Noncommutative Geometry: Quo vadis?

A Hodgepodge of Ideas, Questions and Projects
No Answers

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Extensions of Standard Model Spectral Triple

The following extensions address different short commings of the "classical" almost-commutative Standard Model (list probably not exhaustive...).

- New scalars, fermions, interactions
  Chamseddine, Connes, C.S., Jureit,...
- Pati-Salam extension
  Chamseddine, Connes, van Suijlekom,...
- Grand symmetry extension
  Lizzi, Devastato, Martinetti,...
- Supersymmetry
  van den Broek, van Suijlekom
- Torsion
  Hanisch, Pfäffle, C.S., Iochum, Levy, Vassilevich, Dabrovski, Sitarz,...
- Twisted spectral triples
  Connes, Moscovici, Lizzi, Martinetti, Landi, Devastato, Filaci,...
- B-L extensions
  Besnard, Brouder
- non-associative & Jordan algebras
  Wulkenhaar, Boyle, Farnsworth

BSM Spectral Triples

Spektral triple of Standard Model + Spectral Action is allready a model beyond ΛCDM!

$$\mathcal{S}_{Bos}(D_{SM}) \sim \mathcal{S}_{\Lambda} + \mathcal{S}_{E-H} + \mathcal{S}_{YM} + \mathcal{S}_{Weyl^2} + \mathcal{S}_{conf.} + \mathcal{S}_{top.} + \dots$$

BSM models address many observations, challenges or desiderata from particle physics:

- SM particle content (various classification approaches)
- Higgs potential (sign of quadratic term)
- Higgs mass (with extra scalar)
- top quark mass (at least consistency)
- Seesaw mechanism (individual neutrino masses not yet measured)
- dark matter candidates (no direct observation yet)
- ...

New Observations & Experiments from Particle Physics

- basically none...

New Observations & Experiments from Particle Physics

- basically none...
  At least no deviations from the Standard Model that are widely accepted as significant.

Observations & Anomalies from Cosmology & Astrophysics

Known anomalies at odds with ΛCDM model (without dark matter):

- low-ℓ CMB
  The largest angular scales of the CMB show unusual alignments and asymmetries.
- H_0 tension
  Local distance-ladder measurements give a higher expansion rate than early-universe inferences from the CMB+BAO
- S_8 tension
  Weak-lensing surveys find less matter clustering than Planck data predicts.
- ^7Li abundance
  Standard Big Bang Nucleosynthesis overpredicts the primordial ^7Li abundance compared to what is observed in old stars.
- Baryon asymmetry
  The observed excess of matter over antimatter cannot be explained by Standard Model CP violation alone.

What can NCG contribute in Cosmology & Astrophysics?

- (Dark matter)
  Difficult...no "positive" observation. If dark matter is particle-like, it can be almost anything.
- low-ℓ CMB
  Has been partially addressed by Marcolli, Ball, Guicardi,...with Swiss cheese type cosmologies, fractal structures, etc.
- Baryon asymmetry
  Perhaps within B-L model?
- Other anomalies (H_0, S_8, ^7Li)
  Perhaps torsion, Weyl-term or conformal scalar coupling can help?

Bonus: Many ongoing and new experiments, for example Gaia, LIGO, and others.

Lorentzian Spectral Triples

Possible approaches:

- Axiomatic Lorentzian NCG
  Eckstein, Franco, Besnard, Bizi, Majid...
- Twisted Spectral Triples
  Devastato, Farnsworth, Lizzi, Martinetti,...

Questions:

- Is Lorentzian signature related to finite spectral triples?
- Could there be a connection to Lorentzian length spaces?
  Kunzinger, Sämann,...

My personal mantra: Thou shallt not Wick rotate!

Spectral Action

Musings...

- Can we get an interpretation or better understanding from a statistical or thermodynamic interpretation?
  - Interpretation as an entropy (of what?) using KMS states?
    Chamseddine, Connes, van Suijlekom,...
  - Second quantisation of fermions?
    Dong, Khalkhali, van Suijlekom,...
  - Zeta function regularisation?
    Iochum, Levy, Vassilevich,...
  - Interpretation in terms of the Dirac sea?
- How can we define a Lorentzian "spectral" action?
  - Twisted spectral triples and Fermionic actions?
    Martinetti, Singh,...
  - Bosonic action via Hadamard coefficients?
    Dang, Wrochna,Ronge,...
- Have we defined the "correct" spectral action

Flows & PSC Spectral Triples

Doing geometry on spectral triples.

- Ricci flow, Yamabe flow, mean curvature flow,...
  Can we define and study geometric flows on spectral triples?
  Are there spectral triples that are flexible enough to accommodate flows?
  Bhuyain, Marcolli, Duvenhage, Vacaru, Floricel, Dabrovski, Sitarz, Zalecki,...
- Positive scalar curvature (PSC) spectral triples
  Can one find rigidity or classification results for PSC spectral triples?
  Piazza, Schick,...
- Lorentzian spectral triples and Lorentzian length spaces
  Kunzinger, Sämann,...

Overview of Proof Assistants

Proof assistants have been developed in computer science to check validity of algorithms and proofs.

What is a Proof Assistant?

- Interactive theorem prover
- Assists in developing formal proofs
- Checks correctness of logical steps
- Used in mathematics, computer science, formal verification
- Combines automated and interactive techniques

Examples of Popular Proof Assistants

- **Lean** - Calculus of Constructions (type theory), developed by Microsoft Research, user-friendly, powerful features
- **Rocq (formerly known as Coq)** - Calculus of Constructions (type theory), developed by INRIA, formalizes mathematical proofs, verifies software
- **Isabelle** - Higher-order logic, developed by Tobias Nipkow and Lawrence Paulson, supports various logics, formal verification
- **HOL Light** - Higher-order logic, developed by Mike Gordon at Cambridge University, verifies mathematical proofs, hardware/software

Importance in Mathematics

- **Verification of complex proofs**
  - Four-color theorem (1976, verified 2005 in Rocq by Gonthier)
  - Liquid Tensor Experiment by Scholze (Lean)
- **Formalizing mathematics** (focus on Lean 4)
  - Perfectoid spaces by Scholze, Buzzard, Commelin, Massot
  - Cap-set problem by Dahmen, Hölzl, Lewis
  - Continuum hypothesis by Han, van Doorn
  - PFR conjecture by Tao, Dillies, Mehta
  - Fermat's Last Theorem by Buzzard, Taylor
- **Reducing human error in proofs**
- **Facilitating collaboration**

Lean 4

Technical features of Lean 4:

- Lean 4 is a functional programming language and interactive proof assistant.
- Lean 4 is strongly typed (every variable has a fixed type, think of integers)
- Lean 4 is based on the **Calculus of Constructions** (dependent type theory)
- Fully extensible: users can modify the parser, elaborator, tactics, decision procedures, and more.

Practical features of Lean 4:

- Easily used in VS Code (via extension).
- Has a large library of formalized mathematics (mathlib).
- Has young and active community.
- Lean 4 is open source.

Lean 4 has been developed primarily by Leonardo de Moura at Microsoft Research.

Dependent Type Theory I

"Standard" foundation of mathematics:
Set theory (ZFC) and first-order logic (predicate logic).

Curry-Howard-Lambek correspondence: Mathematical statements and proofs can be equivalently formulated in:

- Set theory + first-order logic,
- (Dependent) type theory, or
- Category theory.

Important for Lean 4:
Curry-Howard correspondence ensures
set theory + first-order logic $\Leftrightarrow$ dependent type theory

Dependent Type Theory II

What is type theory?

1. There are terms and types.
2. Everything is a term. Notation: $a, A, \mathcal{U}, \mathbb{N}, \mathsf{Prop}$, etc.
3. Terms can be types. Notation: $A, B, \mathcal{U}, \mathbb{N}, \mathsf{Prop}$, etc.
4. Every term has a type. Notation $a : A$ (a has type A).
5. Rules how to construct new types from given types.

Some basics:

- If there is a term a of type A, i.e. $a : A$, we say that A is inhabited.
- A type inhabited by types is called a universe.
  Notation: $\mathcal{U}, \mathcal{U}_k, \mathsf{Prop}$, etc.
- $\mathbf{0}/\mathbf{1}$ denote the generic type with no/one element.
  Note: there are more types with no/one element

Propositions and Proofs in Type Theory

Propositions as types

- Prop is the universe of all propositions (true or false).
- Propositions are types $P : \text{Prop}$ with no or one representative.
  Example: $P = \text{Fermat's last theorem}$
- $P : \text{Prop}$ is true if it is inhabited, otherwise it's false.
- Proofs are terms $\text{proof} : P$ but not types.
  Example: $P = \text{Fermat's last theorem}$ is inhabited, i.e. there exists a proof.

Overview of Dependent Type Theory

Types	Logic	Sets
A	proposition	set
$a : A$	proof	element
$B(x)$	predicate	family of sets
$b(x) : B(x)$	conditional proof	family of elements
$\mathbf{0}, \mathbf{1}$	$\bot, \top$	$\emptyset, \{\emptyset\}$
$A + B$	$A \vee B$	disjoint union
$A \times B$	$A \wedge B$	set of pairs
$A \to B$	$A \Rightarrow B$	set of functions
$\sum_{(x:A)} B(x)$	$\exists_{x:A} B(x)$	disjoint sum
$\prod_{(x:A)} B(x)$	$\forall_{x:A} B(x)$	product
Id_A	equality $=$	$\{ (x, x) \mid x \in A \}$

Figure: Overview from the HoTT Book

Connes' Reconstruction Theorem

Why formalising Noncommutative Geometry in Lean 4?

- Algebraic fields of mathematics are easier to formalise.
- NCG may provide back door to formalising Differential Geometry.
- Because I like NCG and want to formalise it.

The Aim: Reconstruction Theorem

Let (A, H, D) be a p-dimensional commutative spectral triple. There exist a compact oriented Riemannian p-manifold X, a Hermitian vector bundle $E \to X$, and an essentially self-adjoint Dirac-type operator D_E on E, such that

$$(A, H, D) \cong (C^{\infty}(X), L^2(X, E), D_E).$$

Where we stand: Functional Analysis & C^*-algebras

Definition/Theorem	In mathlib
Hilbert space and bounded/self-adjoint/compact operators	Yes
Compact operator and compact resolvent	Yes
*-algebra and *-representation on Hilbert space	Partial
GNS construction from a state on a C*-algebra	No
C*-algebra and C*-norm identity	Yes
Gelfand–Naimark theorem for C*-algebras	Partial
Gelfand duality for commutative C*-algebras	Yes
Compact operators $K(H)$	Yes
Fredholm operators and Fredholm modules	Partial
Functional calculus for self-adjoint operators	Partial

Where we stand: Smooth Manifolds & Vector Bundles

Definition/Theorem	**In mathlib**
Smooth manifold (Hausdorff, second-countable, C^∞-atlas)	Yes
Algebra $C^\infty(X)$ of smooth functions	Yes
Complex vector bundle and smooth sections	Partial
Serre–Swan theorem	No
Riemannian metric on a manifold	Partial
Orientation and oriented manifold	No
Spin and spin^c structures	No
Spinor bundle from spin^c structure	No
Dirac-type operator on Hermitian bundle	No
Spectral theorem for self-adjoint elliptic operators	Partial

Where we stand: Spectral Triples

Definition/Theorem	**In mathlib**
Spectral triple (A, H, D) with bounded commutators	No
Summability: $\mathrm{Tr}(\|D\|^{-s}) < \infty$ for $s > p$	No
Dimension spectrum of a spectral triple	No
Regularity: commutators with $\|D\|$ densely defined	No
Smooth domain of D	No
$H_\infty = \bigcap \mathrm{Dom}(D^k)$	No
Finiteness: H_∞ is finitely generated projective A-module	No
Orientability: Hochschild p-cycle with $\pi_D(c) = \gamma$	No
Hochschild complex and Hochschild homology	No
Grading γ with $\gamma D = -D\gamma$	No
Real structure J on H	No
First-order condition $[[D, a], Jb^* J^{-1}] = 0$	No

Formalising Connes' Reconstruction Theorem in Lean 4

Github Repository
(very much under construction)

Current team in Potsdam:

- Florian Hanisch
- Lennart Ronge
- C.S.
- Jonathon Taylor
- Dominik Ulrich

This is a collaborative project, new members are welcome!