C) - 1 \otimes C - C \otimes 1}{2} = \frac{x_i \otimes x^i + x^i \otimes x_i}{2} \in U\mathfrak{g} \otimes U\mathfrak{g}$$

$\Rightarrow \chi$ is ad-invariant and $\Delta(\cdot)\chi = \chi\Delta(\cdot)$.

...for $\mathfrak{g} = \mathrm{sl}_2$ we have $\chi = \frac{1}{4}\left(E \otimes F + F \otimes E + \frac{H \otimes H}{2}\right)$.

Can we promote $\mathcal{R} := e^{\hbar\chi} \in (U\mathfrak{g} \otimes U\mathfrak{g})[[\hbar]]$ to an $\mathcal{R}$-matrix on $(U\mathfrak{g}[[\hbar]], \Delta, \varepsilon)$?

In general, no!

But $(U\mathfrak{g}[[\hbar]], \Delta, \varepsilon, \Phi, e^{\hbar\chi})$ is a quasitriangular quasi-Hopf algebra!

$$(\mathrm{id} \otimes \Delta)\Delta(\cdot) = \Phi(\Delta \otimes \mathrm{id})\Delta(\cdot)\Phi^{-1}$$

$$(\mathrm{id} \otimes \Delta)(\mathcal{R}) = \Phi_{312}^{-1}\mathcal{R}_{13}\Phi_{213}\mathcal{R}_{12}\Phi^{-1}$$

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$\Phi \in U\mathfrak{g}^{\otimes 3}[[\hbar]]$ is called a re-associator.

Most famous example: the re-associator $\Phi_{\mathrm{KZ}}$ constructed from a solution of the Knizhnik-Zamolodchikov differential equation

$$\mathrm{d}w(z_1, \ldots, z_n) = \frac{\hbar}{2\pi \mathrm{i}} \sum_{1 \leq i < j \leq n} \frac{t_{ij}}{z_i - z_j} (\mathrm{d}z_i - \mathrm{d}z_j) w(z_1, \ldots, z_n).$$

Why is this construction interesting?

$$(U\mathfrak{g}[[\hbar]], \Delta, \varepsilon, \Phi_{\mathrm{KZ}}, e^{\hbar\chi}) \sim_{\mathcal{F}} U_\hbar\mathfrak{g}$$

is gauge twist equivalence with (the $\hbar$-adic version of) the famous Drinfel'd-Jimbo quasitriangular Hopf algebra!

**Question:** Can we generalize this construction?

- With no Lie algebra in the background, what is $\chi$?

$$\tilde{\mathcal{R}} = \mathcal{R}(1 \otimes 1 + \hbar\chi + \mathcal{O}(\hbar^2))$$

- Does a similar quantization ansatz $e^{\hbar\chi}$ work, using re-associators?
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# Category-Algebra correspondence

$H$ associative algebra over field $\mathbb{k}$ and ${}_H\mathcal{M}$ category of left $H$-modules.

| **Category** | **Algebra** |
|---|---|
| $({}_H\mathcal{M}, \otimes, \mathbb{k})$ is monoidal | $(H, \Delta, \varepsilon)$ is bialgebra<br>$\Delta \colon H \to H \otimes H$ coproduct<br>$\varepsilon \colon H \to \mathbb{k}$ counit |
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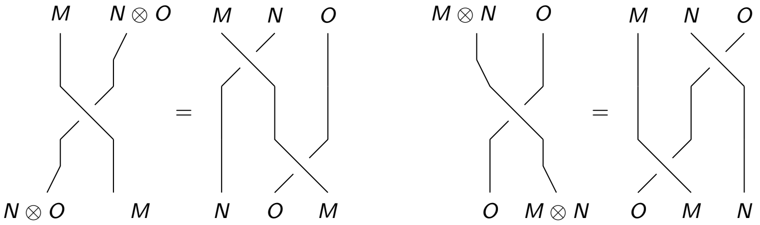
$H$ associative algebra over field $\mathbb{k}$ and ${}_H\mathcal{M}$ category of left $H$-modules.

| **Category** | **Algebra** |
|---|---|
| $({}_H\mathcal{M}, \otimes, \mathbb{k})$ is monoidal | $(H, \Delta, \varepsilon)$ is bialgebra<br>$\Delta \colon H \to H \otimes H$ coproduct<br>$\varepsilon \colon H \to \mathbb{k}$ counit |
| $({}_H\mathcal{M}, \otimes, \mathbb{k}, \sigma)$ is **braided** monoidal<br>$\sigma_{M,N} \colon M \otimes N \to N \otimes M$ braiding | $(H, \Delta, \varepsilon, \mathcal{R})$ is **quasitriangular** bialgebra<br>$\mathcal{R} \in H \otimes H$ universal $\mathcal{R}$-matrix |
| $({}_H\mathcal{M}, \otimes, \mathbb{k}, \sigma, t)$ pre-Cartier category<br>$t_{M,N} \colon M \otimes N \to M \otimes N$ **infinitesimal braiding** | $(H, \Delta, \varepsilon, \mathcal{R}, \chi)$ is pre-Cartier bialgebra<br>$\chi \in H \otimes H$ **infinitesimal $\mathcal{R}$-matrix** |

$$\tilde{\sigma}_{M,N} = \sigma_{M,N} \circ (\mathrm{id} + \hbar t_{M,N} + \mathcal{O}(\hbar^2))$$
$$\tilde{\mathcal{R}} = \mathcal{R}(1 \otimes 1 + \hbar\chi + \mathcal{O}(\hbar^2))$$

# Braided monoidal categories

Category $\mathcal{C}$ with a monoidal structure $\otimes$, the "tensor product", and a natural isomorphism $\sigma_{M,N} \colon M \otimes N \to N \otimes M$ such that



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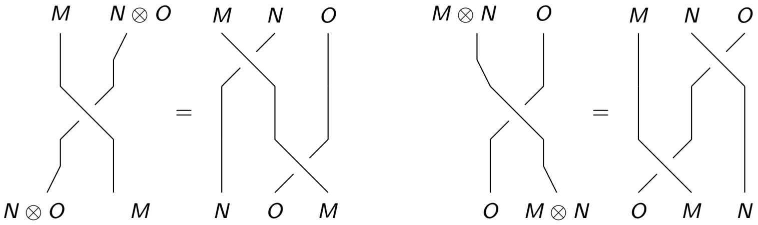


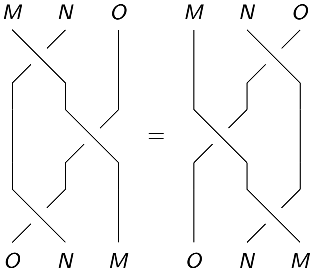

**Braided monoidal category** is $(\mathcal{C}, \otimes, I, a, \ell, r, \sigma)$, with $\mathcal{C}$ category, $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ functor, $I \in \mathcal{C}$, and natural isomorphisms

$$a : \otimes \circ (\otimes \times \mathrm{id}) \Rightarrow \otimes \circ (\mathrm{id} \times \otimes) \qquad r : \otimes \circ (\mathrm{id} \times I) \Rightarrow \mathrm{id}$$

$$\ell : \otimes \circ (I \times \mathrm{id}) \Rightarrow \mathrm{id} \qquad \sigma : \otimes \Rightarrow \otimes^{\mathrm{op}}$$

such that for all objects $X, Y, Z, W$ in $\mathcal{C}$:

i.)

$$\begin{array}{ccc}
(X \otimes (Y \otimes Z)) \otimes W & \xleftarrow{a_{X,Y,Z} \otimes \mathrm{id}_W} & ((X \otimes Y) \otimes Z) \otimes W \\
 & & \big\downarrow a_{X \otimes Y, Z, W} \\
\big\downarrow a_{X, Y \otimes Z, W} & & (X \otimes Y) \otimes (Z \otimes W) \\
 & & \big\downarrow a_{X, Y, Z \otimes W} \\
X \otimes ((Y \otimes Z) \otimes W) & \xrightarrow[\mathrm{id}_X \otimes a_{Y,Z,W}]{} & X \otimes (Y \otimes (Z \otimes W))
\end{array}$$

ii.)

$$\begin{array}{ccc}
(X \otimes I) \otimes Y & \xrightarrow{a_{X,I,Y}} & X \otimes (I \otimes Y) \\
 & {\scriptstyle r_X \otimes \mathrm{id}_Y} \searrow \quad \swarrow {\scriptstyle \mathrm{id}_X \otimes \ell_Y} & \\
 & X \otimes Y &
\end{array}$$

iii.) + hexagons (as before)

# Quasitriangular bialgebras and universal $\mathcal{R}$-matrices

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**Theorem (Drinfel'd-Majid '90)**

*$({}_H\mathcal{M}, \otimes)$ is braided if and only if $H$ is* **quasitriangular**, *i.e. $\exists\, \mathcal{R} \in H \otimes H$ invertible s.t.*

$$(\Delta \otimes \mathrm{id}_H)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{23}, \qquad (\mathrm{id}_H \otimes \Delta)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{12}$$

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Note: $\sigma$ is **symmetric**, i.e. $\sigma^2 = \mathrm{id}$, if and only if $\mathcal{R}$ is **triangular**, i.e. $\mathcal{R}^{-1} = \mathcal{R}^{\mathrm{op}}$.

**Example**

i.) Every cocommutative (i.e. $\Delta = \Delta^{\mathrm{op}}$) bialgebra is triangular with $\mathcal{R} = 1 \otimes 1$. For example $\Bbbk[G]$ for any group $G$ and $U\mathfrak{g}$ for any Lie algebra $\mathfrak{g}$.

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Let $H$ be a bialgebra.

$\Rightarrow$ The formal power series $\tilde{H} = H[[\hbar]]$ with formal parameter $\hbar$ become a **topological bialgebra** with $\hbar$-linearly extended (co)algebra structures.

Consider a quasitriangular structure $\tilde{\mathcal{R}} \in (H \otimes H)[[\hbar]] \cong \tilde{H} \hat{\otimes} \tilde{H}$ on $\tilde{H} = H[[\hbar]]$.

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What are the properties of $\chi \in H \otimes H$?

**Definition (Ardizzoni-Bottegoni-Sciandra-TW '25)**

We call $(H, \mathcal{R}, \chi)$ a **pre-Cartier bialgebra** with infinitesimal $\mathcal{R}$-matrix $\chi \in H \otimes H$ if $(H, \mathcal{R})$ is a quasitriangular bialgebra such that $\chi\Delta(\cdot) = \Delta(\cdot)\chi$ and

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$$(\mathrm{id}_H \otimes \Delta)(\chi) = \chi_{12} + \mathcal{R}_{12}^{-1}\chi_{13}\mathcal{R}_{12}, \qquad (\Delta \otimes \mathrm{id}_H)(\chi) = \chi_{23} + \mathcal{R}_{23}^{-1}\chi_{13}\mathcal{R}_{23}.$$

If also $\mathcal{R}\chi = \chi^{\mathrm{op}}\mathcal{R}$, we call $(H, \mathcal{R}, \chi)$ a **Cartier bialgebra**.

# The first order of an $\mathcal{R}$-matrix

Let $H$ be a bialgebra.

$\Rightarrow$ The formal power series $\tilde{H} = H[[\hbar]]$ with formal parameter $\hbar$
become a **topological bialgebra** with $\hbar$-linearly extended (co)algebra structures.

Consider a quasitriangular structure $\tilde{\mathcal{R}} \in (H \otimes H)[[\hbar]] \cong \tilde{H}\hat{\otimes}\tilde{H}$ on $\tilde{H} = H[[\hbar]]$.

$\Rightarrow$ $\tilde{\mathcal{R}} = \mathcal{R} + \mathcal{O}(\hbar)$ gives quasitriangular bialgebra $(H, \mathcal{R})$ and we can write

$$\tilde{\mathcal{R}} = \mathcal{R}(1 \otimes 1 + \hbar\chi + \mathcal{O}(\hbar^2)).$$

What are the properties of $\chi \in H \otimes H$?

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# The categorical counterpart

**Proposition**

*$(H,\mathcal{R},\chi)$ is a pre-Cartier bialgebra iff ${}_H\mathcal{M}$ is a braided monoidal category and there is a natural transformation $t_{M,N}\colon M\otimes N\to M\otimes N$ in ${}_H\mathcal{M}$ such that*

$$t_{M,N\otimes L}=t_{M,N}\otimes L+(\sigma_{M,N}^{-1}\otimes L)(N\otimes t_{M,L})(\sigma_{M,N}\otimes L)$$
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$$\sigma_{M,N}\circ t_{M,N}=t_{N,M}\circ\sigma_{M,N}.$$

**Definition**

A braided monoidal category $(\mathcal{C},\sigma)$ with a natural transformation $t$ as above is called **(pre-)Cartier category** and we call $t$ an **infinitesimal braiding**.

For $\sigma$ a symmetric braiding ($\sigma^2=\mathrm{id}$) with $\sigma\circ t=t\circ\sigma$ we recover the known notion of **Cartier category**. [Cartier '93, Kassel '95, Heckenberger-Vendramin '23]

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# Let's see some examples...

**Example (Trivial example)**

The infinitesimal $\mathcal{R}$-matrices $\chi$ of a quasitriangular bialgebra $(H, \mathcal{R})$ form a vector space. In particular, $\chi = 0$ is a trivial solution which makes $(H, \mathcal{R})$ Cartier.

**Example**

If $(\mathfrak{g}, [\cdot,\cdot], r)$ is a **quasitriangular Lie bialgebra**, i.e.

- $(\mathfrak{g}, [\cdot,\cdot])$ is Lie algebra and
- $r \in \mathfrak{g} \otimes \mathfrak{g}$ is ad-invariant ($\mathrm{ad}_x^{(2)}(r) = 0$ for all $x \in \mathfrak{g}$) and satisfies the **classical Yang-Baxter equation**

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0$$

then $\chi := r + r^{\mathrm{op}}$ is an infinitesimal $\mathcal{R}$-matrix of $(U\mathfrak{g}, 1 \otimes 1)$.

**Example (The (co)commutative case: primitive elements)**

If $H$ is commutative, then $\chi$ is an infinitesimal $\mathcal{R}$-matrix iff $\chi \in P(H) \otimes P(H)$, where $P(H)$ are the primitive elements of $H$. $(H, \mathcal{R}, \chi)$ is Cartier if furthermore $\chi^{\mathrm{op}} = \chi$.

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# A noncommutative and non-cocommutative example

**Example (The $E(n)$ Hopf algebras, [Bottegoni-Renda-Sciandra, arXiv:2412.02350])**

Generators $g, x_1, \ldots, x_n$ with $g^2 = 1$, $x_j x_i = -x_i x_j$, $x_i g = -g x_i$, $\rightsquigarrow \dim E(n) = 2^{n+1}$, $\Delta(g) = g \otimes g$, $\Delta(x_i) = x_i \otimes 1 + g \otimes x_i$.

All quasitriangular structures on $H$ are of the form

$$\mathcal{R}_{(a_{ij})} = \frac{1}{2}\Big(1 \otimes 1 + g \otimes 1 + 1 \otimes g - g \otimes g\Big) \exp\Big(\sum\nolimits_{i,j} a_{ij} g x_i \otimes x_j\Big)$$

for $(a_{ij}) \in M_n(\mathbb{C})$. $\mathcal{R}_{(a_{ij})}$ triangular $\Leftrightarrow$ $(a_{ij}) \in M_n(\mathbb{C})$ symmetric.

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$E(1)$ is Sweedler's 4-dimensional Hopf algebra. All $\mathcal{R}$-matrices of $E(1)$ are triangular, $\chi = bgx \otimes x$ is pre-Cartier and only $\chi = 0$ is Cartier.

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# The role of Hochschild cohomology

**Theorem (Ardizzoni-Bottegoni-Sciandra-TW '25)**

*Let $(H, \mathcal{R}, \chi)$ be a pre-Cartier quasitriangular bialgebra. Then*

i.) $\chi$ *is a* **Hochschild 2-cocycle**, *i.e.* $\chi_{12} + (\Delta \otimes \mathrm{id})(\chi) = \chi_{23} + (\mathrm{id} \otimes \Delta)(\chi)$.

ii.) $\chi$ *satisfies the* **infinitesimal quantum Yang-Baxter equation**

$$\mathcal{R}_{12}\chi_{12}\mathcal{R}_{13}\mathcal{R}_{23} + \mathcal{R}_{12}\mathcal{R}_{13}\chi_{13}\mathcal{R}_{23} + \mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23}\chi_{23} \\ = \mathcal{R}_{23}\chi_{23}\mathcal{R}_{13}\mathcal{R}_{12} + \mathcal{R}_{23}\mathcal{R}_{13}\chi_{13}\mathcal{R}_{12} + \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}\chi_{12}.$$

This generalizes a result of [Majid '97] for $(U\mathfrak{g}, 1 \otimes 1)$ with $(\mathfrak{g}, [\cdot,\cdot], r)$ quasitriangular Lie bialgebras as before.

If $(H, \mathcal{R}, \chi)$ is a pre-Cartier quasitriangular **Hopf** algebra, define $\gamma := S(\chi^i)\chi_i \in H$, the **Casimir element**, where $\chi = \chi^i \otimes \chi_i$. It follows that $\gamma$ is central!

**Proposition (Ardizzoni-Bottegoni-Sciandra-TW '25)**

*If* $(H, \mathcal{R}, \chi)$ *is a* **Cartier triangular Hopf** *algebra, then* $\chi = b^1(\frac{\gamma}{2})$ *is a Hochschild 2-coboundary, where* $b^1(h) := 1 \otimes h - \Delta(h) + h \otimes 1$.

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$\rightsquigarrow$ [Esposito, C., Rivezzi, A., Schnitzer, J., TW: *Quantization of infinitesimal braidings and pre-Cartier quasi-bialgebras*, arXiv:2505.17729]

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# Quasi-bialgebras

**Definition**

Let $H$ be an associative unital algebra with two algebra morphisms $\Delta\colon H \to H \otimes H$ and $\varepsilon\colon H \to \mathbb{k}$.

i.) We call $H$ a **quasi-bialgebra** if $\exists$ invertible elements $\Phi \in H^{\otimes 3}$ and $\ell, r \in H$ s.t.

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## Definition (Esposito-Rivezzi-Schnitzer-TW '25)

A quasitriangular quasi-bialgebra $(H,\Delta,\varepsilon,\Phi,\ell,r,\mathcal{R})$ is called **pre-Cartier** if $\exists\ \chi\in H\otimes H$, the **infinitesimal $\mathcal{R}$-matrix**, s.t.

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If, in addition, $\mathcal{R}\chi=\chi^{\mathrm{op}}\mathcal{R}$ we call $H$ **Cartier**.

## Theorem

*Let $H$ be an associative unital algebra.*

i.) *The representation category ${}_H\mathcal{M}$ is monoidal (with strong monoidal functor ${}_H\mathcal{M}\to{}_{\Bbbk}\mathrm{Vec}$) iff $(H,\Delta,\varepsilon,\Phi,\ell,r)$ is quasi-bialgebra:*

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### Theorem

*Let $H$ be an associative unital algebra.*

i.) *The representation category ${}_H\mathcal{M}$ is monoidal (with strong monoidal functor ${}_H\mathcal{M} \to {}_{\Bbbk}\mathrm{Vec}$) iff $(H, \Delta, \varepsilon, \Phi, \ell, r)$ is quasi-bialgebra:*

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Let $H$ be an associative unital algebra. Elements $\{A^{ij}\}_{1\leq i<j\leq 4}$ of $H$ satisfy the **infinitesimal braid relations** if

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**Definition**

Let $\mathbb{k}$ be a field s.t. $\mathbb{Q} \subseteq \mathbb{k}$. A **Drinfel'd associator** is a formal power series in two noncommuting variables $\Psi(A, B) \in \mathbb{k}\langle\langle A, B\rangle\rangle$ s.t.

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$$\exp(\hbar/2\Lambda) = \exp(\hbar/2A)\Psi(\hbar C, \hbar A)\exp(\hbar/2C)\Psi(\hbar B, \hbar C)\exp(\hbar/2B)$$

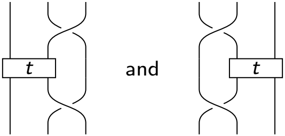
iii.) $\Psi(A,B)^{-1} = \Psi(B,A)$

iv.) $\Psi(A,B)$ is group-like, where one defines $\Delta(A) = A \otimes 1 + 1 \otimes A$.

## Theorem (Drinfel'd '90)

*There exists a Drinfel'd associator. ...think of* $\Phi_{\mathrm{KZ}}$.

$(\mathcal{C}, \otimes, \sigma, t)$ pre-Cartier category, $X, Y, Z$ objects in $\mathcal{C}$. $\rightsquigarrow$ there are (at least) two possible ways to define $t^{13}_{X,Y,Z} \colon X \otimes Y \otimes Z \to X \otimes Y \otimes Z$, namely




**Lemma**

$$t^{13,\mathrm{I}}_{X,Y,Z} := (\mathrm{id}_X \otimes \sigma^{-1}_{Y,Z}) \circ (t_{X,Z} \otimes \mathrm{id}_Y) \circ (\mathrm{id}_X \otimes \sigma_{Y,Z})$$
$$t^{13,\mathrm{II}}_{X,Y,Z} := (\sigma^{-1}_{X,Y} \otimes \mathrm{id}_Z) \circ (\mathrm{id}_Y \otimes t_{X,Z}) \circ (\sigma_{X,Y} \otimes \mathrm{id}_Z)$$
$$t^{13,\mathrm{III}}_{X,Y,Z} := (\sigma_{Y,X} \otimes \mathrm{id}_Z) \circ (\mathrm{id}_Y \otimes t_{X,Z}) \circ (\sigma^{-1}_{Y,X} \otimes \mathrm{id}_X)$$
$$t^{13,\mathrm{IV}}_{X,Y,Z} := (\mathrm{id}_X \otimes \sigma_{Z,Y}) \circ (t_{X,Z} \otimes \mathrm{id}_Y) \circ (\mathrm{id}_X \otimes \sigma^{-1}_{Z,Y})$$

*Then* $t^{13,\mathrm{I}}_{X,Y,Z} = t^{13,\mathrm{II}}_{X,Y,Z} = t^{13,\mathrm{III}}_{X,Y,Z} = t^{13,\mathrm{IV}}_{X,Y,Z} =: t^{13}_{X,Y,Z}$.

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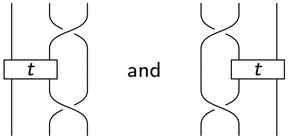


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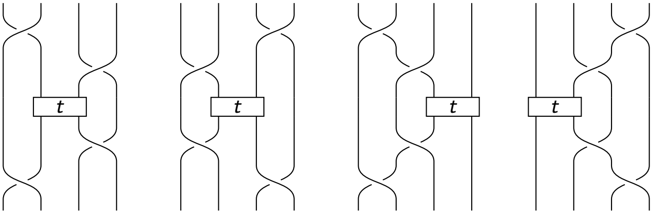


**Lemma**

$$t^{14,\mathrm{I}}_{X,Y,Z,W} = (\sigma^{-1}_{X,Y} \otimes \mathrm{id}_{Z\otimes W})(\mathrm{id}_{Y\otimes X} \otimes \sigma^{-1}_{Z,W})(\mathrm{id}_Y \otimes t_{X,W} \otimes \mathrm{id}_Z)(\mathrm{id}_{Y\otimes X} \otimes \sigma_{Z,W})(\sigma_{X,Y} \otimes \mathrm{id}_{Z\otimes W})$$

$$t^{14,\mathrm{II}}_{X,Y,Z,W} = (\mathrm{id}_{X\otimes Y} \otimes \sigma^{-1}_{Z,W})(\sigma^{-1}_{X,Y} \otimes \mathrm{id}_{W\otimes Z})(\mathrm{id}_Y \otimes t_{X,W} \otimes \mathrm{id}_Z)(\sigma_{X,Y} \otimes \mathrm{id}_{W\otimes Z})(\mathrm{id}_{X\otimes Y} \otimes \sigma_{Z,W})$$

$$t^{14,\mathrm{III}}_{X,Y,Z,W} = (\sigma^{-1}_{X,Y} \otimes \mathrm{id}_{Z\otimes W})(\mathrm{id}_Y \otimes \sigma^{-1}_{X,Z} \otimes \mathrm{id}_W)(\mathrm{id}_{Y\otimes Z} \otimes t_{X,W})(\mathrm{id}_Y \otimes \sigma_{X,Z} \otimes \mathrm{id}_W)(\sigma_{X,Y} \otimes \mathrm{id}_{Z\otimes W})$$

$$t^{14,\mathrm{IV}}_{X,Y,Z,W} = (\mathrm{id}_{X\otimes Y} \otimes \sigma^{-1}_{Z,W})(\mathrm{id}_X \otimes \sigma^{-1}_{Y,W} \otimes \mathrm{id}_Z)(t_{X,W} \otimes \mathrm{id}_{Y\otimes Z})(\mathrm{id}_X \otimes \sigma_{Y,W} \otimes \mathrm{id}_Z)(\mathrm{id}_{X\otimes Y} \otimes \sigma_{Z,W})$$

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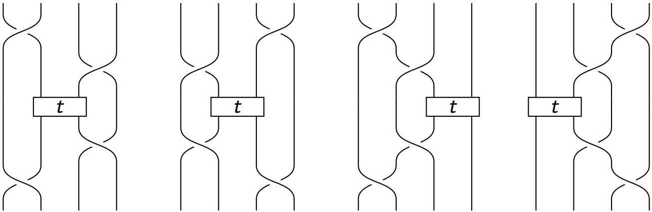


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$$t^{14,\mathrm{II}}_{X,Y,Z,W} = (\mathrm{id}_{X\otimes Y} \otimes \sigma^{-1}_{Z,W})(\sigma^{-1}_{X,Y} \otimes \mathrm{id}_{W\otimes Z})(\mathrm{id}_Y \otimes t_{X,W} \otimes \mathrm{id}_Z)(\sigma_{X,Y} \otimes \mathrm{id}_{W\otimes Z})(\mathrm{id}_{X\otimes Y} \otimes \sigma_{Z,W})$$

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$$t^{14,\mathrm{IV}}_{X,Y,Z,W} = (\mathrm{id}_{X\otimes Y} \otimes \sigma^{-1}_{Z,W})(\mathrm{id}_X \otimes \sigma^{-1}_{Y,W} \otimes \mathrm{id}_Z)(t_{X,W} \otimes \mathrm{id}_{Y\otimes Z})(\mathrm{id}_X \otimes \sigma_{Y,W} \otimes \mathrm{id}_Z)(\mathrm{id}_{X\otimes Y} \otimes \sigma_{Z,W})$$

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## Proposition

*Let $(\mathcal{C}, \otimes, I, a, \sigma, t)$ be a pre-Cartier category. For any objects $X, Y, Z, W$ in $\mathcal{C}$, consider the following endomorphims of $X \otimes Y \otimes Z$*

$$t^{12}_{X,Y,Z} := t_{X,Y} \otimes \mathrm{id}_Z$$
$$t^{23}_{X,Y,Z} := \mathrm{id}_X \otimes t_{Y,Z}$$
$$t^{13}_{X,Y,Z} := (\mathrm{id}_X \otimes \sigma^{-1}_{Y,Z}) \circ (t_{X,Z} \otimes \mathrm{id}_Y) \circ (\mathrm{id}_X \otimes \sigma_{Y,Z})$$

*and the following endomorphisms of $X \otimes Y \otimes Z \otimes W$*

$$t^{12}_{X,Y,Z,W} := t^{12}_{X,Y,Z} \otimes \mathrm{id}_W = t_{X,Y} \otimes \mathrm{id}_{Z \otimes W}$$
$$t^{23}_{X,Y,Z,W} := t^{23}_{X,Y,Z} \otimes \mathrm{id}_W = \mathrm{id}_X \otimes t_{Y,Z} \otimes \mathrm{id}_W$$
$$t^{13}_{X,Y,Z,W} := t^{13}_{X,Y,Z} \otimes \mathrm{id}_W = \left((\mathrm{id}_X \otimes \sigma^{-1}_{Y,Z}) \circ (t_{X,Z} \otimes \mathrm{id}_Y) \circ (\mathrm{id}_X \otimes \sigma_{Y,Z})\right) \otimes \mathrm{id}_W$$
$$t^{14}_{X,Y,Z,W} := (\mathrm{id}_X \otimes \sigma^{-1}_{Y \otimes Z,W}) \circ (t_{X,W} \otimes \mathrm{id}_{Y \otimes Z}) \circ (\mathrm{id}_X \otimes \sigma_{Y \otimes Z,W})$$
$$t^{24}_{X,Y,Z,W} := \mathrm{id}_X \otimes t^{13}_{Y,Z,W} = \mathrm{id}_X \otimes \left((\mathrm{id}_Y \otimes \sigma^{-1}_{Z,W}) \circ (t_{Y,W} \otimes \mathrm{id}_Z) \circ (\mathrm{id}_Y \otimes \sigma_{Z,W})\right)$$
$$t^{34}_{X,Y,Z,W} := \mathrm{id}_X \otimes t^{23}_{Y,Z,W} = \mathrm{id}_{X \otimes Y} \otimes t_{Z,W}$$

*Then the morphisms $t^{ij}_{X,Y,Z}$ and $t^{ij}_{X,Y,Z,W}$* **satisfy the infinitesimal braid relations**.

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$$
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\end{aligned}
$$

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$$
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t^{12}_{X,Y,Z,W} &:= t^{12}_{X,Y,Z} \otimes \mathrm{id}_W = t_{X,Y} \otimes \mathrm{id}_{Z \otimes W} \\
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t^{13}_{X,Y,Z,W} &:= t^{13}_{X,Y,Z} \otimes \mathrm{id}_W = \Big( (\mathrm{id}_X \otimes \sigma^{-1}_{Y,Z}) \circ (t_{X,Z} \otimes \mathrm{id}_Y) \circ (\mathrm{id}_X \otimes \sigma_{Y,Z}) \Big) \otimes \mathrm{id}_W \\
t^{14}_{X,Y,Z,W} &:= (\mathrm{id}_X \otimes \sigma^{-1}_{Y \otimes Z, W}) \circ (t_{X,W} \otimes \mathrm{id}_{Y \otimes Z}) \circ (\mathrm{id}_X \otimes \sigma_{Y \otimes Z, W}) \\
t^{24}_{X,Y,Z,W} &:= \mathrm{id}_X \otimes t^{13}_{Y,Z,W} = \mathrm{id}_X \otimes \Big( (\mathrm{id}_Y \otimes \sigma^{-1}_{Z,W}) \circ (t_{Y,W} \otimes \mathrm{id}_Z) \circ (\mathrm{id}_Y \otimes \sigma_{Z,W}) \Big) \\
t^{34}_{X,Y,Z,W} &:= \mathrm{id}_X \otimes t^{23}_{Y,Z,W} = \mathrm{id}_{X \otimes Y} \otimes t_{Z,W}
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**Theorem (Esposito-Rivezzi-Schnitzer-TW '25)**

$(\mathcal{C}, \otimes, \mathbb{k}, a, \ell, r, \sigma, t)$ *$\mathbb{k}$-linear Cartier category, $\Psi$ Drinfeld associator.*

*Then $\mathcal{C}_{\Psi,t} = (\mathcal{C}_{\Psi,t}, \tilde{\otimes}, \mathbb{k}, a^{\Psi}, \tilde{\ell}, \tilde{r}, \sigma^{\Psi})$ is a braided monoidal category, where*

$$
\begin{aligned}
a^{\Psi}_{X,Y,Z} &= \tilde{a}_{X,Y,Z} \circ \Psi\big(\hbar t_{X,Y} \otimes \mathrm{id}_Z, \hbar a^{-1}_{X,Y,Z} \circ (\mathrm{id}_X \otimes t_{Y,Z}) \circ a_{X,Y,Z}\big) \\
\sigma^{\Psi}_{X,Y} &= \tilde{\sigma}_{X,Y} \circ e^{\frac{\hbar}{2} t_{X,Y}}
\end{aligned}
\tag{2}
$$

*are the deformed associativity constraint and braiding, respectively.*

**Theorem (Esposito-Rivezzi-Schnitzer-TW '25)**

*Let $(\mathcal{C}, \sigma, t)$ be a $\mathbb{k}$-linear pre-Cartier category satisfying*

$$[t^{12}_{X,Y,Z}, t^{23}_{X,Y,Z}] = 0 \tag{3}$$

*for any objects $X, Y, Z$. Then $(\mathcal{C}, \mathbb{k}, \tilde{\otimes}, \tilde{a}, \tilde{\ell}, \tilde{r}, \hat{\sigma})$ is a braided monoidal category, where $\hat{\sigma}_{X,Y} = \tilde{\sigma}_{X,Y} \circ e^{\frac{\hbar}{2} t_{X,Y}}$.*

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$(H, \Delta, \varepsilon, \Phi, \ell, r, \mathcal{R}, \chi)$ pre-Cartier quasi-bialgebra.

**Lemma**

$$\Phi^{-1}\mathcal{R}_{23}^{-1}\Phi_{132}\chi_{13}\Phi_{132}^{-1}\mathcal{R}_{23}\Phi = \mathcal{R}_{12}^{-1}\Phi_{213}^{-1}\chi_{13}\Phi_{213}\mathcal{R}_{12} =: \overline{\chi_{13}}$$

**Theorem (Esposito-Rivezzi-Schnitzer-TW '25)**

$$\Theta^{12} := \chi_{12} \qquad \Upsilon^{12} := \Phi\chi_{12}\Phi^{-1}$$
$$\Theta^{23} := \Phi^{-1}\chi_{23}\Phi \qquad \Upsilon^{23} := \chi_{23}$$
$$\Theta^{13} := \overline{\chi_{13}} \qquad \Upsilon^{13} := \Phi\overline{\chi_{13}}\Phi^{-1}$$

*Then* $\{\Theta^{12}, \Theta^{23}, \Theta^{13}\}$ *and* $\{\Upsilon^{12}, \Upsilon^{23}, \Upsilon^{13}\}$ *satisfy the infinitesimal braid relations.*

**Theorem (Esposito-Rivezzi-Schnitzer-TW '25)**

*The* **quantum Yang-Baxter equation**

$$\mathcal{R}_{12}\Phi_{231}\mathcal{R}_{13}\Phi_{132}^{-1}\mathcal{R}_{23}\Phi = \Phi_{321}\mathcal{R}_{23}\Phi_{312}^{-1}\mathcal{R}_{13}\Phi_{213}\mathcal{R}_{12}$$

*and the* **infinitesimal quantum Yang-Baxter equation** *hold*

$$\mathcal{R}_{12}\chi_{12}\Phi_{231}\mathcal{R}_{13}\Phi_{132}^{-1}\mathcal{R}_{23}\Phi + \mathcal{R}_{12}\Phi_{231}\mathcal{R}_{13}\chi_{13}\Phi_{132}^{-1}\mathcal{R}_{23}\Phi + \mathcal{R}_{12}\Phi_{231}\mathcal{R}_{13}\Phi_{132}^{-1}\mathcal{R}_{23}\chi_{23}\Phi$$
$$= \Phi_{321}\mathcal{R}_{23}\chi_{23}\Phi_{312}^{-1}\mathcal{R}_{13}\Phi_{213}\mathcal{R}_{12} + \Phi_{321}\mathcal{R}_{23}\Phi_{312}^{-1}\mathcal{R}_{13}\chi_{13}\Phi_{213}\mathcal{R}_{12} + \Phi_{321}\mathcal{R}_{23}\Phi_{312}^{-1}\mathcal{R}_{13}\Phi_{213}\mathcal{R}_{12}\chi_{12}$$

## Theorem (Esposito-Rivezzi-Schnitzer-TW '25)

*Let $(H, \Phi, \mathcal{R}, \chi)$ be a Cartier quasi-bialgebra and let $\Psi$ be a Drinfeld associator. Then there is a topological quasitriangular quasi-bialgebra structure on $\tilde{H} = H[[\hbar]]$ such that*

$$\tilde{\mathcal{R}} = \mathcal{R}\big(1 \otimes 1 + \hbar\chi + \mathcal{O}(\hbar^2)\big).$$

*Explicitly, its re-associator and universal $\mathcal{R}$-matrix are*

$$\tilde{\Phi} = \Psi(\hbar\chi_{12}, \hbar\Phi^{-1}\chi_{23}\Phi) \qquad \text{and} \qquad \tilde{\mathcal{R}} = \mathcal{R}e^{\frac{\hbar}{2}\chi}.$$

## Theorem (Esposito-Rivezzi-Schnitzer-TW '25)

*For a pre-Cartier bialgebra $(H, \mathcal{R}, \chi)$ satisfying one of the equivalent conditions*

$$[\chi_{12}, \mathcal{R}_{12}^{-1}\chi_{13}\mathcal{R}_{12}] = 0$$
$$[\chi_{23}, \mathcal{R}_{23}^{-1}\chi_{13}\mathcal{R}_{23}] = 0$$
$$[\chi_{12}, \chi_{23}] = 0$$

*there is a quasitriangular structure*

$$\tilde{\mathcal{R}} := \mathcal{R}\exp(\hbar\chi)$$

*on the trivial topological bialgebra $\tilde{H} := H[[\hbar]]$.*

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*there is a quasitriangular structure*

$$\tilde{\mathcal{R}} := \mathcal{R}\exp(\hbar\chi)$$

*on the trivial topological bialgebra $\tilde{H} := H[[\hbar]]$.*

$$\begin{array}{ccc} (E(n),\mathcal{R},\chi) & \xrightarrow{\text{gauge twist}} & (E(n)_{\mathcal{F}},\mathcal{R}_{\mathcal{F}},\chi_{\mathcal{F}}) \\ \text{deform}\downarrow & & \downarrow\text{deform} \\ (\widetilde{E(n)},\tilde{\mathcal{R}}) & \xrightarrow{\text{gauge twist}} & (\widetilde{E(n)_{\mathcal{F}}},\tilde{\mathcal{R}_{\mathcal{F}}}) \end{array}$$

$\rightsquigarrow$ we obtain a deformation of the pre-Cartier bialgebras $(E(n),\mathcal{R},\chi)$, namely,

$$\tilde{\mathcal{R}}=\tau(\hat{\sigma}_{H,H}(1\otimes 1))=\frac{1}{2}\big(1\otimes 1+1\otimes g+g\otimes 1-g\otimes g\big)e^{a_{ij}gx_i\otimes x_j+\frac{\hbar}{2}b_{k\ell}gx_k\otimes x_\ell}$$

is a quasitriangular structure for the topological bialgebra $\widetilde{E(n)}=E(n)[[\hbar]]$ and

$$\tilde{\mathcal{R}}_{\mathcal{F}}=\tau(\hat{\sigma}^{\mathcal{F}}_{H,H}(1\otimes 1))=-\frac{1}{2}\big(1\otimes 1-1\otimes g-g\otimes 1-g\otimes g\big)e^{-\sum_{i,j=1}^n a_{ij}gx_i\otimes x_j-\frac{\hbar}{2}\sum_{k,\ell=1}^n b_{k\ell}gx_k\otimes x_\ell}$$

is an $\mathcal{R}$-matrix for the topological quasi-bialgebra $\widetilde{E(n)_{\mathcal{F}}}=(E(n)[[\hbar]],\tilde{\Delta}_{\mathcal{F}},\varepsilon)$ with re-associator $1\otimes 1\otimes g$.

The previous two $\mathcal{R}$-matrices are by construction related via the gauge transformation $\mathcal{F}=1\otimes g$.

$$\begin{array}{ccc}
(E(n),\mathcal{R},\chi) & \xrightarrow{\text{gauge twist}} & (E(n)_\mathcal{F},\mathcal{R}_\mathcal{F},\chi_\mathcal{F}) \\
{\scriptstyle\text{deform}}\big\downarrow & & \big\downarrow{\scriptstyle\text{deform}} \\
(\widetilde{E(n)},\tilde{\mathcal{R}}) & \xrightarrow{\text{gauge twist}} & (\widetilde{E(n)_\mathcal{F}},\tilde{\mathcal{R}_\mathcal{F}})
\end{array}$$

$\rightsquigarrow$ we obtain a deformation of the pre-Cartier bialgebras $(E(n),\mathcal{R},\chi)$, namely,

$$\tilde{\mathcal{R}}=\tau(\hat{\sigma}_{H,H}(1\otimes 1))=\frac{1}{2}\big(1\otimes 1+1\otimes g+g\otimes 1-g\otimes g\big)e^{a_{ij}gx_i\otimes x_j+\frac{\hbar}{2}b_{k\ell}gx_k\otimes x_\ell}$$

is a quasitriangular structure for the topological bialgebra $\widetilde{E(n)}=E(n)[[\hbar]]$ and

$$\tilde{\mathcal{R}}_\mathcal{F}=\tau(\hat{\sigma}^\mathcal{F}_{H,H}(1\otimes 1))=-\frac{1}{2}\big(1\otimes 1-1\otimes g-g\otimes 1-g\otimes g\big)e^{-\sum_{i,j=1}^n a_{ij}gx_i\otimes x_j-\frac{\hbar}{2}\sum_{k,\ell=1}^n b_{k\ell}gx_k\otimes x_\ell}$$

is an $\mathcal{R}$-matrix for the topological quasi-bialgebra $\widetilde{E(n)_\mathcal{F}}=(E(n)[[\hbar]],\tilde{\Delta}_\mathcal{F},\varepsilon)$ with re-associator $1\otimes 1\otimes g$.

The previous two $\mathcal{R}$-matrices are by construction related via the gauge transformation $\mathcal{F}=1\otimes g$.

$$\begin{array}{ccc} (E(n),\mathcal{R},\chi) & \xrightarrow{\text{gauge twist}} & (E(n)_{\mathcal{F}},\mathcal{R}_{\mathcal{F}},\chi_{\mathcal{F}}) \\ \text{deform}\downarrow & & \downarrow\text{deform} \\ (\widetilde{E(n)},\tilde{\mathcal{R}}) & \xrightarrow{\text{gauge twist}} & (\widetilde{E(n)}_{\mathcal{F}},\tilde{\mathcal{R}_{\mathcal{F}}}) \end{array}$$

$\rightsquigarrow$ we obtain a deformation of the pre-Cartier bialgebras $(E(n),\mathcal{R},\chi)$, namely,

$$\tilde{\mathcal{R}} = \tau(\hat{\sigma}_{H,H}(1\otimes 1)) = \frac{1}{2}\big(1\otimes 1 + 1\otimes g + g\otimes 1 - g\otimes g\big)e^{a_{ij}gx_i\otimes x_j + \frac{\hbar}{2}b_{k\ell}gx_k\otimes x_\ell}$$

is a quasitriangular structure for the topological bialgebra $\widetilde{E(n)} = E(n)[[\hbar]]$ and

$$\tilde{\mathcal{R}}_{\mathcal{F}} = \tau(\hat{\sigma}^{\mathcal{F}}_{H,H}(1\otimes 1)) = -\frac{1}{2}\big(1\otimes 1 - 1\otimes g - g\otimes 1 - g\otimes g\big)e^{-\sum_{i,j=1}^{n} a_{ij}gx_i\otimes x_j - \frac{\hbar}{2}\sum_{k,\ell=1}^{n} b_{k\ell}gx_k\otimes x_\ell}$$

is an $\mathcal{R}$-matrix for the topological quasi-bialgebra $\widetilde{E(n)}_{\mathcal{F}} = (E(n)[[\hbar]],\tilde{\Delta}_{\mathcal{F}},\varepsilon)$ with re-associator $1\otimes 1\otimes g$.

The previous two $\mathcal{R}$-matrices are by construction related via the gauge transformation $\mathcal{F} = 1\otimes g$.

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Thank you for your attention!

### Definition

Let $(\mathcal{C}, \otimes, \Bbbk, a, \ell, r, \sigma)$ be a $\Bbbk$-linear braided monoidal category. A **deformation** of $\mathcal{C}$ is a $\Bbbk[[\hbar]]$-linear braided monoidal category $(\hat{\mathcal{C}}, \hat{\otimes}, \Bbbk, \hat{a}, \hat{\ell}, \hat{r}, \hat{\sigma})$ such that $\mathrm{Ob}(\hat{\mathcal{C}}) = \mathrm{Ob}(\mathcal{C})$, $\mathrm{Hom}_{\hat{\mathcal{C}}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(X, Y)[[\hbar]]$, the tensor product $\hat{\otimes}$ coincides with $\otimes$ on objects and it is the $\hbar$-adic completion of $\otimes$ on morphisms, and

$$\begin{aligned} \hat{a} \quad \mathrm{mod} \quad \hbar &= a \\ \hat{\ell} \quad \mathrm{mod} \quad \hbar &= \ell \\ \hat{r} \quad \mathrm{mod} \quad \hbar &= r \\ \hat{\sigma} \quad \mathrm{mod} \quad \hbar &= \sigma. \end{aligned}$$

In particular, the new concatenation rule of morphisms is given by

$$\Big(\sum_{k=0}^{\infty} \phi_k \hbar^k\Big) \circ \Big(\sum_{k=0}^{\infty} \psi_k \hbar^k\Big) := \sum_{k=0}^{\infty} \Big(\sum_{i=0}^{k} \phi_i \circ \psi_{k-i}\Big) \hbar^k.$$

### Example

The trivial deformation of a $\Bbbk$-linear monoidal category $\mathcal{C}$ is the one obtained by $\hbar$-linearly extending the isomorphisms $a, \ell, r, \sigma$. We shall denote it by $\tilde{\mathcal{C}}$.