Spectral functionals for geometric invariants

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Given Laplace type operator and using Wodzicki residue I will recall certain spectral functionals of vector fields, the densities of which reproduce volume (integral), metric, scalar curvature and Einstein tensors. Alternatively, given Dirac type operator, I will describe analogous functionals of differential forms which yield the dual tensors. In the latter setup we recently introduced also spectral *torsion* functional which recovers the torsion tensor T for the canonical Dirac operator coupled to T. These functionals often generalize to noncommutative geometry. In particular, the conformally rescaled noncommutative torus, Einstein-Yang-Mills and quantum SU(2)-group spectral triples are torsion free, while the quantum 2-sheeted space and the almost commutataive geometry of the Standard Model do have torsion. I will comment on relation to the algebraic notion of torsion and Levi-Civita connection, and present impact on the other spectral functionals.
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(with A. Sitarz, P. Zalecki, A. Bochniak, Y. Liu, S. Mukhopadhyay and F. Požar).

Cracow, 30 September 2025

Torsion in Spectral Geometry:

For a (pseudo) Riemannian M, $\exists!$ connection ∇ on TM preserving metric g with a prescribed torsion $T(X,Y) = \nabla_X Y - \nabla_Y X$; in particular Levi-Civita connection with $T = 0$. ♠ Φ, Beggs-Majid …

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An eminent spectral scheme that yields geometric quantities on M dim$M = n\,(= 2m)$, such as volume, scalar curvature ..., is $t \searrow 0$ asymptotic expansion of the trace of heat kernel

$$\mathrm{Tr}\, e^{-t\Delta} \approx \sum_{\ell=0}^{\infty} t^{\frac{\ell-n}{2}} a_\ell.$$

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The coefficients a_ℓ can be expressed via Wodzicki residue $\mathcal{W}$ ♠

$$\mathcal{W}(P) := \frac{1}{vol(S^{n-1})} \int_M \left(\int_{|\xi|=1} tr\, \sigma_{-n}(P)(x,\xi)\ \mathcal{V}_\xi \right)\ d^n x.$$

Namely

$$a_{2k}(\triangle) \propto \mathcal{W}(\triangle^{k-m}) \quad \text{for} \quad k < m > 1.$$

Then, on closed oriented M

Geometry from $\mathcal{W}$

noncommutative integral *functional (of $f \in C^\infty(M)$)*

$$\mathcal{V}^\Delta(f) \equiv ⨍(f) := \mathcal{W}(f\Delta^{-m}) = \int_M f \; vol_g \tag{1}$$

scalar curvature *functional on $C^\infty(M)$*

$$\mathcal{R}^\Delta(f) := \mathcal{W}(f\Delta^{-m+1}) = \frac{n-2}{12}\int_M f R \, vol_g, \tag{2}$$

where $R = R(g) = g^{jk}R_{jk} = g^{jk}R_{\ell j \ell k}$ is the scalar curvature.

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↬ When $f \equiv 1$, (1) is equivalent to the Weyl formula
and (2) is $\propto$ to the Einstein-Hilbert action functional (of g) for Riemannian general relativity (in vacuum) & also 2nd expansion coefficient of the spectral action functional.
↬ Related to the asymptotic growth of eigenvalues of Δ;
clear e.g. from the 'trace theorem' of Connes that $\mathcal{W} = \mathrm{Tr}^+$.

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↬ We've uncovered a few new spectral 'localised' functionals, by placing some differential operators in place of f.

→ new fnls

New functionals

E.g. for two vector fields V, W on M (derivations of $C^\infty(M)$):

Def/Thm: Metric functional

The functional $\quad \mathrm{g}^{\Delta}(\mathrm{V},\mathrm{W}) := \mathcal{W}\big(\mathrm{VW}\Delta^{-\mathrm{m}-1}\big)$

is a bilinear, symmetric map, whose density is proportional to the metric g evaluated on V, W

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Pf. By tedious symbol calculus in normal coordinates.

Laplace-type, Spin Laplacian, squared Dirac

We computed also more general Laplace-type operators

$$\Delta_{T,E} = -g^{ab}(\nabla_a \nabla_b - \Gamma^c_{ab} \nabla_c) + E$$

on a vector bundle Ξ with connection ∇ and $E \in End\ \Xi$.

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A particular interesting case is a $spin_c$ manifold M with Ξ a spinor bundle Σ of rank 2^m and the *spin Laplacian*

$$\Delta^{(s)} := \nabla^{(s)*}\nabla^{(s)} = -\nabla^{(s)}_{e_i}\nabla^{(s)}_{e_i} + \nabla^{(s)}_{\nabla_{e_i} e_i},$$

where $\nabla^{(s)}$ is the spin connection and e_j is ON frame:

$$\mathrm{g}^{\Delta^{(s)}}(V,W) := \mathcal{W}\big(\nabla^{(s)}_V \nabla^{(s)}_W (\Delta^{(s)})^{-m-1}\big) = 2^m \mathrm{g}^{\Delta}(V,W),$$

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or *squared Dirac* (coupled do $U(1)$-gauge 1-form A):

$$\mathrm{g}^{\not{D}_A^2}(V,W) := \mathcal{W}(\nabla^{(s)}_V \nabla^{(s)}_W |\not{D}_A|^{-n-2}) = 2^m \mathrm{g}^{\Delta}(V,W),$$

$$\mathrm{G}^{\not{D}_A^2}(V,W) := \mathcal{W}(\nabla^{(s)}_V \nabla^{(s)}_W |\not{D}_A|^{-n})$$

$$= 2^m\Big(\mathrm{G}^{\Delta}(V,W) + 2^{-3}\int_M R\, g(V,W) vol_g\Big).$$

Can go quantum (= noncommutative)

Example: noncommutative torus with smooth algebra $A = C^\infty(\mathbb{T}^n_\theta)$, generated by n unitaries U_j,

$$U_j U_k = e^{i\theta_{jk}} U_k U_j.$$

It has faithful state τ invariant under derivations δ_j, $\delta_j U_k = \delta_{jk} U_k$, which are interpreted as noncommutative vector fields.

One regards $\Delta = \sum_j \delta_j^2$ on $H = L^2(\mathbb{T}^2_\theta, \tau)$ as 'flat' Laplace operator. This generalises to the (non-flat) conformally rescaled geometry:

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For simplicity consider the *strictly irrational* $\mathbb{T}^n_\theta$ (i.e., $\mathcal{Z}(A) = \mathbb{C}$) with τ extended to $\hat{A} := A \otimes A^o$ as $\tau(a \otimes b^o) = \tau(a)\tau(b^o)$, where A^o is a copy of A in the commutant A' of A in $B(H)$. Such τ is still invariant under the extended derivations. We use it to define the tracial state $\mathcal{W}$ on $\hat{A}$-valued symbols $\sigma(\xi)$ (where $\delta_a \mapsto \xi_a$ much the same as for M).

Rescaled NC 2-torus: vector fields

Given $0 < h \in A = C^\infty(\mathbb{T}^2_\theta)$, by *conformally rescaled* Δ on $\mathbb{T}^2_\theta$ we mean the selfadjoint operator on $H = L^2(\mathbb{T}^2_\theta, \tau)$:

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$$\Delta_h = h^{-1} \Delta \, h^{-1}.$$

Accordingly, as vector fields we take

$$V_h = \sum_{a=1,2} V^a h \delta_a h^{-1}, \quad V^a \in A^o.$$

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Proposition

$$g^{\Delta_h}(V_h, W_h) = \mathcal{W}\big(V_h W_h \Delta_h^{-2}\big) = \pi \tau(h^4) V^a W^a,$$

whereas

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We have also computed $\mathbb{T}^4_\theta$.

Can do also θ-deformed spaces, or NC spaces with derivations.

Alternatively & better. . .

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use $\not{D}$ on spinors in Σ in a deeper way to get (via $\mathcal{W}$)
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which yield contravariant tensors (with "raised indices").

For that use the Clifford representation of $v \in \Omega^1(M)$
as 0-order differential operators $\hat{\nu} \in \mathrm{End}(\Sigma)$.
They form a $C^\infty(M)$-bimodule $\Omega^1_{D\!\!\!/} \simeq \Omega^1(M)$ generated by $[D\!\!\!/, f]$.
Thus the spinorial Dirac operator is 'self-sufficient' for our purposes
(and NCG-ready when assembled to a spectral triple of A. Connes).

Metric and Einstein functionals on 1-forms

Thm

The spectral functionals of one-forms on M

$$\mathrm{g}_{\not{D}}(v,w) := \mathcal{W}\big(\hat{v}\hat{w}\not{D}^{-n}\big)$$

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$$= \mathcal{W}\big((D\!\!\!/\hat{v} + \hat{v}D\!\!\!/)\hat{w}D\!\!\!/^{-n+1}\big) = \frac{2^m}{6}\int_M G(v,w)\ vol_g,$$

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They perfectly (dually) match g^{Δ} and G^{Δ} up to a coefficient.

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Actually,

$$\mathrm{Ric}_{\not D}(v,w) := \mathcal{W}\big(\hat{v}(\not D\hat{w} + \tfrac{n-4}{n-2}\hat{w}\not D)\not D^{-n+1}\big) = \tfrac{2^m}{6}\int_M Ric(v,w)\ vol_g.$$

Rescaled noncommutative 2-torus: 1-forms

The above functionals extend to NC tori $A = C^\infty(\mathbb{T}^n_\theta)$ with the conformal rescaling D_k of the 'flat' Dirac operator $D = \sum_j \gamma^j \delta_j$ on $H = L^2(\mathbb{T}^2_\theta, \tau) \otimes \mathbb{C}^{2^m}$ which we take on H as

$$D_k = kDk,$$

following Connes-Moscovici, however with $0 < k \in A^o \subset A'$.
This assures that (A, D_k, H) is a spectral triple and $\exists$ an A-bimodule (of 1-forms) $\Omega^1_{D_k}(A)$ generated by $[D_k, A]$. ♠
In effect, $\Omega^1_{D_k}(A)$ is freely generated by $k^2\gamma^j$.

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For $n = 2$, $\gamma^j = \sigma^j$, and for $\mathbb{T}^2_\theta$ we have

Proposition

For $v = k^2 v^j \sigma^j$ and $w = k^2 w^j \sigma^j$, $v^j, w^j \in A$,

$$\mathrm{g}_{D_k}(v, w) = \tau(v^j w^j),$$

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→ torsion

Spectral Torsion

These functionals extend to any n-summable regular $(\mathcal{A}, D, \mathcal{H})$, using as $\mathcal{W}$ the tracial state ($\text{Res}|_{s=n}\text{Tr}|D|^{-s}$) on the ΨDO calculus by Connes-Moscovici'95. Thus, even though such abstract D or Δ may not refer in principle to any *connection* (covariant derivative), thanks to our g_D we can now 'control' the *metricity* condition.
How about the *zero-torsion* condition ?
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Clearly torsion can 'contaminate' $\mathcal{V}$, $\mathcal{R}$, g & G (it does R & G !). Fortunately:

Def/Thm: Torsion functional

Torsion functional *is a trilinear functional of* $u, v, w \in \Omega^1_D(\mathcal{A})$,

$$\mathcal{T}_D(u, v, w) := \mathcal{W}(uvwD|D|^{-n}).$$

We say that D *is torsion-free if* $\mathcal{T}_D \equiv 0$. *For the Dirac operator* $\not{D}_T$ *with torsion* T *on a closed spin manifold of dimension* n

$$\mathcal{T}_{\not{D}_T}(u, v, w) = -2^{[\frac{n}{2}]} i \int_M T(u, v, w)\, vol_g. \quad (3)$$

→ exa

Examples with $\mathcal{T} = 0$

- Hodge-de Rham: $\big(C^\infty(M), L^2(\Omega^\bullet_M), d + d^*\big)$.
- Einstein-Yang-Mills: $\Big(C^\infty(M) \otimes M_N(\mathbb{C}), L^2(\Sigma) \otimes M_N(\mathbb{C}), \widetilde{D}\Big)$,

where $\widetilde{D} = \not{D} \otimes \mathrm{id}_N + A + JAJ^{-1}$ with $A = A^* \in \Omega^1_{\widetilde{D}}$ and $J = C \otimes *$, with C being the charge conjugation on spinors in Σ.

- conformally rescaled noncommutative tori.

- quantum $SU(2)$: $\big(\mathcal{A}(SU_q(2)), \mathcal{H}, D\big)$,

where $\mathcal{H}$ and D are isomorphic to the classical case $q = 1$.

Examples with $\mathcal{T} \neq 0$

- quantum $\mathbb{Z}_2$: $(A,H,D)=\left(\mathbb{C}^2,\mathbb{C}^2,\begin{pmatrix}0 & \phi\\ \phi^* & 0\end{pmatrix}\right)$ (for $n=0$, $\mathcal{W}$=Tr)
- quantum 2-sheeted space $M\times\mathbb{Z}_2$:

$$(\mathcal{A},\mathcal{H},\mathcal{D}) = \left(C^\infty(M)\otimes\mathbb{C}^2, L^2(\Sigma)\otimes\mathbb{C}^2, \begin{pmatrix}\not{D} & \chi\phi\\ \chi\phi^* & \not{D}\end{pmatrix}\right)$$

where χ is grading on Σ and $\phi\in\mathbb{C}$.

Now, $\Omega^1_{\mathcal{D}} \ni \omega = \begin{pmatrix} w^+ & \phi\chi f^+\\ \phi^*\chi f^- & w^-\end{pmatrix}$ for $w^\pm\in\Omega^1(M)$, $f^\pm\in C^\infty(M)$.

Then, $\mathcal{T}_{\mathcal{D}}(\omega_1^o,\omega_2^o,\omega_3^o) = |\phi|^4\int_M(f_1^+f_2^-f_3^+ + f_1^-f_2^+f_3^-)vol_g$.

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Now, $\Omega^1_{\mathcal{D}} \ni \omega = \begin{pmatrix} w^+ & \phi\chi f^+ \\ \phi^*\chi f^- & w^- \end{pmatrix}$ for $w^\pm \in \Omega^1(M)$, $f^\pm \in C^\infty(M)$.

Then, $\mathcal{T}_{\mathcal{D}}(\omega_1^o, \omega_2^o, \omega_3^o) = |\phi|^4 \int_M (f_1^+ f_2^- f_3^+ + f_1^- f_2^+ f_3^-) vol_g$.

- internal quantum geometry of the Standard Model: (A, H, D), where $A = (\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}))$, $H = \mathbb{C}^{96}$ and D is built from CKM & PMNS mixing matrices Υ. Then eg. for $u = [D, (0, 1_2, 0)]$,

$$\mathcal{T}_D(u, u, u) = \text{Tr}\left(|\Upsilon_e|^4 + |\Upsilon_\nu|^4 + 3|\Upsilon_u|^4 + 3|\Upsilon_d|^4\right) > 0.$$

Examples with $\mathcal{T} \neq 0$

• quantum $\mathbb{Z}_2$: $(A,H,D) = \left(\mathbb{C}^2, \mathbb{C}^2, \begin{pmatrix} 0 & \phi \\ \phi^* & 0 \end{pmatrix}\right)$ (for $n=0$, $\mathcal{W}$=Tr)

• quantum 2-sheeted space $M \times \mathbb{Z}_2$:

$$(\mathcal{A},\mathcal{H},\mathcal{D}) = \left(C^\infty(M)\otimes\mathbb{C}^2, L^2(\Sigma)\otimes\mathbb{C}^2, \begin{pmatrix} \not{D} & \chi\phi \\ \chi\phi^* & \not{D} \end{pmatrix}\right)$$

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• full almost commutative geometry of the Standard Model

Relation to 'algebraic' torsion

A priori quite different. Common territory is the algebraic 1stODC realized in terms of operators associated to the spectral triple, but must supply a 2ndODC and choose a connection.

Easy for the inner triple $(A, H, D) = \left(\mathbb{C}^2, \mathbb{C}^2, \begin{bmatrix} 0 & \phi \\ \phi^* & 0 \end{bmatrix}\right)$ on $\mathbb{Z}_2$:

an arbitrary (left) connection is

$$\nabla : \Omega^1 \to \Omega^1 \otimes_A \Omega^1, \quad de \mapsto \begin{bmatrix} c^+ & 0 \\ 0 & c^- \end{bmatrix} de \otimes de, \quad c_\pm \in \mathbb{C},$$

where $e = (1, 0) \in \mathbb{C}^2$ is represented as $\mathrm{diag}(1, 0)$ on $\mathbb{C}^2$.

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Its torsion reads

$$T^\nabla := m \circ \nabla - d : \Omega^1 \to \Omega^2, \quad \mapsto - \begin{bmatrix} c^+ \phi^* \phi & 0 \\ 0 & c^- \phi \phi^* \end{bmatrix}.$$

Then, for a unique connection $\nabla^{(1,-1)}$ with $c^\pm = \pm 1$, we reproduce the (discrete) spectral torsion, i.e. $\forall u, v, w \in \Omega^1$:

$$\mathcal{T}^{\nabla^{(1,-1)}}(u, v, w) := \mathit{Tr}\left(uv T^{\nabla^{(1,-1)}}(w)\right) = \mathit{Tr}(uvwD) =: \mathcal{T}^D(u, v, w),$$

where in the last eq. we used matrix Tr as $\mathcal{W}$ (!) [Connes].

Relation to 'algebraic' torsion: $M \times \mathbb{Z}_2$

The full torsionful case $M \times \mathbb{Z}_2$ requires some adjustments...

We first work with Connes' differential calculus, but quotient by the so called 'junk' differential ideal kills the torsion generated from $\mathbb{Z}_2$ and agreement can be only got with an altered spectral torsion.

To overcome this problem we adopt a recent modification in Mesland-Rennie'24 (see also [BM,BGM,BGJ]) of the algebraic approach and provide its ingredients appropriate for $M \times \mathbb{Z}_2$.

Since it is a metric product we use a product-type connection, which however realizes only part of the spectral torsion.

To overcome this unexpected *impasse* we resort to a non-product connection by adding a suitable mixing perturbation term. With this we ultimately achieve a perfect agreement with the spectral approach

Some details

Extend $\pi : \mathcal{A} \to B(\mathcal{H})$ to $\pi_{\mathcal{D}} : \Omega^1_u(\mathcal{A}) \to \Omega^1_{\mathcal{D}}(\mathcal{A})$ via $\delta \mapsto [\mathcal{D}, \cdot]$.

For an idempotent Ψ on $\Omega^1_{\mathcal{D}}(\mathcal{A}) \otimes \Omega^1_{\mathcal{D}}(\mathcal{A})$ which satisfies

$$JT^2_{\mathcal{D}} \subseteq \mathrm{Im}(\Psi) \subseteq \mu^{-1}(J^2_{\mathcal{D}}),$$

where

$$JT^2_{\mathcal{D}} = \{\pi_{\mathcal{D}} \otimes \pi_{\mathcal{D}}(\delta(w)) \,|\, w \in \ker \pi_{\mathcal{D}}\}, \quad J^2_{\mathcal{D}} = \mu(JT^2_{\mathcal{D}}),$$

there is a 2ndODC

$$\mathcal{A} \xrightarrow{[\mathcal{D},\cdot]} \Omega^1_{\mathcal{D}}(\mathcal{A}) \xrightarrow{\mathrm{d}_\Psi} \Omega^1_{\mathcal{D}}(\mathcal{A}) \otimes \Omega^1_{\mathcal{D}}(\mathcal{A}),$$

where

$$\mathrm{d}_\Psi(a[\mathcal{D}, b]) := (1 - \Psi)\left([\mathcal{D}, a] \otimes_{\mathcal{A}} [\mathcal{D}, b]\right), \ \forall a, b \in \mathcal{A}.$$

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We have found Ψ and S such that the torsion

$$T^{\widetilde{\nabla}}_{\Psi} := (1 - \Psi) \circ \widetilde{\nabla} - \mathrm{d}_\Psi : \Omega^1_{\mathcal{D}}(\mathcal{A}) \to \Omega^1_{\mathcal{D}}(\mathcal{A})\otimes\Omega^1_{\mathcal{D}}(\mathcal{A})$$

of the connection

$$\widetilde{\nabla} = \nabla^{LC} \otimes 1 + \chi \otimes \nabla^{(1,-1)} + S,$$

satisfies $\forall u, v, w \in \Omega^1_{\mathcal{D}}(\mathcal{A})$

$$\mathcal{T}_\Psi(u, v, w) := \mathcal{W}\Big(uv\mu(T^{\widetilde{\nabla}}_{\Psi}(w))\Big) = \mathcal{W}(uvw\mathcal{D}|\mathcal{D}|^{-n}) =: \mathcal{T}_{\mathcal{D}}(u, v, w).$$

$\mathcal{V}$, $\mathcal{R}$, g and G in presence of torsion

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$\mathcal{V}$, $\mathcal{R}$, g and G in presence of torsion

We worked out wide class of Dirac-type operators.

In particular, NC integral $\mathcal{V}_{\not{D}_T}$ and metric $\mathrm{g}_{\not{D}_T}$ are blind for T, but

$$\mathcal{R}_{\not{D}_T}(f) = \frac{2^m(n-2)}{24}\int_M f\left[-R + \frac{9}{4}T^0_{abc}T^0_{abc}\right] vol_g,$$

✓ Pfäffle-Stephan, ✗ usual R_T;

$$\mathrm{G}_{\not{D}_T}(u,w) = \mathrm{G}_{\not{D}}(u,w) +$$

$$3\cdot 2^{m-1}\int_M \left[-u_a\, w_{bc}\, T^0_{abc} + \frac{1}{8}u_a w_b\left(\delta^{ab}T^0_{ijk}T^0_{ijk} - 4\, T^c_{ab} - 6\, T^0_{ajk}T^0_{bjk}\right)\right] vol_g$$

✗ impediment to torsion ?!, ✗ usual G_T.

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HAPPY BIRTHDAY ANDRZEJ !