Towards Gibbs semigroups on Banach spaces

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Why this study?

A *Gibbs semigroup* $(e^{-tA})_{t\geq 0}$ on a Hilbert space is:

a C_0-semigroup (i.e. strongly continuous)

$\mathrm{Tr}\, e^{-tA} < \infty$ for $t > 0$

D. A. Uhlenbrock in 1971 motivated by Quantum Statistical Mechanics:
The canonical quantum Gibbs state of a finite system is defined by the density matrix operator: $Z_\Lambda(\beta)^{-1} e^{-\beta H_\Lambda}$

$\beta^{-1} \geq 0$ is the temperature

H_Λ is its Hamiltonian

Canonical partition function: $Z_\Lambda(\beta) = \mathrm{Tr}\, e^{-\beta H_\Lambda}$

Few motivations

Quantum mechanics

Notion of action

Improve theory of semigroups

I Motivation: Quantum mechanics

Feymann

”I think I can safely say that nobody understands quantum mechanics“

Ok, but: quantum mechanics in a nutshell:

Axiom 1: any object is a quantum object, so its a wave controlled by a wave function ψ

Axioms 2: ψ is a solution of Schrödinger equation:

$H = -\Delta + V$

$i\frac{\partial}{\partial t}\psi = H\psi$

$H\psi = E\psi$

Problem 1: Why this equation?

I Motivation: Quantum mechanics

Pseudo answer:

Theorem

A linear differential operator of order 2 in $\mathbb{R}^{n+1}$ which is invariant by isometry on the space and translation in time is of the following type:

Elliptic, parabolic, hyperbolic or Schrödinger

If it is real, then it is:

$$a_0 + \Delta_{(x,t)}\ ,\ \ a_0 + \Box_{(x,t)}\ ,\ \ \partial_t - \Delta_x$$

Problem 2: What is the space of solutions?

A standard answer: Hilbert space

Induced question: justify the scalar product

justify the closure (which is the linear normed structure on the equivalence classes of Cauchy sequences, knowing that $\mathbb{R}$ is the "unique" complete Archimedean field)

Spectrum of Schrödinger Operator for a finite well in various realizations

Let $H = -\Delta + V$, where V is a finite potential well:

$$V(x) = \begin{cases} 0 & \text{for } |x| > a \\ -V_0 & \text{for } |x| \leq a \end{cases}, \quad V_0 > 0$$

1. On $L^2(\mathbb{R})$ (Hilbert space)

- **Domain:** $\mathcal{D}(H) = H^2(\mathbb{R})$
- **Spectrum:**

$$\sigma(H) = \{E_n\}_{n=1}^{N} \cup [0, \infty)$$

where $E_n \in (-V_0, 0)$ are finitely many bound states

I Motivation: Quantum mechanics in a nutshell

2. On $C^\infty(\mathbb{R})$ with L^2 inner product

- Dense in L^2, elliptic regularity applies
- **Spectrum:** Same as $L^2(\mathbb{R})$

3. On $C_0^\infty(\mathbb{R})$ with Fréchet topology

- **Topology:** Seminorms $p_{m,K}(\phi) = \sup_{|\alpha|\leq m} \sup_{x\in K} \left|D^\alpha \phi(x)\right|$
- **Spectrum:** $\sigma(H) = \mathbb{C}$

I Motivation: Quantum mechanics in a nutshell

Why $\mathbb{C}$?

$C_0^\infty(\mathbb{R})$ is *not normable*

The operator cannot have a bounded inverse on C^∞, because the fundamental solution E_λ of $H - \lambda$ always has non-compact support and grows in derivatives; applying it to a compactly supported smooth function produces a smooth function outside any compact, hence outside C_0^∞ as a Fréchet space

Caution:

To recover a meaningful spectral picture, one typically considers the *distributional spectrum* in $\mathcal{D}'(\mathbb{R}^n)$, which is real:

$$\sigma_{\mathcal{D}'}(H) = \overline{\{V(x) : x \in \mathbb{R}\} + [0, \infty)} \subset \mathbb{R}$$

4. On $C_0(\mathbb{R})$ (supremum norm)

- **Domain:** $\{u \in C_0(\mathbb{R}) \cap C^2(\mathbb{R}) : \Delta u \in C_0(\mathbb{R})\}$
- **Topology:** $\|u\|_\infty = \sup_{x\in\mathbb{R}} |u(x)|$
- **Spectrum:**
$$\sigma(H) = \{E_j\}_{j=1}^N \cup [0, \infty)$$
- **Reason:** At infinity $V \to 0$, so essential spectrum is $[0, \infty)$. Bound states E_j persist since L^2-eigenfunctions lie in C_0. No residual spectrum

Side remark:

Distributional aspects suggest to move from Hilbert spaces to Gelfand triples (rigged Hilbert space), but …

I Motivation: Quantum mechanics in a nutshell

Space	Spectrum	Finite well case
$L^2(\mathbb{R})$	$\{E_j\}_{j=1}^N \cup [0,\infty)$	Finitely many negative bound states $E_j \in (-V_0, 0)$ plus $[0,\infty)$
C_0^∞ (pre-Hilbert L^2)	$\{E_j\}_{j=1}^N \cup [0,\infty)$	Closure recovers L^2 spectrum
H^s, $s < 2$	$\{E_j\}_{j=1}^N \cup [0,\infty)$	Same as L^2 under relative compactness of V
C^∞ (Fréchet)	$\mathbb{C}$	Spectrum ill-defined; resolvent discontinuous
C_0^∞ (Fréchet)	$[-V_0, \infty)$ (distributional)	Continuous negative spectrum interval, no discrete eigenvalues
C_0	$\{E_j\}_{j=1}^N \cup [0,\infty)$	Same bound states as in L^2, plus $[0,\infty)$ essential spectrum
L^p, $1 \leq p \leq \infty$	$\{E_j\}_{j=1}^N \cup [0,\infty)$	By L^p-independence principle (Hempel–Voigt/Davies)

L^P cases with $1 \leq p \leq \infty$

H_p = realization of H on $L^p(\mathbb{R})$ with domain
$W^{2,p}(\mathbb{R})$ = Sobolev space of L^p-functions with first, second derivatives in L^p
For L^∞ take the *maximal* realization

$$\mathrm{Dom}_{\max}(H_p) = \{u \in L^\infty(\mathbb{R}) : \Delta u \in L^\infty(\mathbb{R})\}$$

p -independence of the spectrum:
Schrödinger semigroup $(e^{-tH})_{t \geq 0}$ is positivity preserving and extends to an analytic semigroup on every L^p
By the Hempel–Voigt/Davies L^p-independence principle, the spectrum is independent of p

$$\begin{aligned} \sigma(H_p) &= \sigma(H_2) \qquad \text{for all } p \in [1, \infty[\\ &= \{E_1, \dots, E_N\} \cup [0, \infty) \end{aligned}$$

I Motivation: Quantum mechanics

Same result for the Hamiltonian of hydrogen atom in $\mathbb{R}^3$

$$H = -\Delta - \frac{1}{|x|}$$

Space	**Point Spectrum**	**Continuous Spectrum**	**Residual Spectrum**	**Notes**
$L^p(\mathbb{R}^3)$ $(p \in]1,\infty[$	$\{-\frac{1}{4n^2}\}_{n=1}^{\infty}$	$[0,\infty)$	$\emptyset$	L^p-independence
$L^1(\mathbb{R}^3)$	$\{-\frac{1}{4n^2}\}_{n=1}^{\infty}$	$[0,\infty)$	$\emptyset$	Non-reflexive space
$L^\infty(\mathbb{R}^3)$	$\{-\frac{1}{4n^2}\}_{n=1}^{\infty}$	$[0,\infty)$	$\emptyset$	Non-reflexive space
$C^\infty(\mathbb{R}^3)$	$\emptyset$	$\emptyset$	$\mathbb{C}$	Fréchet topology
$C_0^\infty(\mathbb{R}^3)$	$\emptyset$	$\emptyset$	$\mathbb{C}$	Fréchet topology
$\mathcal{S}'(\mathbb{R}^3)$	$\{-\frac{1}{4n^2}\}_{n=1}^{\infty}$	$[0,\infty)$	$\emptyset$	Tempered distributio
$\mathcal{D}'(\mathbb{R}^3)$	$\{-\frac{1}{4n^2}\}_{n=1}^{\infty}$	$[0,\infty)$	$\emptyset$	General distributions

I Motivation: Quantum mechanics

Spectrum of the Harmonic Oscillator $H = -\Delta + |x|^2$

Space	**Point Spectrum**	**Continuous Spectrum**	**Residual Spectrum**	**Notes**
$L^p(\mathbb{R}^n)$ $(1 < p < \infty)$	$\{2\|\alpha\| + n\}_{\alpha\in\mathbb{N}^n}$	$\emptyset$	$\emptyset$	L^p-independence
$L^1(\mathbb{R}^n)$	$\{2\|\alpha\| + n\}_{\alpha\in\mathbb{N}^n}$	$\emptyset$	$\emptyset$	Standard domain choice
$L^\infty(\mathbb{R}^n)$	$\{2\|\alpha\| + n\}_{\alpha\in\mathbb{N}^n}$	$\emptyset$	$\emptyset$	Standard domain choice
$C^\infty(\mathbb{R}^n)$	$\emptyset$	$\emptyset$	$\mathbb{C}$	Fréchet top.
$C_0^\infty(\mathbb{R}^n)$	$\emptyset$	$\emptyset$	$\mathbb{C}$	Fréchet top.
$\mathcal{S}'(\mathbb{R}^n)$	$\{2\|\alpha\| + n\}_{\alpha\in\mathbb{N}^n}$	$\emptyset$	$\emptyset$	Tempered distributions
$\mathcal{D}'(\mathbb{R}^n)$	$\{2\|\alpha\| + n\}_{\alpha\in\mathbb{N}^n}$	$\emptyset$	$\emptyset$	General distributions

I Motivation: Quantum mechanics

Pseudo conclusion:

Potential problems

- In the closure process

- Within Banach space theory: $\mathcal{B}$ be a Banach space
Hilbert spaces are characterized by
Parallelogram Law (Jordan–von Neumann).

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2), \qquad \forall x, y \in \mathcal{B}$$

Geometric characterization by projections (Lindenstrauss–Tzafriri)
For every closed subspace $M \subset \mathcal{B}$, there exists a linear projection $P : \mathcal{B} \to M$ with

$$\|P\| = 1 \quad \text{and} \quad \|x\|^2 = \|Px\|^2 + \|(\mathbb{1} - P)x\|^2 \qquad x \in \mathcal{B}$$

I Motivation: Banach space theory

Characterization via orthogonality (Birkhoff–James)

$$x \perp y \iff \|x + \lambda y\| \geq \|x\| \quad \forall \lambda \in \mathbb{C}$$

Characterization via operator ideals (König, Pietsch)
There exists $C > 0$ s.t. for every compact operator $T \in \mathcal{L}(\mathcal{B})$,

$$\sum_{n=1}^{\infty} a_n(T) \; \leq \; C \sum_j |\lambda_j(T)|$$

where $a_n(T)$ are the approximation numbers and $\lambda_j(T)$ the eigenvalues of T

Probabilistic characterization (Kwapień) It has both type 2 and cotype 2

Type p and cotype q

$1 \leq p \leq 2$ and $2 \leq q \leq \infty$

There exist constants C_1, C_2 such that for every finite sequence $(x_j) \subset \mathcal{B}$

$$\Big(\mathbb{E}\Big\|\sum_j \varepsilon_j x_j\Big\|^p\Big)^{1/p} \;\leq\; C_1\Big(\sum_j \|x_j\|^p\Big)^{1/p}$$

and

$$\Big(\sum_j \|x_j\|^q\Big)^{1/q} \;\leq\; C_2\Big(\mathbb{E}\Big\|\sum_j \varepsilon_j x_j\Big\|^q\Big)^{1/q}$$

The *expectation* $\mathbb{E}$ of *independent Rademacher random variables* (ϵ_k) *taking values in* $\{-1, +1\}$ with equal probability:
$\mathbb{P}(\epsilon_k = +1) = \mathbb{P}(\epsilon_k = -1) = \frac{1}{2}$
$\mathbb{E}[\epsilon_k] = (+1) \cdot \frac{1}{2} + (-1) \cdot \frac{1}{2} = 0$

I Motivation: Banach space theory

But

Weak Hilbert spaces (Pisier) with type 2 and cotype $2 + \varepsilon$ for any $\varepsilon > 0$

There exists Banach spaces $\mathcal{B} \sim \mathcal{B}^{**}$ not reflexive (James)

II Motivation: Action

Feynman approach of propagator expressed as an "integral over all paths" $\gamma : [0, t] \to \mathbb{R}^n$ joining y to x:

$$K(t, x; y) = \langle x | e^{-itH} | y \rangle \;\sim\; \int_{\gamma(0)=y}^{\gamma(t)=x} \exp\Big(\tfrac{i}{\hbar} S[\gamma]\Big)\; \mathcal{D}\gamma$$

with *action functional*

$$S[\gamma] \;=\; \int_0^t L(\gamma(s), \dot\gamma(s))\, ds\,, \qquad L(x, \dot x) = \tfrac{m}{2} |\dot x|^2 - V(x)$$

Short-time Gaussian approximation

$$K(\varepsilon, z; y) \approx \Big(\tfrac{m}{2\pi i \hbar \varepsilon}\Big)^{n/2} \exp\Big(\tfrac{i}{\hbar} \tfrac{m|z-y|^2}{2\varepsilon} - \tfrac{i}{\hbar} \varepsilon V\big(\tfrac{z+y}{2}\big)\Big)$$

Perform a stationary-phase integration in z about $z = y$, expanding $K(t, x; z)$ to second order in $(z - y)$:

$$K(t + \varepsilon, x; y) = K(t, x; y) - \tfrac{i\varepsilon}{\hbar}\Big(- \tfrac{\hbar^2}{2m}\Delta_y + V(y)\Big) K(t, x; y) + o(\varepsilon)$$

II Motivation: Action

Equivalently,

$$i\hbar\,\partial_t K(t,x;y) = \Big(-\tfrac{\hbar^2}{2m}\Delta_y + V(y)\Big)K(t,x;y)$$

Since $\psi(t,x) = \int K(t,x;y)\,\psi(0,y)\,dy$

$\rightarrow$ Appearance Schrödinger equation

$$i\hbar\,\partial_t\psi(t,x) = \Big(-\frac{\hbar^2}{2m}\Delta + V(x)\Big)\psi(t,x)$$

Similarly, a derivation of
- Newton dynamics obtained from least action principle $\delta S = 0$ with

$$\mathcal{S}:\ L \rightarrow \int_{t_0}^{t_1} L[x(t),\dot{x}(t)]\,dt \in \mathbb{C}$$

II Motivation: Action

- Einstein equations is obtained from the Einstein–Hilbert action

- More generally the spectral action of $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ (Chamseddine–Connes)

$$S(\mathcal{D}, f) := \mathrm{Tr}\, f(|\mathcal{D}|)$$

If $f(x) = \int_0^\infty e^{-tx}\,\mathrm{d}\phi(t)$, then

$$S(\mathcal{D}, f) = \int_0^\infty \mathrm{Tr}\, e^{-t\,|\mathcal{D}|}\,\mathrm{d}\phi(t)$$

Key remark

$(e^{-itA})_{t\in\mathbb{R}}$ unitary group $\implies$ $(e^{-t(\pm iA)})_{t\geq 0}$ are contraction semigroups

All existing actions are based on the analysis of a semigroup e^{-tA} for $t \to 0$

III Gibbs semigroup on Banach space

Goal: extend this when $\mathcal{H}$ is replaced by a Banach space $\mathcal{B}$

Theory of C_0-semigroups on $\mathcal{B}$ has a long history: Hille–Phillips, Yosida,…

$(U(t)_{t\geq 0}$ has a generator A
i.e. $U(t) = e^{-tA}$
which is a closed and densely defined operator

But

No general trace theory on Banach spaces

About traces

Natural trace on an ideal of $\mathcal{L}(\mathcal{H})$

$$\operatorname{tr}(A) = \sum_i \langle e_i, Ae_i \rangle$$

for an orthonormal basis $\{e_i\}_i$ of $\mathcal{H}$
V. Lidskiĭ proved in 1959 that it coincides with the spectral functional

$$\operatorname{tr}_\sigma(A) = \sum_i \lambda_i(A)$$

Main difficulty: prove that tr_σ is additive

But there exists $\mathcal{B}$ without Schauder basis
$(e_i)_i \subset \mathcal{B}$: every $x \in \mathcal{B}$ has a unique decomposition $x = \sum_i c_i\, e_i$, $c_i \in \mathbb{R}, \mathbb{C}$

Just few definitions!

From now on, $\mathcal{B}$ is a separable Banach space

Uniqueness $\implies$ a corresponding sequence $\{e_i^*\} \subset \mathcal{B}^*$, called the *coordinate functionals*: $\langle e_i^*, e_j \rangle = \delta_{ij}$ and $c_i = \langle e_i^*, x \rangle$ for all $x \in \mathcal{B}$

A *biorthogonal sytem* on $\mathcal{B}$ is given by a family $\{(e_i, f_i^*)\}_{i\in\mathbb{N}} \subset \mathcal{B} \times \mathcal{B}^*$ such that $\langle f_i^*, e_j \rangle = \delta_{ij}$

Such system is called

- *fundamental* if $\overline{\text{span}\{e_i\}_i}^{\|\cdot\|_{\mathcal{B}}} = \mathcal{B}$
- *shringking* whenever $\overline{\text{span}\{f_i^*\}_i}^{\|\cdot\|_{\mathcal{B}^*}} = \mathcal{B}^*$
- *bounded (and $\{e_i\}_i$ uniformly minimal)* if $\{e_i\}_i$ and $\{f_i^*\}_i$ are both bounded,
- A *Steinitz basis* if $\forall x \in \mathcal{B},\ g^* \in \mathcal{B}^*$, there exists a permutation $\{\pi(i)\}_i$

$$\langle g^*, x \rangle = \sum_{i=1}^{\infty} \langle f_{\pi(i)}^*, x \rangle \langle g^*, e_{\pi(i)} \rangle$$

Just few definitions!

A *Markushevich basis (M-basis)* on $\mathcal{B}$ is given by a biorthogonal system $\{(e_i, f_i^*)\}_{i\in\mathbb{N}} \subset \mathcal{B} \times \mathcal{B}^*$ satisfying:

- $\overline{\text{span}\{e_i\}_i}^{\|\cdot\|_{\mathcal{B}}} = \mathcal{B}$
- $\overline{\text{span}\{f_i^*\}_i}^{\text{weak}^*} = \mathcal{B}^*$

An M-basis $\{(e_i, f_i^*)\}_{i\in\mathbb{N}}$ is

- *strong* if $x \in \overline{\text{span}\{\langle f_i^*, x\rangle\, e_i\}}^{\|\cdot\|_{\mathcal{B}}}$ for all $x \in \mathcal{B}$,
- *a strongly series summable* if there exists a triangular matrix of scalars (λ_{ij}) such that

$$x = \|\cdot\|\text{-}\lim_n \sum_{i=1}^{n} \lambda_{ni} \langle f_i^*, x\rangle\, e_i$$

$\mathcal{B}$ has the AP (MAP) if $\mathbb{1}_{\mathcal{B}}$ is approximated by finite rank operators (of norm 1) uniformly on compact sets in $\mathcal{B}$

Just few results!

- Every $\mathcal{B}$ admits an M-basis that is not a Schauder basis under any permutation
- Johnson: every $\mathcal{B}$ admits an M-basis that is not strong
- Terenzi: every $\mathcal{B}$ admits an M-basis $\{(e_i, f_i^*)\}_{i\in\mathbb{N}}$ with an increasing sequence of finite subset F_n of $\mathbb{N}$ such that

$$x = \|\cdot\|\text{-}\lim_n \sum_{i\in F_n} \langle f_i^*, x\rangle\, e_i \quad \text{for all } x \in \mathcal{B}$$

- Pełczyński: every $\mathcal{B}$ admits a $(1+\varepsilon)$-bounded M-basis
- Enflo: there exist a reflexive $\mathcal{B}$ without the (AP) and without a Schauder basis
- Szarek: there exists a reflexive $\mathcal{B}$ with the (AP) but without a Schauder basis
- Grothendieck: every reflexive $\mathcal{B}$ with the (AP) is such that both $\mathcal{B}$ and $\mathcal{B}^*$ have the (MAP)

Just few results!

Let $\mathcal{B}$ be a separable Banach space

Theorem

The following are equivalent:
(i) $\mathcal{B}$ is reflexive
(ii) There is an M-basis $\{(e_i, f_i^*)\}_{i\in\mathbb{N}}$ that is both shrinking and boundedly complete

Theorem (Terenzi)

Every $\mathcal{B}$ has a bounded (convex) strong M-basis which is also a Steinitz basis

Grothendieck approach

For $0 < p \leq 1$, we define on $\mathcal{B}^* \otimes \mathcal{B}$ the p-quasi-norm as

$$\|A\|_p := \inf \{ \big(\sum_{i=1}^{n} \|x_i^*\|^p \, \|y_i\|^p \big)^{1/p} \mid A = \sum_{i=1}^{n} x_i^* \otimes y_i \in \mathcal{B}^* \otimes \mathcal{B} \} \,,$$

and denote by $\mathcal{B}^* \widehat{\otimes}_p \mathcal{B}$ the completion of $\mathcal{B}^* \otimes \mathcal{B}$ for $\|\cdot\|_p$
When $p = 1$, we denote by $\|\cdot\|_\pi := \|\cdot\|_1$ the usual projective norm

Any $A \in \mathcal{B}^* \widehat{\otimes}_p \mathcal{B}$ can be represented as $A = \sum_{i=1}^{\infty} x_i^* \otimes y_i$ with $\|A\|_p < \infty$

Action of $x^* \otimes y$ on $z \in \mathcal{B}$ defined as $(x^* \otimes y)(z) := \langle x^*, z \rangle \, y \in \mathcal{B}$
thus one can define a map

$$\widetilde{\ }: A \in \mathcal{B}^* \widehat{\otimes}_p \mathcal{B} \mapsto \widetilde{A} \in \mathcal{L}(\mathcal{B}) \quad \text{by } \widetilde{A}\, z = \sum_{i=1}^{\infty} \langle x_i^*, \, z \rangle \, y_i \text{ for } z \in \mathcal{B}$$

Grothendieck approach

An operator in $\mathcal{L}(\mathcal{B})$ is a *p-nuclear operator* when it is of the form $\widetilde{A}$ with $A \in \mathcal{B}^* \widehat{\otimes}_p \mathcal{B}$

The nuclear p-quasi-norm (or norm if $p = 1$) of an operator $X \in \mathcal{L}(\mathcal{B})$:

$$\|X\|_{\mathcal{N}_p} := \inf\{\|A\|_p \,|\, A \in \mathcal{B}^* \widehat{\otimes}_p \mathcal{B}\,,\ \widetilde{A} = X\}$$

Note the p-triangle inequality: $\|X + Y\|_{\mathcal{N}_p}^p \leq \|X\|_{\mathcal{N}_p}^p + \|Y\|_{\mathcal{N}_p}^p$

$\mathcal{N}(\mathcal{B}) \subset \mathcal{K}(\mathcal{B})$

Delicate point: $\widetilde{\ }$ is injective on $\mathcal{B}^* \widehat{\otimes}_\pi \mathcal{B}$ if and only if $\mathcal{B}$ has the approximation property

Definition: *(quasi-)Banach operator ideal* $\mathcal{U}(\mathcal{B})$ is a subspace of $\mathcal{L}(\mathcal{B})$ with a (quasi-)norm $\|\ \|_{\mathcal{U}(\mathcal{B})}$ on $\mathcal{U}(\mathcal{B})$ such that for $A \in \mathcal{U}(\mathcal{B})$ and $B, C \in \mathcal{L}(\mathcal{B})$

$\mathcal{F}(\mathcal{B}) \subset \mathcal{U}(\mathcal{B})$ with normalization

$BAC \in \mathcal{U}(\mathcal{B})$ with $\|BAC\|_{\mathcal{U}(\mathcal{B})} \leq \|B\|_{\mathcal{L}(\mathcal{B})} \|A\|_{\mathcal{U}(\mathcal{B})} \|C\|_{\mathcal{L}(\mathcal{B})}$

$\mathcal{U}(\mathcal{B})$ is complete for $\|\cdot\|_{\mathcal{U}(\mathcal{B})}$

$\Longrightarrow \|A\| \leq \|A\|_{\mathcal{U}}$

Grothendieck approach

Trace on $\mathcal{U}(\mathcal{B})$:

linear map $\tau : \mathcal{U} \to \mathbb{C}$ with $\tau(AB) = \tau(BA)$ for all $A \in \mathcal{U}$ and $B \in \mathcal{L}(\mathcal{B})$

Grothendieck: $\mathcal{N}(\mathcal{B})$ is an Banach operator ideal and

$$\mathrm{tr}: \; A = \sum_{i=1}^{\infty} x_i^* \otimes y_i \mapsto \mathrm{tr}(A) := \sum_{i=1}^{\infty} \langle x_i^*, y_i \rangle$$

is independent of the representations of A in $\mathcal{N}(\mathcal{B})$ if and only if the space $\mathcal{B}$ has the (A.P.)

In this case, it is a trace (by extension of finite sum) with $|\mathrm{tr}\, A| \leq \|A\|_{\mathcal{N}}$

Problem: if $\mathcal{B}$ is not isomorphic to a Hilbert space, there exists $A \in \mathcal{N}(\mathcal{B})$ such that $\sum_j |\lambda_j(A)| = \infty$

Key property: we need good spectral properties to work on generators of semigroups, like $\mathrm{tr} = \mathrm{tr}_\sigma$

$$\longrightarrow \qquad \mathcal{N}_{\ell^1}(\mathcal{B}) := \{A \in \mathcal{N}(\mathcal{B}) \; \text{with} \; \lambda(A) \in \ell^1\}$$

Problem: $\mathcal{N}_{\ell^1}(\mathcal{B})$ is not necessarily a vector space

Grothendieck approach

König: $\mathcal{N}_{\ell^1}(\mathcal{B}) = \mathcal{N}(\mathcal{B})$ if and only if $\mathcal{B}$ is a Hilbert space

Grothendieck:

if $A \in \mathcal{N}(\mathcal{B})$ then $\lambda(A) \in \ell^2(\mathbb{N})$ since

$$\sum_i |\lambda_i(A)|^2 \leq \|A\|_{\mathcal{N}}^2 \quad A \in \mathcal{N}(\mathcal{B})$$

Strategy

Several possibilities to get

$$\operatorname{tr} A = \operatorname{tr}_\sigma A \text{ for any } A \in \mathcal{N}_{\ell^1}(\mathcal{B})$$

(1) Find quasi-Banach operator ideals on which $\operatorname{tr} = \operatorname{tr}_\sigma$ for any Banach space $\mathcal{B}$

(2) To look at Banach spaces $\mathcal{B}$ (called Lidskiĭ spaces) where this holds for all $A \in \mathcal{N}_{\ell^1}(\mathcal{B})$

Strategy for (1)

Theorem (Blunck–Weis)

Let $(\mathcal{U}, \|\cdot\|_{\mathcal{U}})$ be a normed operator ideal in $\mathcal{L}(\mathcal{B})$. Then, given a differentiable C_0-semigroup $(e^{-tA})_{t\geq 0}$, if $(A - \lambda \mathbb{1})^{-1} \in \mathcal{U}$ for some $\lambda \in \rho(A)$, then $e^{-tA} \in \mathcal{U}$ for all $t > 0$

Application: $\mathcal{U} = \mathcal{N}(\mathcal{B})$ and let A be a generator of differentiable semigroup on a Banach space $\mathcal{B}$ such that $A^{-1} \in \mathcal{N}(\mathcal{B})$. Then $e^{-tA} \in \mathcal{N}(\mathcal{B})$ for $t > 0$

Possible extensions to quasi-Banach operator ideal for analytic semigroups

Pietsch approach

A. Pietsch (1974) introduced on $\mathcal{B}$ a substitute to singular values of an operator on a Hilbert space

$s: A \in \mathcal{L}(\mathcal{B}) \to \{s_n(A)\}_{n\in\mathbb{N}} \in [0,\infty[$ is a sequence of *s-numbers* if

$$\|A\| = s_1(A) \geq s_2(A) \geq \cdots \geq 0$$

$$s_{m+n-1}(A+B) \leq s_m(A) + s_n(B), \text{ for } A, B \in \mathcal{L}(\mathcal{B})$$

$s_n(XAY) \leq \|X\|\, s_n(A)\, \|Y\|$ for arbitrary $X \in \mathcal{L}(\mathcal{B}_0, \mathcal{B})$, $Y \in \mathcal{L}(\mathcal{B}, \mathcal{B}_1)$ and arbitrary Banach spaces $\mathcal{B}_0$, $\mathcal{B}_1$

if rank $A < n$, then $s_n(A) = 0$

$$s_n(\mathrm{Id}: \ell^2 \to \ell^2) = 1$$

Pietsch approach

Examples of s-numbers for any $A \in \mathcal{L}(\mathcal{B})$ and $n \in \mathbb{N}$

Approximation numbers:

$$a_n(A) := \inf \{ \|A - F\| \mid F \in \mathcal{F}(\mathcal{B}) \text{ with rank } F < n\}$$

Weyl numbers:

$$x_n(A) := \sup \{a_n(AB) \text{ with } B \in \mathcal{L}(\ell^2, \mathcal{B}),\ \|B\| \leq 1\}$$

Gelfand numbers:

$$c_n(A) := \sup\{a_n(BA) \mid B \in \mathcal{L}(\mathcal{B}, \ell^\infty),\ \|B\| \leq 1\}$$

Kolmogorov numbers:

$$d_n(A) := \sup\{a_n(AB) \mid B \in \mathcal{L}(\ell^1, \mathcal{B}),\ \|B\| \leq 1\}$$

Entropy numbers:

$$e_n(A) := \inf \Big\{\varepsilon > 0 : \exists\, x_1, \ldots, x_{2^{n-1}} \in \mathcal{B} \text{ with } A(B_{\mathcal{B}}) \subseteq \textstyle\bigcup_{j=1}^{2^{n-1}} B(x_j, \varepsilon)\Big\}$$

Pietsch: **They coincide with singular values for a Hilbert space**

Pietsch approach

Define

$$\mathcal{L}^1_{(s)}(\mathcal{B}) := \{A \in \mathcal{L}(\mathcal{B}) \,|\, (s_n(A)_n \in \ell^1\} \text{ for s=a, x, c, d, e}$$

Pietsch: All these $\mathcal{L}^1_{(s)}(\mathcal{B})$ are quasi-normed operator ideals

Theorem

(i) $\mathcal{L}^1_{(a)}(\mathcal{B}) \subset \mathcal{N}_{\ell^1}(\mathcal{B})$
(ii) Equality if and only if $\mathcal{B}$ *is isomorphic to a Hilbert space*

A good list of operator ideals

A list of operator ideals with $\mathrm{tr} = \mathrm{tr}_\sigma$

Case $\mathfrak{P}_2 \circ \mathfrak{P}_2 \subset \mathcal{N}$: Here $\mathfrak{P}_2$ is the set of 2-summing operators on $\mathcal{B}$ such that there exists $c \geq 0$ satisfying

$$\Big(\sum_{k=1}^{n} \| A x_k \|^2\Big)^{1/2} \leq c \sup_{\|f^*\|_{\mathcal{B}^*} \leq 1} \Big(\sum_{k=1}^{n} |\langle f^*, x_k \rangle|^2\Big)^{1/2}$$

for every finite family $x_1, \cdots, x_n$ in $\mathcal{B}$

Case $(\mathfrak{P}_2)_{(a)}^{2,1} \subset \mathcal{N}$:

$$(\mathfrak{P}_2)_{(a)}^{2,1} := \{A \in \mathfrak{P}_2 \mid \|A\|_{(a)}^2 := \textstyle\sum_{n=1}^{\infty} n^{-1/2} a_n(A) < \infty\}$$

Case $\mathcal{L}_{(a)}^1 \cup \mathcal{L}_{(c)}^1 \subseteq (\mathfrak{P}_2)_{(a)}^{2,1}$ (approximation and Gelfand numbers)

Case $\mathcal{N}_{(r,s),p}$: introduced by O. Reinov: for $0 < r, s \leq 1$, $1/r = 1/p + 1/2$ with $1 \leq p \leq 2$, and $A \in \mathcal{L}(\mathcal{B})$

$$\|A\|_{\mathcal{N}_{(r,s),\,p}} := \inf\Big\{ \|(\alpha_i)_{i=1}^{\infty}\|_{\ell^{r,s}} \, \|(x_i)_{i=1}^{\infty}\|_{\ell^\infty(\mathcal{B})} \, \|(y_i)_{i=1}^{\infty}\|_{\ell_w^{p'}(\mathcal{B})} \;\Big|\; A = \sum_{i=1}^{\infty} \alpha_i \, x_i \otimes y_i \Big\}$$

A good list of operator ideals

Generalization

Case $\mathcal{L}^1_{(x)}$ and $\mathcal{L}^1_{(e)}$ (Weyl and entropy numbers)
have always spectral traces but possibly no nuclear traces:

Proposition

There exist $A \in \mathcal{L}^1_{(x)}(\mathcal{B})$ for a reflexive Banach space $\mathcal{B}$ with the (M.A.P.) which is not nuclear
$\mathcal{L}^1_{(x)} \not\subset \mathcal{N}$

Strategy for (2)

$\mathcal{B}$ (with the AP) is called a *Lidskiĭ space* when the two traces coincide:

$$\operatorname{tr} A = \operatorname{tr}_\sigma\ A\,, \quad \text{for any } A \in \mathcal{N}_{\ell^1}(\mathcal{B})$$

Theorem (Figiel–Johnson)

The following are equivalent for a Banach space $\mathcal{B}$ with the AP
(i) $\mathcal{B}$ is a Lidskiĭ space
(ii) Every quasi-nilpotent nuclear operator on $\mathcal{B}$ has zero trace
(iii) $\mathcal{B}$ has the nest approximation property

Remark:

A Lidskiĭ space has the hereditary approximation property

This exclude all L^p spaces for $2 \neq p \in [1, \infty]$ (Davie–Figiel–Szankowski)

This include a large set of $\mathcal{B}$ wich include weak Hilbert spaces

Strategy for (2)

One can work within the class

$$\mathbb{L} = \{\mathcal{B} = \text{ reflexive Banach space with the AP}\}$$

since more stable than the class of Lidskiĭ spaces

Theorem

Let $\mathcal{B} \in \mathbb{L}$
(i) There exists a strongly series summable M-basis $\{(e_i, f_i^*)\}_{i=1}^{\infty}$ for $\mathcal{B}$ associated to a lower-triangular matrix of scalars (λ_{mi})
(ii) If $A \in \mathcal{N}(\mathcal{B})$, then its nuclear trace is
$\operatorname{tr} A = \lim_{m\to\infty} \sum_{i=1}^{m} \lambda_{mi} \langle f_i^*,\, A\, e_i\rangle < \infty$

Remark

Let $\mathcal{B} \in \mathbb{L}$ endowed with an M-basis $\{e_i, f_i^*\}_{i=1}^{\infty}$. Then the following are equivalent:
(i) $\operatorname{tr} A = \sum_{i=1}^{\infty} \langle f_i^*,\, A\, e_i\rangle < \infty$ for any nuclear operator A
(ii) $\{e_i\}_i$ is a Schauder basis

What else?

Project

With all this machinery:

Compute the behaviour of a Gibbs semigroup $(e^{-tA})_{t\geq 0}$ when $t \to 0$

Link with the Trotter formula, which for Hilbert space looks like (with hypothesis on A and B!)

$$\|e^{-t(A+B)} - \big(e^{-tA/n}\, e^{-tB/n}\big)^n\|_1 \asymp \mathcal{O}(n^{-1})$$

What else?

Whish me good luck, and thank you