Loop models on causal dynamical triangulations and height-coupled trees

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1. Loop models on causal triangulations

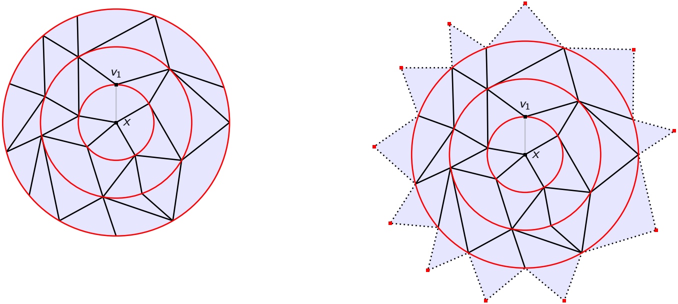
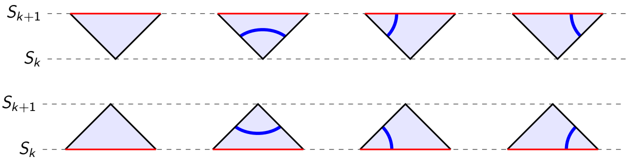


Figure: Causal triangulation C (Ambjørn+Loll, 1998) of the disk (left) with distinguished edge $\{x, v_1\}$, augmented to a triangulation of the sphere (right).

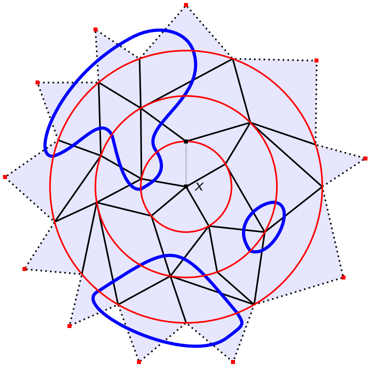
Spacelike slices: $S_k = \{v \mid d(x, v) = k\}$ $k = 0, 1, \ldots, h(C)$

(red circles)

The loop model



Decorations of elementary triangles by arcs.



Loop configuration on

a causal triangulation of the sphere.

$\mathcal{L}_m = \{\text{loop configurations } L \text{ of height } m\}$

$|L| = \sharp\{\text{spacelike edges in } L\} = \frac{1}{2}\sharp\{\text{triangles in } L\} = \frac{1}{2}(\text{area of } L)$

$t(L) = \sharp\{\text{timelike arcs in } L\}$ $\quad$ $s(L) = \sharp\{\text{spacelike arcs in } L\}$

$\ell(L) = t(L) + s(L) = \text{length of } L$

Triangle weight : $g^{\frac{1}{2}}$

Timelike arc weight: α

Spacelike arc weight: 1

Partition functions:

$$Z_m(g,\alpha) = \sum_{L\in\mathcal{L}_m} g^{|L|}\alpha^{t(L)}$$

$$Z(g,\alpha) = \sum_{m=1}^{\infty} Z_m(g,\alpha)\,.$$

Note:

1) $$Z(g,\alpha=0)=\sum_{m=1}^{\infty}2^m Z_m^{CDT}(g^{\frac{1}{2}})\,,$$

where

$$Z_m^{CDT}(g)=\sum_{C\in\mathcal{C}_m} g^{|C|}$$

$\mathcal{C}_m=\{\text{causal triangulations } C \text{ of height } m\}\,,\quad |C|=\text{area of C.}$

2) $$Z(g,\alpha)=Z^{Ising}(g,\alpha,\beta=1)\,,$$

where Z^{Ising} is the partition function for the (anisotopic) Ising model on a CDT defined by

$$Z^{Ising}(g,\alpha,\beta)=\sum_{L\in\mathcal{L}} g^{|L|}(\alpha\beta)^{t(L)}\beta^{s(L)}\,,$$

corresponding to the Hamiltonian

$$H_C(\sigma)=-J_s\sum_{\langle ij\rangle_{\text{spacelike}}}\sigma_i\sigma_j-J_t\sum_{\langle ij\rangle_{\text{timelike}}}\sigma_i\sigma_j$$

for an Ising spin configuration $\sigma=\{\sigma_i\}_{i\in C}$ on C and where

$$\alpha=e^{-J_s}\,,\quad \beta=e^{-J_t}\,,\qquad J_s,J_t\geq 0\,.$$

2. Correspondence with trees

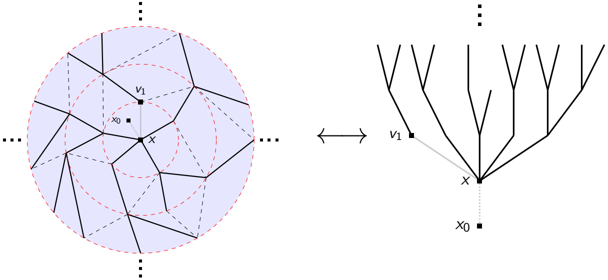


Figure: Causal triangulation of the disk C and the corresponding tree $\psi(C)$.

$\mathcal{C}_m(N) = \{\text{causal triangulations } C \text{ of height } m \text{ with } N \text{ vertices}\}$

$\mathcal{T}_m(N) = \{\text{planar trees } T \text{ of height } m+1 \text{ with } N \text{ edges}\}$

• Bijection $\psi : \mathcal{C}_m(N) \to \mathcal{T}_m(N)$ given by deleting spacelike edges and the rightmost outgoing edge from every vertex $v \neq x$ and adding a root edge (x_0, x) in triangle to the left of marked edge in C.

Note: $V(C) = V(\psi(C)) \setminus x_0$

Labelled trees:

$\widetilde{\mathcal{T}}_m(N) = \{(T, \delta) \mid T \in \mathcal{T}_m(N), \sum_{v \in S_k} \delta(v) = \text{even}\}$

• $\psi : \mathcal{C}_m(N) \to \mathcal{T}_m(N)$ extends to $\widetilde{\psi} : \mathcal{L}_m(N) \to \widetilde{\mathcal{T}}_m(N)$ by setting $\delta(v) = 1$ if spacelike edge to the left of v is intersected by an arc, else $\delta(v) = 0$.

Taking into account that

$$\widetilde{\psi} \text{ is } 2^m \text{ to } 1 \quad \text{and} \quad |\delta| = \frac{1}{2} t(L)\,,$$

we have

$$Z_m(g,\alpha) = 2^m \sum_{(T,\delta)\in\widetilde{\mathcal{T}}_m} g^{|T|-1}\alpha^{2|\delta|}\,,$$

where $|T| = \sharp\{\text{edges in } T\} :=$ size of T and $\widetilde{\mathcal{T}}_m = \cup_N \widetilde{\mathcal{T}}_m(N)$.

Loop models on causal triangulations

Correspondence with trees

Phase structure

Tree models

Questions

Tree partition functions

Unlabelled tree:

$$W(g) = \sum_{T \in \mathcal{T}} g^{|T|-1} = \frac{1-\sqrt{1-4g}}{2g}, \quad g_{crit} = \frac{1}{4}$$

Labelled tree:

$$W(g,\alpha) = \sum_{(T,\delta)\in\widetilde{\mathcal{T}}} 2^{h(T)} g^{|T|-1} \alpha^{|\delta|}$$

$$= \sum_{m=1}^{\infty} \sum_{T \in \mathcal{T}_m} g^{|T|-1} \prod_{i=1}^{m} \left[(1+\alpha)^{n_i(T)} + (1-\alpha)^{n_i(T)}\right],$$

where $n_i(T) = \sharp\{\text{vertices in } S_i\}$ and $\mathcal{T} = \cup_m \mathcal{T}_m$.

Height-coupled tree:

$$W(g;k) = \sum_{T\in\mathcal{T}} k^{h(T)} g^{|T|-1}$$

$$g_{crit}(k) = \begin{cases} \frac{1}{4} & \text{if } 0 \le k \le 1 \\ \frac{k}{(k+1)^2} & \text{if } k > 1 \text{ (simple pole!)} \end{cases}$$

<u>Note</u>:

$$W(g,\alpha^2) = Z(g;\alpha) \qquad (*)$$
$$W(g,\alpha=0) = W(g;k=2) \qquad (**)$$
$$W(g,\alpha=1) = W(2g) = Z^{CDT}((2g)^{\frac{1}{2}}) \qquad (***)$$

3. Results on phase structure

• Hausdorff dimension d_h: $|B_R(C)| \sim R^{d_h}$ for $R \to \infty$ where $B_R(C)$ is the ball of radius R around the centre x.

a) From $(***)$ follows that

$$Z^{CDT}(g) \sim Z^{CDT}(g_{crit}) - cst.(g_{crit} - g)^{\frac{1}{2}}$$
$$= 2 - 2(\frac{1}{2} - g)^{\frac{1}{2}},$$

and Hausdorff dimension $d_h(\alpha = 1) = 2$.

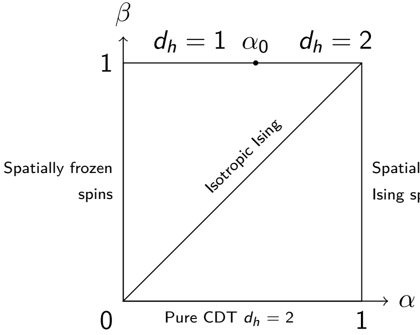
b) From $(**)$ follows that the loop model for $\alpha = 0$ is in a different universality class than pure CDT:

$$Z(g, \alpha = 0) \sim \frac{cst}{g_{crit}(0) - g} \qquad (\star)$$

and $d_h(\alpha = 0) = 1$.

Thm. The behaviour ($\star$) and $d_h(\alpha) = 1$ persist for α sufficiently small.

Two possible scenarios:
i) $d_h(\alpha) = 1$ for $0 \leq \alpha < 1$ and $d_h(\alpha = 1) = 2$.
ii) There exists $\alpha_0 < 1$ such that $d_h(\alpha) = 1$ for $\alpha < \alpha_0$ and $d_h(\alpha) = 2$ for $\alpha > \alpha_0$.

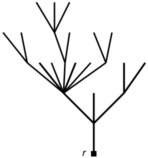


4. Tree models

Generic planar trees

$\mathcal{T}(N) = \{$planar rooted trees with N edges$\}$; root r of degree 1
$\sigma_i =$ degree of vertex $i \in T$
$B_R(T) =$ ball of radius R in T around the root

$$\overline{\mathcal{T}} = \mathcal{T} \cup \mathcal{T}(\infty)$$
$$d(T, T') = \inf\{\tfrac{1}{R} \mid B_R(T) = B_R(T')\}$$
$$(\overline{\mathcal{T}}, d)\,, \text{complete separable metric space}$$



Convergence of measures:

$$\mu_N \to \mu \quad \text{if} \quad \mu_N(\mathcal{B}_a(T)) \to \mu(\mathcal{B}_a(T))\,, T \in \mathcal{T}, a > 0,$$

where $\mathcal{B}_a(T)$ is the ball (in $\overline{\mathcal{T}}$) of radius a around T.

Branching (or offspring) probabilities:

$$p_n \geq 0,\ n = 0, 1, 2, \ldots \quad \sum_n p_n = 1$$

define a (Galton-Watson) branching process with average offspring

$$m = \sum_n n p_n$$

- Critical case: $m = 1$ Subcritical case: $m < 1$

Finite size partition function: $Z_N = \sum_{T \in \mathcal{T}_N} \prod_{i \in T \setminus r} p_{\sigma_i - 1}$

Generating function: $Z(g) = \sum_{N \geq 1} Z_N g^N$

Finite size measure: $\mu_N(T) = \frac{\prod_{i \in T \setminus r} p_{\sigma_i - 1}}{Z_N},\ T \in \mathcal{T}(N)$

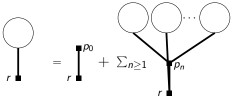
Example (uniform tree): $\quad p_n = 2^{-(n+1)},\ n \geq 0,$

$$\prod_{i \neq r} p_{\sigma_i - 1} = 2^{\sum_{i \neq r} \sigma_i} = 2^{-2|T|+1}, \qquad Z_N = 2^{-2N+1} \sharp \mathcal{T}(N),$$

$$\mu_N(T) = \frac{1}{\sharp \mathcal{T}(N)},\ \ T \in \mathcal{T}(N)$$

Basic equation for partition functions:

$$\boxed{\mathrm{Z(g) = g\ f(Z(g))}}$$

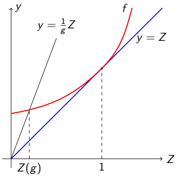


where $\quad f(z) = \sum_{n=0}^{\infty} p_n z^n$

Generic assumption:

a) (p_n) is critical, i.e. $\sum_{n=0}^{\infty} n p_n = 1$
(valid for uniform tree)

b) Convergence radius for f is $\rho > 1$
($\rho = 2$ for uniform tree)

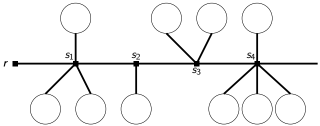


Then

$$Z(g) = 1 - \sqrt{\frac{2}{f''(1)}}(1-g)^{\frac{1}{2}} + O(1-g)$$

$$Z_N = (2\pi f''(1))^{-\frac{1}{2}} N^{-\frac{3}{2}}(1 + O(N^{-1}) \ (N \to \infty)$$

Theorem a) The limit $\mu = \lim_{N\to\infty} \mu_N$ exists and is concentrated on trees with a single spine and with branches at spine vertices s_i that are i.i.d. Galton-Watson trees with offspring probabilities p_n and σ_{s_i} are i.i.d. with probability $(\sigma_{s_i} - 1)p_{\sigma_{s_i}-1}\,,\ \sigma_{s_i} \geq 2$.



b) Hausdorff dimension $d_h = 2$, i.e.
$|B_R(T)| \ \sim \ R^2$ for $R \to \infty$ for almost all T.

• The infinite size limit of causal dynamical triangulations exists and $d_h = 2$ almost surely.

Trees with height dependent weights (with M. Ünel, 2023)

Set $A_{m,N} = \sharp\mathcal{T}_m(N)$ and define

$$\mu_N^{(\zeta)}(T) = \frac{e^{-\zeta h(T)}}{Z_N^{(\zeta)}}\,,\;\; T \in \mathcal{T}(N)$$

,
where $Z_N^{(\zeta)} = \sum_{T\in\mathcal{T}(N)} e^{-\zeta h(T)} = \sum_{m=1}^{\infty} e^{-\zeta m} A_{m,N}$

Results:

a) For $\zeta = 0$ the measure $\mu_N^{(0)}$ coincides with the uniform measure on $\mathcal{T}(N)$.

b) For $\zeta < 0$ generic trees still have a single spine, but branches become subcritical, yielding $d_h = 1$.

c) $\zeta > 0$: We have $A_{m,N} = B_{m,N} - B_{m-1,N}$, where

$$B_{m,N} = 4^N \sum_{k=1}^{\lfloor \frac{m}{2} \rfloor} \frac{1}{m+1} \tan^2 \frac{\pi k}{m+1} \Big(1 + \tan^2 \frac{\pi k}{m+1}\Big)^{-N},\;\; N \geq 2$$

The asymptotic form of $Z_N^{(\zeta)}$ is obtained by a saddlepoint approximation:

$$Z_N^{(\zeta)} = (e^{\zeta} - 1)\sqrt{\frac{\pi}{B}}\frac{\zeta}{2}e^{-AN^{\frac{1}{3}}}N^{-\frac{5}{6}}4^N(1 + N^{-\delta})\ (\delta > 0)$$

$$A = 3\left(\frac{\pi\zeta}{2}\right)^{\frac{2}{3}}, \qquad B = 3\left(\frac{\zeta^2}{4\pi}\right)^{\frac{2}{3}}$$

Theorem Assume $\zeta > 0$. The $\lim_{N\to\infty}\mu_N^{(\zeta)} = \mu^{(\zeta)}$ exists. It is concentrated on $\mathcal{T}(\infty)$ and determined by

$$\mu^{(\zeta)}(\mathcal{B}_{\frac{1}{r}}(T)) = e^{-\zeta(r-1)}4^{-|T|}2^{K+1}\sum_{R=1}^{K}\binom{K}{R}\frac{\zeta^{R-1}}{(R-1)!}$$

for any finite tree T of height r and with K vertices at height r.

Description of $\mu^{(\zeta)}$:

The *spine* T^s of $T \in \mathcal{T}_\infty$ is defined as the subtree of T spanned by vertices of *infinite type*, i.e. vertices having infinitely many descendants in T.

Let $\mathcal{T}^s = \{T \in \mathcal{T}_\infty \mid$ all vertices in T are of infinite type$\}$. Then the spine map $\chi : T \to T^s$ maps $\mathcal{T}_\infty$ to $\mathcal{T}^s$ and $\mu^{(\zeta)}$ gives rise to a measure $\tilde{\mu}^{(\zeta)}$ on $\mathcal{T}^s$ by

$$\tilde{\mu}^{(\zeta)}(E) = \mu^{(\zeta)}(\chi^{-1}(E))\,,\ E \subseteq \mathcal{T}^s$$

.

Theorem $\tilde{\mu}^{(\zeta)}$ is a *Poisson tree* defined by

$$\tilde{\mu}^{(\zeta)}(\mathcal{B}^s_{\frac{1}{r}}(T)) = e^{-\zeta(r-1)}\frac{\zeta^{R-1}}{(R-1)!}\,,\ r \geq 1$$

for any finite tree T of height r with R leaves, all at height r.

Corollary The random variables

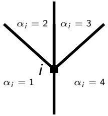
$$\tau_r(T^s) = |B_{r+1}(T^s)| + |B_{r-1}(T^s)| - 2|B_r(T^s)|$$

are i.i.d. with distribution

$$\tilde{\mu}^{(\zeta)}(\tau_r = n) = e^{-\zeta}\frac{\zeta^n}{n!}\,, \quad n \geq 0.$$

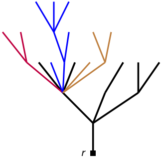
- This implies $d_h(T^s) = 2$ for $\tilde{\mu}^{(\zeta)}$ - a.e. T^s.

Vertex i in T^s has σ_i angular sectors:

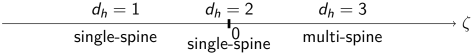


Vertex i of degree 4
with 4 angular sectors.

Theorem $\mu^{(\zeta)}$ is obtained from $\tilde{\mu}^{(\zeta)}$ by decorating T^s with branches T_{i,α_i}, $\alpha_i = 1, \ldots, \sigma_i$, in each angular sector, independently and identically distributed as Galton-Watson trees with offspring probability $p_n = 2^{-(n+1)}$



Theorem $d_h(T) = 3$ holds $\mu^{(\zeta)}$ - a.s.



5. Questions

- Spectral dimensions of $\mu^{(\zeta)}$ and $\tilde{\mu}^{(\zeta)}$ for $\zeta > 0$ not known.
- Generalisations: introduce height-dependence for general generic GW-trees.
- Implications for randomly triangulated models, in particular CDT coupled to e.g. spin systems or to loop models.