Solutions to the quantum YB equation and related deformations

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a few recent papers with

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Background

A correspondence (a duality) between

spaces and algebras of functions on these spaces

The idea of noncommutative geometry:

forget the commutativity of the algebras of functions and replace them by appropriate classes of noncommutative associative algebras. These are considered as algebras of functions on some (virtual) noncommutative spaces

For instance, the natural algebras of functions on finite-dimensional vector spaces are the algebras of polynomial functions generated by the coordinates; in these polynomial algebras the coordinates commute.

Given a class of noncommutative associative algebras generalizing the polynomial algebras, one may consider regular algebras generated by coordinates in which they satisfy relations other than the commutation between them and defining thereby, by duality, noncommutative vector spaces.

In the following,

a noncommutative $\mathbb{R}^N$ correspond,

by duality,

to a regular complex $*$-algebra generated by

hermitian elements x^k, $k \in \{1, \ldots, N\}$.

Abstract

- noncommutative generalizations of $\mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$

- noncommutative generalizations of $\mathbb{S}^{N_1+N_2-1}$

- noncommutative generalizations of $\mathbb{S}^{N_1-1} \times \mathbb{S}^{N_2-1}$

- a quaternionic noncommutative torus $\mathbb{S}^3 \times_{\mathbf{u}} \mathbb{S}^3$, $\mathbf{u} \in \mathbb{S}^2 = \mathrm{SU}(2)/\mathrm{U}(1)$

- spherical manifolds : volume forms from top Chern–Connes characters

- generalized Clifford algebras $C\ell(\mathcal{A}_R)$

- noncommutative principal bundles

everything algebraic

The θ-deformation symmetry : $\mathbb{T}^2 = U(1) \times U(1)$

$\mathbb{C}^2_\theta$ = Noncommutative space dual to the

$*$-algebra $\mathcal{A}_\theta$ generated by normal elements z_1, z_2 with relations

$$z_1 z_2 = e^{i\theta} z_2 z_1, \qquad z_1 z_2^* = e^{-i\theta} z_2^* z_1 \tag{1}$$

$\Rightarrow$ center generated by $z_1 z_1^* = \|z_1\|^2$ and $z_2 z_2^* = \|z_2\|^2$

2-torus and 3-sphere

$$\mathbb{C} \times_\theta \mathbb{C}/(\|z_1\|^2 - 1, \|z_2\|^2 - 1) = \mathbb{T}^2_\theta \qquad \|z_1\|^2 = 1, \quad \|z_2\|^2 = 1$$

$$\mathbb{C} \times_\theta \mathbb{C}/(\|z_1\|^2 + \|z_2\|^2 - 1) = \mathbb{S}^3_\theta \qquad \|z_1\|^2 + \|z_2\|^2 = 1$$

nice examples of singular spaces

associated with noncommutative tori

and an action of the commutative torus $\mathbb{T}^2$

Hermitian generators: real version $\mathbb{C}^2_\theta = (\mathbb{R}^2)^2_\theta = \mathbb{R}^2 \times_\theta \mathbb{R}^2$

$$z_1 = x_1^1 + \mathrm{i} x_1^2, \qquad z_2 = x_2^1 + \mathrm{i} x_2^2, \qquad (x_k^\lambda)^* = x_k^\lambda$$

and relation (1) + normality of $z_k \Longleftrightarrow$ (2)

$$\begin{cases} x_1^\lambda x_1^\mu = x_1^\mu x_1^\lambda \quad ; \qquad x_2^\lambda x_2^\mu = x_2^\mu x_2^\lambda \\ x_1^\lambda x_2^\mu = R^{\lambda\mu}_{\nu\rho} x_2^\nu x_1^\rho \end{cases} \tag{2}$$

with

$$R^{\lambda\mu}_{\nu\rho} = \cos(\theta)\, \delta^\lambda_\rho \delta^\mu_\nu + \mathrm{i} \sin(\theta)\, C^\lambda_\rho D^\mu_\nu \qquad C = -D = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

The quadratic $*$-algebra $\mathcal{A}_R$

The $*$-algebra $\mathcal{A}_R$ is generated by two sets of hermitian elements x_1^λ with $\lambda \in \{1, \dots, N_1\}$ and x_2^α with $\alpha \in \{1, \dots, N_2\}$ with relations

$$\begin{cases} x_1^\lambda x_1^\mu = x_1^\mu x_1^\lambda \quad ; \qquad x_2^\alpha x_2^\beta = x_2^\beta x_2^\alpha \\ x_1^\lambda x_2^\alpha = R^{\lambda\alpha}_{\beta\mu} x_2^\beta x_1^\mu \end{cases} \tag{3}$$

for suitable $R^{\lambda\alpha}_{\beta\mu} \in \mathbb{C}$. In view of the hermiticity of the x_1^λ, x_2^α we impose

$$\overline{R}^{\lambda\alpha}_{\beta\mu} R^{\mu\beta}_{\gamma\nu} = \delta^\lambda_\nu \delta^\alpha_\gamma \tag{4}$$

Thus $\mathcal{A}_R$ is a graded quadratic algebra $\mathcal{A}_R = \oplus_n \mathcal{A}_R^n$

which is connected: $\mathcal{A}_R^0 = \mathbb{C}\mathbb{1}$;

the elements x_1^λ, x_2^α form a basis of $\mathcal{A}_R^1$;

by the requirement (4):
the elements $x_1^\lambda x_1^\mu$ with $\lambda \leq \mu$, $x_2^\alpha x_2^\beta$ with $\alpha \leq \beta$ and $x_1^\lambda x_2^\alpha$ form a basis of $\mathcal{A}_R^2$.

The Koszul dual $\mathcal{A}^!_R$

A (homogeneous) quadratic algebra is an algebra $\mathcal{A}$ of the form

$$\mathcal{A} = T(E)/(\mathcal{R})$$

E is a finite-dim vector space and $(\mathcal{R})$ is the ideal generated by $\mathcal{R} \subset E \otimes E$.

The algebra $\mathcal{A} = A(E, R)$ is a connected (i.e. $\mathcal{A}_0 = \mathbb{C}\mathbb{1}$) graded algebra $\mathcal{A} = \bigoplus_{n\in\mathbb{N}} \mathcal{A}_n$ generated by the degree 1 part, $\mathcal{A}_1 = E$.

The Koszul dual $\mathcal{A}^!$ of $\mathcal{A}$ is the quadratic algebra

$$\mathcal{A}^! = T(E^*)/(\mathcal{R}^\perp)$$

where $\mathcal{R}^\perp \subset E^* \otimes E^*$ is the orthogonal of $\mathcal{R}$: $R^\perp = \{\omega \in E^* \otimes E^* \, ; \, \langle \omega, r \rangle = 0\}$.

Our $\mathcal{A}^!_R$ is generated by the dual bases $\theta^1_\lambda, \theta^2_\alpha$ of the x^λ_1, x^α_2 with relations

$$\begin{cases} \theta^1_\lambda \theta^2_\mu = -\theta^2_\mu \theta^1_\lambda \quad ; \qquad \theta^2_\alpha \theta^2_\beta = -\theta^2_\beta \theta^2_\alpha \\ \theta^2_\beta \theta^1_\mu = -R^{\lambda\alpha}_{\beta\mu} \theta^1_\lambda \theta^2_\alpha \end{cases} \tag{5}$$

More on Koszul algebras

With *Koszul dual* $\mathcal{A}^!$, $\mathcal{A}^! = A(E^*, R^\perp)$,

The sequence of free left $\mathcal{A}$-modules

$$K(\mathcal{A}) \qquad \cdots \xrightarrow{b} \mathcal{A} \otimes \mathcal{A}^{!*}_{n+1} \xrightarrow{b} \mathcal{A} \otimes \mathcal{A}^{!*}_{n} \to \cdots \to \mathcal{A} \otimes \mathcal{A}^{!*}_{2} \xrightarrow{b} \mathcal{A} \otimes E \xrightarrow{b} \mathcal{A} \to 0$$

where $b : \mathcal{A} \otimes \mathcal{A}^{!*}_{n+1} \to \mathcal{A} \otimes \mathcal{A}^{!*}_{n}$ defined by

$$b(a \otimes (x_0 \otimes x_1 \otimes \cdots \otimes x_n)) = ax_0 \otimes (x_1 \otimes \cdots \otimes x_n)$$

is such that $b^2 = 0$, is a chain complex $K(\mathcal{A})$ of free left $\mathcal{A}$-modules called the *Koszul complex* of the quadratic algebra $\mathcal{A}$.

The quadratic algebra $\mathcal{A}$ is said to be a Koszul algebra whenever its Koszul complex is acyclic in positive degrees, that is,

$$H_n(K(\mathcal{A})) = 0 \qquad \text{for} \quad n \geq 1.$$

If $\mathcal{A} = A(E,R)$ quadratic Koszul is such that $\mathcal{A}^!_D \neq 0$ and $\mathcal{A}^!_n = 0$ for $n > D$,

then the trivial (left) module $\mathbb{C}$ has projective dimension D which implies that $\mathcal{A}$ has global dimension D.

A cochain complex $L(\mathcal{A})$ of right $\mathcal{A}$-modules

$$0 \to \mathcal{A} \xrightarrow{b'} \cdots \xrightarrow{b'} \mathcal{A}^!_n \otimes \mathcal{A} \xrightarrow{b'} \mathcal{A}^!_{n+1} \otimes \mathcal{A} \xrightarrow{b'} \cdots . \tag{6}$$

with b' left multiplication by $\sum_k \theta^k \otimes e_k$ in $\mathcal{A}^! \otimes \mathcal{A}$ and (e_k, θ^j) are dual bases.

The algebra $\mathcal{A}$ is said to be Koszul-Gorenstein if it is Koszul of finite global dimension D as above and if $H^n(L(\mathcal{A})) = \mathbb{C}\,\delta^n_D$.

This implies that $\mathcal{A}^!_n \simeq \mathcal{A}^{!*}_{D-n}$ as vector spaces (a version of Poincaré duality).

Finally, a graded algebra $\mathcal{A} = \oplus_n \mathcal{A}_n$ is said to have *polynomial growth* whenever there are a positive C and $N \in \mathbb{N}$ such that, for any $n \in \mathbb{N}$,

$$\dim(\mathcal{A}_n) \leq C n^{N-1}.$$

An $\mathcal{R}$-matrix

Define x^a for $a \in \{1, \dots, N_1 + N_2\}$ by $\quad x^\lambda = x_1^\lambda, \quad x^{\alpha+N_1} = x_2^\alpha.$

Then the relations for $\mathcal{A}_R$ in (3) together with $x_2^\alpha x_1^\lambda = \overline{R}^{\lambda\alpha}_{\beta\mu} x_1^\mu x_2^\alpha$ reads

$$x^a x^b = \mathcal{R}^{ab}_{cd} x^c x^d \tag{7}$$

In view of (3) and (4) one gets:

$$\begin{aligned} \mathcal{R}^{\lambda\mu}_{\tau\rho} &= \delta^\lambda_\rho \, \delta^\mu_\tau, & \mathcal{R}^{\gamma\delta}_{\alpha\beta} &= \delta^\gamma_\beta \, \delta^\delta_\alpha \\ \mathcal{R}^{\lambda\alpha}_{\beta\mu} &= R^{\lambda\alpha}_{\beta\mu}, & \mathcal{R}^{\alpha\lambda}_{\mu\beta} &= \overline{R}^{\lambda\alpha}_{\beta\mu} \end{aligned}$$

$$\begin{aligned} \mathcal{R}^{\lambda\mu}_{\alpha\nu} &= \mathcal{R}^{\lambda\mu}_{\alpha\beta} = \mathcal{R}^{\lambda\mu}_{\nu\beta} = 0, \\ \mathcal{R}^{\alpha\gamma}_{\lambda\beta} &= \mathcal{R}^{\alpha\gamma}_{\lambda\mu} = \mathcal{R}^{\alpha\gamma}_{\beta\mu} = 0, \\ \mathcal{R}^{\lambda\alpha}_{\mu\nu} &= \mathcal{R}^{\lambda\alpha}_{\beta\gamma} = \mathcal{R}^{\lambda\alpha}_{\mu\beta} = 0, \\ \mathcal{R}^{\alpha\lambda}_{\mu\nu} &= \mathcal{R}^{\alpha\lambda}_{\beta\gamma} = \mathcal{R}^{\alpha\lambda}_{\beta\mu} = 0. \end{aligned}$$

Involutive and the Yang-Baxter condition

The matrix $\mathcal{R}$ is involutive, i.e. $\mathcal{R}^2 = \mathbb{1} \otimes \mathbb{1}$ or

$$\mathcal{R}^{ab}_{cd}\,\mathcal{R}^{cd}_{ef} = \delta^a_e \delta^b_f$$

We next impose the Yang-Baxter condition for $\mathcal{R}$

$$(\mathcal{R} \otimes \mathbb{1})(\mathbb{1} \otimes \mathcal{R})(\mathcal{R} \otimes \mathbb{1})^{abc} = (\mathbb{1} \otimes \mathcal{R})(\mathcal{R} \otimes \mathbb{1})(\mathbb{1} \otimes \mathcal{R})^{abc}$$

This breaks in a series of conditions


$R_{12}R_{23}R_{12} = R_{23}R_{12}R_{23}$

Noncommutative product of $\mathbb{R}^{N_1}$ and $\mathbb{R}^{N_2}$

The classical (commutative) solution $R = R_0$ is

$$(R_0)^{\lambda\alpha}_{\beta\mu} = \delta^{\lambda}_{\mu}\delta^{\alpha}_{\beta}$$

and the corresponding algebra $\mathcal{A}_{R_0}$ is the algebra of polynomial functions on the product $\mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$

This is the reason we define the noncommutative product of $\mathbb{R}^{N_1}$ and $\mathbb{R}^{N_2}$

$$\mathbb{R}^{N_1} \times_R \mathbb{R}^{N_2}$$

to be the “dual” of the algebra $\mathcal{A}_R$ for general R.

Regularity properties of $\mathcal{A}_R$ and $\mathcal{A}^!_R$

Since the relations of $\mathcal{A}_R$ can be written an $\mathcal{R}$ which is involutive and satisfies the Yang-Baxter equation,

it follows from general results (Gurevich, Wambst) that $\mathcal{A}_R$ is very regular.

In particular $\mathcal{A}_R$ is a Koszul algebra of global dimension $N_1 + N_2$ having the Gorenstein property (an appropriate version of the Poincaré duality property)

In our case, this implies that, in terms of the x^a and the dual basis θ_a:

the $x^{a_1} \dots x^{a_p}$ for $a_1 \leq \dots \leq a_p$ and $p \in \mathbb{N}$ is a basis of $\mathcal{A}_R$

while the $\theta_{a_1} \dots \theta_{a_p}$ for $a_1 < \dots < a_p$ and $p \in \{1, \dots, N_1 + N_2\}$ is a basis of $\mathcal{A}^!_R$.

As a consequence the Poincaré series are classical :

$$P_{\mathcal{A}_R}(t) := \sum_n \mathrm{dim}(\mathcal{A}^n_R)t^n = \left(\frac{1}{1-t}\right)^{N_1+N_2} \quad \text{and} \quad P_{\mathcal{A}^!_R}(t) = (1+t)^{N_1+N_2} \quad (8)$$

Regularity properties cont.d

The algebra $\mathcal{A}_R$ is even a Calabi–Yau algebra:

any generator of the top one-dimensional space $((\mathcal{A}^!_R)_{N_1+N_2})^*$

is a cyclic potential for the algebra $\mathcal{A}_R$ (Ginzburg)

(a cyclic pre-regular multilinear form)

Noncommutative product of Euclidean spaces

Theorem. The following conditions (i) (ii) and (iii) are equivalent :

$$\text{(i)}\ \sum_{a=1}^{N_1+N_2}(x^a)^2=\sum_{\lambda=1}^{N_1}(x_1^\lambda)^2+\sum_{\alpha=1}^{N_2}(x_2^\alpha)^2 \qquad \text{is central in} \quad \mathcal{A}_R,$$

$$\text{(ii)}\ \sum_{\lambda=1}^{N_1}(x_1^\lambda)^2 \quad \text{and} \quad \sum_{\alpha=1}^{N_2}(x_2^\alpha)^2 \qquad \text{are in the center of} \quad \mathcal{A}_R, \tag{9}$$

$$\text{(iii)}\ \sum_{\lambda=1}^{N_1} R^{\lambda\gamma}_{\beta\nu}R^{\lambda\beta}_{\alpha\mu}=\delta^\gamma_\alpha\delta_{\mu\nu} \qquad \text{and} \qquad \sum_{\alpha=1}^{N_2} R^{\lambda\alpha}_{\beta\rho}R^{\rho\alpha}_{\gamma\mu}=\delta^\lambda_\mu\delta_{\beta\gamma}$$

We take R to satisfy also (9) and define the the noncommutative product of the Euclidean space $\mathbb{R}^{N_1}$ with the Euclidean space $\mathbb{R}^{N_2}$ to be dual of $\mathcal{A}_R$.

Clearly, the relations (11) are satisfied by the classical $R=R_0$;

$\mathcal{A}_R$ generalizes the algebra of polynomial functions on the product $\mathbb{R}^{N_1}\times\mathbb{R}^{N_2}$

Restrictions on the structure of R

By using (4) and (9) one obtains realations

$$R^{\lambda\beta}_{\alpha\mu} = R^{\mu\alpha}_{\beta\lambda} = \overline{R}^{\mu\beta}_{\alpha\lambda} = (R^{-1})^{\beta\mu}_{\lambda\alpha} \tag{10}$$

as well as

$$R^{\lambda\alpha}_{\beta\rho} R^{\rho\delta}_{\gamma\mu} = R^{\lambda\delta}_{\gamma\rho} R^{\rho\alpha}_{\beta\mu} \qquad \text{and} \qquad R^{\lambda\alpha}_{\gamma\nu} R^{\mu\gamma}_{\beta\rho} = R^{\mu\alpha}_{\gamma\rho} R^{\lambda\gamma}_{\beta\nu} \tag{11}$$

Corollary.
Relations (3) define (the algebra of) a noncommutative product of a N_1-dimensional with a N_2-dimensional Euclidean spaces if and only if the $R^{\lambda\alpha}_{\beta\mu}$ satisfy relations (10) and (11).

Noncommutative product of spheres

The elements

$\sum_{\lambda=1}^{N_1}(x_1^\lambda)^2 = \|x_1\|^2$ and $\sum_{\alpha=1}^{N_2}(x_2^\alpha)^2 = \|x_2\|^2$ of $\mathcal{A}_R$ being central

one may consider the quotient algebra

$$\mathcal{A}_R/(\{\|x_1\|^2 - \mathbb{1}, \|x_2\|^2 - \mathbb{1}\}) \qquad \leftrightarrow \qquad \mathbb{S}^{N_1-1} \times_R \mathbb{S}^{N_2-1}$$

This defines by duality:
the noncommutative product of the classical spheres $\mathbb{S}^{N_1-1}$ and $\mathbb{S}^{N_2-1}$.

Indeed, for $R = R_0$, the above quotient is the restriction to $\mathbb{S}^{N_1-1} \times \mathbb{S}^{N_2-1}$ of the polynomial functions on $\mathbb{R}^{N_1+N_2}$.

Noncommutative spheres

With $\|x\|^2$ denoting the central element $\sum_{a=1}^{N_1+N_2}(x^a)^2 = \|x_1\|^2 + \|x_2\|^2$, one may also consider the quotient of $\mathcal{A}_R$

$$\mathcal{A}_R/(\|x\|^2 - \mathbb{1}) \qquad \leftrightarrow \qquad (\mathbb{S}^{N_1+N_2-1})_R$$

This defines (dualy) the noncommutative $(N_1 + N_2 - 1)$-sphere $(\mathbb{S}^{N_1+N_2-1})_R$ (a subspace of the noncommutative product of $\mathbb{R}^{N_1}$ with $\mathbb{R}^{N_2}$)

This is a noncommutative spherical manifold in the sense of Connes–Landi and Connes–Dubois-Violette (see later).

The (generalized) Clifford algebra $C\ell(\mathcal{A}_R)$

The $*$-algebra $C\ell(\mathcal{A}_R)$ is generated by two sets of hermitian elements Γ^1_λ with $\lambda \in \{1,\dots,N_1\}$ and Γ^2_α with $\alpha \in \{1,\dots,N_2\}$ with relations

$$\begin{cases} \Gamma^1_\lambda\Gamma^1_\mu + \Gamma^1_\mu\Gamma^1_\lambda = 2\delta_{\lambda\mu}\mathbb{1} \\ \Gamma^2_\alpha\Gamma^2_\beta + \Gamma^2_\beta\Gamma^2_\alpha = 2\delta_{\alpha\beta}\mathbb{1} \\ \Gamma^2_\beta\Gamma^1_\mu + R^{\lambda\alpha}_{\beta\mu}\Gamma^1_\lambda\Gamma^2_\alpha = 0 \end{cases} \tag{12}$$

Proposition In the algebra $C\ell(\mathcal{A}_R) \otimes \mathcal{A}_R$ one has :

$$(\Gamma^1_\lambda \otimes x^\lambda_1)^2 = \mathbb{1} \otimes \|x_1\|^2, \qquad (\Gamma^2_\alpha \otimes x^\alpha_2)^2 = \mathbb{1} \otimes \|x_2\|^2$$

and

$$(\Gamma^1_\lambda \otimes x^\lambda_1)(\Gamma^2_\alpha \otimes x^\alpha_2) + (\Gamma^2_\alpha \otimes x^\alpha_2)(\Gamma^1_\lambda \otimes x^\lambda_1) = 0$$

Structure of $C\ell(\mathcal{A}_R)$

Last proposition is equivalent to

$$(\Gamma(x))^2 = \mathbb{1} \otimes \|x\|^2 \tag{13}$$

with $\Gamma(x) = \Gamma_a \otimes x^a = \Gamma^1_\lambda \otimes x^\lambda_1 + \Gamma^2_\alpha \otimes x^\alpha_2$ and $\|x\|^2 = \sum_{a=1}^{N_1+N_2} (x^a)^2$.

The algebra $C\ell(\mathcal{A}_R)$ is nonhomogeneous quadratic with $\mathcal{A}^!_R$ as homogeneous part. It is not $\mathbb{N}$-graded but only $\mathbb{Z}_2$-graded and filtered with

$$\mathcal{F}^n = F^n(C\ell(\mathcal{A}_R)) = \{\text{elements of degree in } \Gamma \leq n\}$$

One has a surjective canonical homomorphism of graded algebra

$$\text{can} : \mathcal{A}^!_R \to \text{gr}(C\ell(\mathcal{A}^!_R)) = \oplus_{n\in\mathbb{N}} \mathcal{F}^n/\mathcal{F}^{n-1} \tag{14}$$

which induce the isomorphism of vector spaces

$$(\mathcal{A}^!_R)^1 \simeq \mathcal{F}^1/\mathcal{F}^0$$

Structure of $C\ell(\mathcal{A}_R)$ - cont.d

The fact that $\|x\|^2 = \sum(x^a)^2$ is central in $\mathcal{A}_R$ and the Koszulity of $\mathcal{A}_R^!$ imply the following PBW property (via the duality of Positselski).

Proposition The homomorphism (14) is an isomorphism of graded algebras.

Thus $C\ell(\mathcal{A}_R)$ is a Koszul nonhomogeneous quadratic algebra since $\mathcal{A}_R^!$ is Koszul, (cf. Dubois-Violette). This implies

$$\dim C\ell(\mathcal{A}_R) = \dim \mathcal{A}_R^! = 2^{N_1+N_2}$$

One has the following isomorphisms :

$$\begin{cases} C\ell(N_1) \simeq \text{ subalgebra of } C\ell(\mathcal{A}) \text{ generated by the } \Gamma^1_\lambda \\ C\ell(N_2) \simeq \text{ subalgebra of } C\ell(\mathcal{A}) \text{ generated by the } \Gamma^2_\alpha \end{cases}$$

and

$$C\ell(\mathcal{A}_R) \simeq C\ell(N_1 + N_2) \tag{15}$$

The general solution for the megtrix $\mathcal{R}$

Let us introduce a two-index $\binom{\lambda}{\alpha}$ with $\lambda \in \{1, \ldots, N_1\}$ and $\alpha \in \{1, \ldots, N_2\}$. Then the conditions (10) can be read as symmetry conditions:

$$R^{\binom{\lambda}{\alpha}}{}_{\binom{\mu}{\beta}} = R^{\binom{\mu}{\beta}}{}_{\binom{\lambda}{\alpha}} = \overline{R}^{\binom{\mu}{\alpha}}{}_{\binom{\lambda}{\beta}} = (R^{-1})_{\binom{\lambda}{\beta}}{}^{\binom{\mu}{\alpha}}.$$

While, the quadratic conditions in (11) can be written as

$$R^{\binom{\lambda}{\alpha}}{}_{\binom{\rho}{\beta}} R^{\binom{\rho}{\gamma}}{}_{\binom{\mu}{\delta}} = R^{\binom{\lambda}{\gamma}}{}_{\binom{\rho}{\delta}} R^{\binom{\rho}{\alpha}}_{\binom{\mu}{\beta}} \qquad \text{and} \qquad R^{\binom{\lambda}{\alpha}}{}_{\binom{\nu}{\gamma}} R^{\binom{\mu}{\gamma}}{}_{\binom{\rho}{\beta}} = R^{\binom{\mu}{\alpha}}{}_{\binom{\rho}{\gamma}} R^{\binom{\lambda}{\gamma}}{}_{\binom{\nu}{\beta}}.$$

Theorem The general solution is of the form

$$R^{\lambda\beta}_{\alpha\mu} = S^{\lambda\beta}_{\mu\alpha} + \mathrm{i}\, A^{\lambda\beta}_{\mu\alpha}$$

$$\text{with} \qquad [S, A] = 0 \qquad S^2 + A^2 = \mathbb{1}_{N_1} \otimes \mathbb{1}_{N_2}.$$

S and A are real matrices with S standing for symmetric while A antisymmetric for exchange in indices $\lambda \leftrightarrow \mu$ and $\alpha \leftrightarrow \beta$

The ansatz $A\ B\ C\ D$

Nontrivial realizations of the $R^{\lambda\alpha}_{\beta\mu}$ are given by the following.

Proposition Let A and C be two commuting real $N_1 \times N_1$-matrices with A symmetric and C antisymmetric and let B and D be two commuting real $N_2 \times N_2$-matrices with B symmetric and D antisymmetric. Assume that

$$A^2 \otimes B^2 + C^2 \otimes D^2 = \mathbb{1}_{N_1} \otimes \mathbb{1}_{N_2}$$

then the $R^{\lambda\alpha}_{\beta\mu}$ given by

$$R^{\lambda\alpha}_{\beta\mu} = A^{\lambda}_{\mu} B^{\alpha}_{\beta} + \mathrm{i}\, C^{\lambda}_{\mu} D^{\alpha}_{\beta} \tag{16}$$

satisfy the conditions of the theorem.

Enter the quaternions

An $SO(4)$-invariant decomposition of $M_4(\mathbb{R})$

$$q = x^0 1 + x^k e_k \in \mathbb{H} \qquad \longleftrightarrow \qquad x = (x^0, x^1, x^2, x^3) \in \mathbb{R}^4 \qquad \text{Euclidean}$$

a right and a left action

$$e_k q \leftrightarrow J_k^{(+)} x, \qquad q e_k \leftrightarrow -J_k^{(-)} x$$

$$(J_k^{(\pm)})_{\mu\nu} = \mp(\delta_{0\mu}\delta_{k\nu} - \delta_{0\nu}\delta_{k\mu}) - \varepsilon_{k\ell m}\delta_\mu^\ell \delta_\nu^m$$

$$J_k^{(\pm)} J_\ell^{(\pm)} = -\delta_{k\ell}\mathbb{1} + \sum_m \varepsilon_{k\ell m} J_m^{(\pm)}, \qquad J_k^{(+)} J_\ell^{(-)} = J_\ell^{(-)} J_k^{(+)}$$

$$M_4(\mathbb{R}) = \mathbb{R}^4 \otimes \mathbb{R}^4 = \mathbb{R}\mathbb{1} \oplus \wedge^2_{(+)}\mathbb{R}^4 \oplus \wedge^2_{(-)}\mathbb{R}^4 \oplus \mathbb{S}^2_0\mathbb{R}^4$$

Orthornormal basis $\mathbb{1}$, $J_k^{(+)}$, $J_\ell^{(-)}$, $J_r^{(+)} J_s^{(-)}$

$(+)$ = antisymmetric self-dual, $\qquad (-)$ = antisymmetric anti-self dual.

Noncommutative quaternionic planes

Use last theorem for $N_1 = N_2 = 4$

$$A = \mathbb{1}, \qquad B = u^0 \mathbb{1}, \qquad C = J_1^{(\pm)}, \qquad D = u^1 J_1^{(\pm)} + u^2 J_2^{(\pm)}$$

with $(u^0)^2 + (u^1)^2 + (u^2)^2 = 1$. This gives:

$$R^{\lambda\alpha}_{\beta\mu} = u^0 \delta^\lambda_\mu \delta^\alpha_\beta + \mathrm{i}(J_1^{(\pm)})^\lambda_\mu (u^1 J_1^{(\pm)} + u^2 J_2^{(\pm)})^\alpha_\beta \tag{17}$$

By using the $J_k^{(\mp)}$ one defines an action of $\mathbb{H}$.

The choice of the direction 1 and of the plane (1 2) is immaterial since one can change them into an arbitrary direction $\vec{n}$ and an arbitrary plane which contains $\vec{n}$ by a rotation of $SO_3^{(\pm)}$.

The exchange $(+) \leftrightarrow (-)$ is induced for instance by the exchange $x^0 \leftrightarrow -x^0$ and therefore does not change the algebra $\mathcal{A}_R$ for R given by (17).

Noncommutative quaternionic planes cont.d

The solution given by (17) generalizes $\mathbb{C}^2_\theta$ for $\mathbb{C} \to \mathbb{H}$;

For the θ-deformation the parameter is in fact

$$\mathbb{S}^1/\mathbb{S}^0 = U_1(\mathbb{C})/U_1(\mathbb{R}) = P_1(\mathbb{R}).$$

The parameter here

$$\mathbf{u} \in \mathbb{S}^2 = \mathbb{S}^3/\mathbb{S}^1 = U_1(\mathbb{H})/U_1(\mathbb{C}) = P_1(\mathbb{C})$$

and for $u^0 = 1$ ($\Rightarrow u^1 = u^2 = 0$), this gives the classical $\mathbb{H}^2$.

Noncommutative quaternionic tori

The N.C. product of $\mathbb{H}$ by $\mathbb{H}$ corresponding to $\mathcal{A}_R$ is denoted $\mathbb{H}^2_{\mathbf{u}}$.

Tori obtained by the quotient by the ideal generated by $\{\|x_1\|^2 - 1, \|x_2\|^2 - 1\}$:

$$\mathcal{A}(\mathbb{T}^{\mathbb{H}}_{\mathbf{u}}) = \mathcal{A}(\mathbb{H}^2_{\mathbf{u}}) / < ||x_1||^2 - 1, ||x_2||^2 - 1 >$$

$$\mathbb{T}^{\mathbb{H}}_{\mathbf{u}} \simeq \mathbb{S}^3 \times_{\mathbf{u}} \mathbb{S}^3$$

an $SU(2) \times SU(2)$ action

Additional strata: other N.C. products of 4-dim. Euclidean spaces

And higher solutions ?

$$\mathbb{H} \times_{\mathbf{u}} \mathbb{H} \qquad \mathbf{u} \in \mathbb{S}^2 = P_1(\mathbb{C})$$

$$\mathbb{O} \times_{\mathbf{u}} \mathbb{O} \qquad \mathbf{u} \in \mathbb{S}^4 = P_1(\mathbb{H})$$

$$\mathbb{S}\mathrm{ed} \times_{\mathbf{u}} \mathbb{S}\mathrm{ed} \qquad \mathbf{u} \in \mathbb{S}^8 = P_1(\mathbb{O})$$

Spherical conditions

A projection $p \in M_{2^n}(\mathcal{A}(S_R^{2n}))$,

$$p = \frac{1}{2}(\mathbb{1} + \Gamma_a x^a + \Gamma x) \qquad \mathrm{tr}(p) = 2^{n-1} = \mathrm{rank}(p)$$

$$\mathrm{ch}_k(p) = 0, \qquad 1 \leq k \leq n-1 \qquad \qquad \mathrm{ch}_n(p) \quad \text{the volume form:}$$

$$b(\mathrm{ch}_n(p)) = 0 \qquad \qquad \mathrm{ch}_n(p) \in HH_n(\mathcal{A}(S_R^{2n}))$$

b is the Hochschild boundary operator

The (Bott) projection p generates the K-theory of $\mathcal{A}(S_R^{2n})$

$$< p[D,p]^{2n} > = \gamma_{2n+1} \qquad \text{the volume form}$$

D a Dirac operator

$$\nabla = p \circ d \qquad \text{an anti-self dual connection}$$

with top Chern number

$$\text{nctr}(< p[D,p]^{2n} >) = -1$$

The odd version

A unitary $U \in M_{2^{n-1}}(\mathcal{A}(S_R^{2n-1}))$,

$$U = \mathbb{1}\, x^0 + \Sigma_j\, x^j$$

$$\Gamma_\mu = \begin{pmatrix} 0 & \Sigma_\mu \\ \Sigma_\mu, & 0 \end{pmatrix}$$

$$\mathrm{ch}_{k-\frac{1}{2}}(U) = 0, \qquad 1 \leq k \leq n-1 \qquad\qquad \mathrm{ch}_{n-\frac{1}{2}}(U) \quad \text{the volume form}$$

Now

$$b(\mathrm{ch}_{n-\frac{1}{2}}(U) = 0 \qquad\qquad \mathrm{ch}_{n-\frac{1}{2}}(U) \in HH_{n-\frac{1}{2}}(\mathcal{A}(S_R^{2n-1}))$$

b is the Hochschild boundary operator

Principal bundles (M. Dubois-Violette, Xiao Han, G.L.)

Consider the two quaternions.

$$x_1 = x_1^\mu e_\mu \quad x_2 = x_2^\alpha e_\alpha$$

with commutation relations governed by a matrix $R^{\lambda\alpha}_{\beta\mu}$. When restricting to the sphere $\mathbb{S}^7_R$, we get a normalised vector valued function

$$|\psi\rangle = \begin{pmatrix} x_2 \\ x_1 \end{pmatrix} \qquad \langle\psi,\psi\rangle = ||x_1||^2 + ||x_2||^2 = \mathbb{1}$$

and thus a projection: ($p = p^* = p^2$)

$$p = |\psi\rangle\langle\psi| = \begin{pmatrix} x_2 x_2^* & x_2 x_1^* \\ x_1 x_2^* & x_1 x_1^* \end{pmatrix},$$

Define coordinate functions $Y = Y^0 e_0 + Y^k e_k$ and Y^4 by

$$Y^4 = ||x_2||^2 - ||x_1||^2 \quad \text{and} \quad \tfrac{1}{2} Y = x_2 x_1^*$$

so that
$$p = |\psi\rangle\langle\psi| = \tfrac{1}{2}\begin{pmatrix} 1 + Y^4 & Y \\ Y^* & 1 - Y^4 \end{pmatrix}.$$

The condition $p^2 = p$ leads to

$$YY^* + (Y^4)^2 = 1 \quad \text{and} \quad Y^*Y + (Y^4)^2 = 1$$

$$YY^4 = Y^4Y \quad \text{and} \quad Y^*Y^4 = Y^4Y^*.$$

Thus the coordinate function Y^4 is central while comparing the first two conditions requires $YY^* = Y^*Y$ and that this is a multiple of the identity. These translates to the following conditions

$$\sum_{\mu=0}^{3}(Y^{\mu *}Y^{\mu} - Y^{\mu}Y^{\mu *}) = 0,$$
$$-(Y^{0*}Y^{k} - Y^{k*}Y^{0}) + \varepsilon_{kmn}Y^{m*}Y^{n} = 0 = Y^{0}Y^{k*} - Y^{k}Y^{0*} + \varepsilon_{kmn}Y^{m}Y^{n*},$$

for $k, r, m = 1, 2, 3$ and totally antisymmetric tensor ε_{krm}.

Then the first condition at top of the page reduces to a four-sphere relation

$$\sum_{\mu=0}^{3} Y^{\mu *}Y^{\mu} + (Y^4)^2 = 1 = \sum_{\mu=0}^{3} Y^{\mu *}Y^{\mu} + (Y^4)^2 = 1. \tag{18}$$

The elements Y^μ generate the $*$-algebra $\mathcal{A}(\mathbb{S}^4_R)$ of a 4-sphere $\mathbb{S}^4_R$. One has

$$Y^{\mu *} = \Lambda^\mu{}_\nu Y^\nu \qquad \mu, \nu = 0, 1, 2, 3$$

for Λ a symmetric unitary matrix

For the commutation relations (17) for the x's:

$$\begin{pmatrix} Y^{0*} \\ Y^{3*} \end{pmatrix} = \begin{pmatrix} u^0 + \mathrm{i}u^1 & \mathrm{i}u^2 \\ \mathrm{i}u^2 & u^0 - \mathrm{i}u^1 \end{pmatrix} \begin{pmatrix} Y^0 \\ Y^3 \end{pmatrix}$$

and

$$\begin{pmatrix} Y^{1*} \\ Y^{2*} \end{pmatrix} = \begin{pmatrix} u^0 + \mathrm{i}u^1 & \mathrm{i}u^2 \\ \mathrm{i}u^2 & u^0 - \mathrm{i}u^1 \end{pmatrix} \begin{pmatrix} Y^1 \\ Y^2 \end{pmatrix}$$

The 2×2 matrix being symmetric and unitary, can be diagonalized by a real rotation S and with a further normalization put in the form

$$\begin{pmatrix} 1 & 0 \\ 0 & e^{\mathrm{i}\theta} \end{pmatrix}, \qquad e^{\mathrm{i}\theta} = \frac{u^0 + \mathrm{i}\sqrt{(u^1)^2 + (u^2)^2}}{u^0 - \mathrm{i}\sqrt{(u^1)^2 + (u^2)^2}} = \left(u^0 + \mathrm{i}\sqrt{(u^1)^2 + (u^2)^2}\,\right)^2.$$

The sphere $\mathbb{S}^4_R$ is (isomorphic to) a θ-deformation.

The algebra inclusion $\mathcal{A}(\mathbb{S}^4_R) \hookrightarrow \mathcal{A}(\mathbb{S}^7_R)$ is a principal SU(2) bundle:

A unit quaternion $w \in \mathbb{H}_1 \simeq \mathrm{SU}(2)$ act on $\mathbb{S}^7_R$ as

$$\alpha_w(|\psi\rangle) = |\psi\rangle\, w = \begin{pmatrix} x_2 w \\ x_1 w \end{pmatrix}.$$

leaving the projection p and then the algebra $\mathcal{A}(\mathbb{S}^4_R)$ invariant.

If $H = \mathcal{A}(\mathrm{SU}(2))$ we have dually a co-action δ of H on $\mathcal{A}(\mathbb{S}^7_R)$ with algebra of co-invariant element again the subalgebra $\mathcal{A}(\mathbb{S}^4_R)$.

Then the canonical map

$$\chi : \mathcal{A}(\mathbb{S}^7_R) \otimes_{\mathcal{A}(\mathbb{S}^4_R)} \mathcal{A}(\mathbb{S}^7_R) \to \mathcal{A}(\mathbb{S}^7_R) \otimes H, \qquad \chi(p' \otimes p) = p'\delta(p)$$

is bijective.

Indeed,

$$\chi\big(\langle\psi| \otimes_{\mathcal{A}(\mathbb{S}^4_R)} |\psi\rangle \big) = \langle\psi|\, \delta(|\psi\rangle) = \langle\psi, \psi\rangle \otimes w = \mathbb{1} \otimes w,$$

showing surjectivity of χ. This is enough since ... (H is classical).

The next fibration

A similar (if more involved) algebra inclusion

$$\mathcal{A}(\mathbb{S}^8_R) \hookrightarrow \mathcal{A}(\mathbb{S}^{15}_R)$$

(coming from an octonionic matrix $R^{\lambda\alpha}_{\beta\mu}$)

which is a $\mathbb{S}^7$-bundle

a 'quasi' group

Summing up

- noncommutative generalizations of $\mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$
- noncommutative generalizations of $\mathbb{S}^{N_1+N_2-1}$
- noncommutative generalizations of $\mathbb{S}^{N_1-1} \times \mathbb{S}^{N_2-1}$
- a quaternionic noncommutative torus $\mathbb{S}^3 \times_{\mathbf{u}} \mathbb{S}^3$, $\mathbf{u} \in \mathbb{S}^2 = \mathrm{SU}(2)/\,\mathrm{U}(1)$
- spherical manifolds : volume forms from top Chern–Connes characters
- Generalized Clifford algebras $C\ell(\mathcal{A}_R)$
- noncommutative principal bundles

thank you