Quantum One-Cross Bundles for Quantum Homogeneous Spaces

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Noncommutative geometry

Turning spaces into algebras

Compact Hausdorff spaces X, Y and continuous maps $f : X \to Y$

Compact Lie groups $(G, e, m : G \times G \to G, \cdot^{-1} : G \to G)$

Compact Riemannian spin manifolds (M, g)

Gelfand—Naimark

Unital, commutative C*-algebras $C(X), C(Y)$ and *-homomorphisms $\varphi : C(Y) \to C(X)$

Unital commutative C*-algebras $C(G)$ with coproduct $\Delta : C(G) \to C(G) \otimes C(G)$ and a density condition.

Spectral triples $(C(M), D, L^2(\mathcal{S}))$

Noncommutative geometry

Forgetting about the spaces

Compact Hausdorff spaces X, Y
and continuous maps $f : X \to Y$

Compact Lie groups
$(G, e, m : G \times G \to G, \cdot^{-1} : G \to G)$

Compact Riemannian spin manifolds
(M, g)

Generalising to the
noncommutaive setting

Unital C*-algebras A, B with
*-homomorphisms $\varphi : A \to B$

Compact quantum groups and
their dense Hopf *-subalgebras

Spectral triples
$(A, D, \mathscr{H})$

Question: *How do we construct spectral triples for quantum groups?*

Hopf Algebras and Quantum Homogeneous Spaces

Quantum homogeneous spaces

- Let A and U be a dual pair of Hopf $*$-algebras.
- Let $W \subset U$ be a Hopf $*$-subalgebra of U.
- Denote the space of invariants $B := {}^{W}A = \{a \in A \mid a_{(1)}\langle a_{(2)}, w\rangle = \epsilon(w)a, \text{ for all } w \in W\}$.
- If $A \otimes_B - : {}_B\mathrm{Mod} \to \mathrm{Vect}_{\mathbb{C}}$ is a faithfully flat functor then we say that B is a *quantum homogeneous space*.

Weak one-cross bundles

Definition: Let A and U be a dual pair of Hopf $*$-algebras, and let $W \subseteq U$ be a Hopf $*$-subalgebra such that the space of invariants $B := {}^{W}A$ is a quantum homogeneous space. A *weak one-cross bundle* for B is an element $X \in U \backslash W$ such that

- $\Delta(X) = X \otimes H + 1 \otimes X$, where $Y \in U$ is an element satisfying $\mathrm{Res}_W(Y) = 1_B$
- $X \triangleleft W$ is a finite-dimensional simple right W-module.

The definition separates into two possible cases:

1. $X^* \in X \triangleleft W$ (real case),

2. $X^* \notin X \triangleleft W$ (complex case).

Here we focus on the complex case, but for the real case things progress analogously.

So what can this strange structure do for quantum homogeneous spaces?

Differential graded algebras and differential calculi

Definition: A *differential graded algebra* over an algebra B is an $\mathbb{N}_0$-graded algebra

$$\Omega^\bullet \cong \bigoplus_{k \in \mathbb{N}_0} \Omega^k,$$

where $\Omega^0 = B$, together with a degree one homogeneous map $\mathrm{d} : \Omega^\bullet \to \Omega^\bullet$ such that

- $\mathrm{d}^2 = 0$, and
- $\mathrm{d}(\omega \wedge \nu) = \mathrm{d}\omega \wedge \nu + (-1)^{|\omega|} \omega \wedge \mathrm{d}\nu,$ for all $\omega, \nu \in \Omega^\bullet$.

If $\Omega^\bullet$ is generated as an algebra by those elements of degree 0 and 1, then we say that it is a *differential calculus* over B.

Lemma: *For a weak one-cross bundle* $(A, W \subseteq U, X)$, *the direct sum (internal to* U*)* $T := (X^* \triangleleft W) \oplus (X \triangleleft W)$ *is a quantum tangent space for* A, *that is,*

$$\Delta(T) \subseteq T \otimes A^\circ.$$

We denote the associated covariant $*$-differential calculus on A by $\Omega^1(A)$, and note that it is W-invariant.

Explicitly, the space of 1-forms is given by the free A-module

$$\Omega^1(A) := A \otimes \Lambda^1,$$

where we have denoted

$$I := \{y \in A^+ \,|\, X(y) = 0, \text{ for all } X \in T\}, \qquad \text{and} \qquad \Lambda^1 := \big(A^+/I\big).$$

The exterior derivative acts explicitly as

$$db = \sum_i (X_i \triangleright b) \otimes e_i, \qquad \text{where } \{X_i\}_i \text{ is a basis of } T, \text{ and } \{e_i\}_i \text{ is a dual basis of } \Lambda^1,$$

or equivalently,

$$db = b_{(1)} \otimes [b^+_{(2)}],$$

where $[b_{(2)}]$ denotes the coset of $b_{(2)}$ in $\Lambda^1 := A^+/I$

There exists an A-bimodule decomposition $\Omega^1(A) \simeq \Omega^{(1,0)} \oplus \Omega^{(0,1)}$, corresponding to the decomposition of the tangent space.

We denote the restriction of $\Omega^1(A)$ to B by $\Omega^1(B)$. By simplicity of $X \lhd W$, the dimension of the two calculi will be the same.

Moreover, we have an induced decomposition

$$\Omega^1(B) \simeq \Omega^{(1,0)} \oplus \Omega^{(0,1)}$$

Theorem: *A quantum principal bundle is given by the triple*

$$\left(B = {}^{W}A,\ \Omega^1(A),\ 0\right)$$

In particular, we have a noncommutative Atiyah sequence

$$0 \to A\Omega^1(B)A \to \Omega^1(A) \to A \otimes \Lambda_{W^\circ} = 0 \to 0.$$

Corollary: *The zero map is a **strong** left A-covariant principal connection for the quantum principal bundle.*

Connections

Definition: Let $\Omega^1(B)$ be a first-order differential calculus for an algebra B. Given a left B-module $\mathscr{F}$, a *connection* is a linear map

$$\nabla : \mathscr{F} \to \Omega^1 \otimes_B \mathscr{F}$$

such that

$$\nabla(bf) = \mathrm{d}b \otimes f + b\nabla(f), \qquad \text{for every } b \in B \text{ and } f \in \mathscr{F}.$$

Definition: A connection ∇ for a B-bimodule $\mathscr{F}$ is said to be a *bimodule connection* if

$$\nabla(fb) = \nabla(f)b + \sigma(f \otimes \mathrm{d}b) \text{ for every } b \in B \text{ and } f \in \mathscr{F},$$

for a *necessarily unique* bimodule map

$$\sigma : \mathscr{F} \otimes_B \Omega^1(B) \to \Omega^1(B) \otimes_B \mathscr{F}.$$

Classically, over a homogeneous space, how do we construct vector bundles?

We use the associated bundle approach!

For a homogeneous space G/K,

$$\mathscr{V} := G \times_K V := G \times V / \{(gk, v) = (g, \rho(k)v) \,|\, g \in G, k \in K\},$$

where $\rho : K \to \mathrm{End}(V)$ is a representation of K.

Fact/Exercise: It holds that

$$\Gamma(\mathscr{V}) \simeq O(G) \,\square_{U(\mathfrak{k})}\, V,$$

where the right hand side denotes all those sums

$$\sum_i f_i \otimes v_i \text{ in } \mathcal{O}(G) \otimes V$$

satisfying

$$\sum_i f_i \triangleleft X \otimes v_i = \sum_i f_i \otimes X \triangleright v_i, \text{ for all } X \in U(\mathfrak{k}).$$

So in the noncommutative setting, we think of the cotensor product

$$\mathscr{F} := A \,\square_W V\,,$$

for some finite-dimensional W-module V, as a *noncommutative associated vector bundle* over B.

Such modules are automatically finitely generated and projective, fitting into the Swan—Serre philosophy of noncomutative vector bundles.

Each such $\mathscr{F}$ comes equipped with a canonical right B-action, given by

$$\left(\sum_i a_i \otimes v_i\right) b = \sum_i a_i b \otimes v_i\,.$$

Lemma: *There exists a covariant connection for all relative Hopf modules.*

Theorem: *Moreover, any covariant connection for $\mathscr{F}$ is automatically a bimodule connection!*

Metrics

Definition: A ***metric*** for Ω^1 is a pair of bilinear maps, called the *evaluation* and *coevaluation* maps,

$$\mathrm{ev}: \Omega^1 \otimes_B \Omega^1 \to B, \text{ and } \mathrm{coev}: B \to \Omega^1 \otimes_B \Omega^1$$

such that the following identities hold

$$\mathrm{id} = (\mathrm{id} \otimes \mathrm{ev}) \circ (\mathrm{coev} \otimes \mathrm{id}) : \Omega^1 \mapsto \Omega^1 \otimes_B \Omega^1 \otimes_B \Omega^1 \to \Omega^1,$$

and

$$\mathrm{id} = (\mathrm{ev} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mathrm{coev}) : \Omega^1 \to \Omega^1 \otimes_B \Omega^1 \otimes_B \Omega^1 \mapsto \Omega^1.$$

Theorem: *There there exists a covariant metric for* $\Omega^1(B)$ *if and only if* $X \triangleleft W$ *and* $X^* \triangleleft W$ *are dual* W*-modules. In this case, the set of covariant metrics* g_λ *is parameterised by a scalar* $\lambda \in \mathbb{C}$.

Corollary: *Moreover, if we require* g_λ *to be quantum symmetric, that is, we require*

$$\wedge \circ g_\lambda = 0\,,$$

then λ *is uniquely defined.*

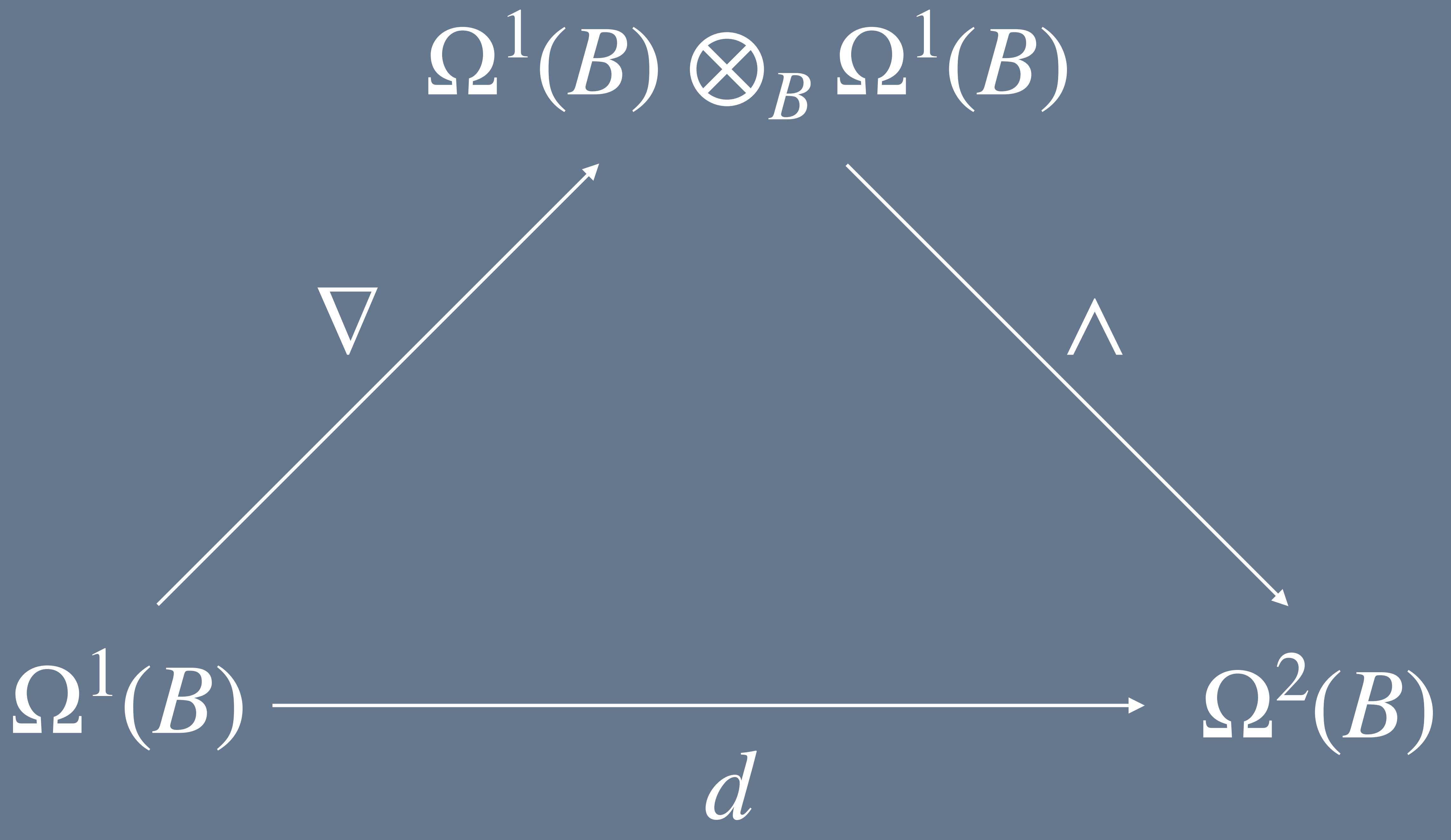
ONE-CROSS BUNDLES

Torsion-free connections

Definition: A connection $\nabla : \Omega^1 \to \Omega^1 \otimes_B \Omega^1$ is *torsion free* if

$$\mathrm{Tor} := \wedge \circ \nabla - d = 0,$$

that is, if the following diagram commutes:



Compatible connections

Definition: We say that a bimodule connection ∇ is *metric compatible* if

$\nabla'(g) = 0$, where we ∇' denotes the map

$$\nabla' := \nabla \otimes \mathrm{id} + (\sigma \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \nabla) : \Omega^1 \otimes_B \Omega^1 \to \Omega^1 \otimes_B \Omega^1 \otimes_B \Omega^1.$$

A *Levi-Civita connection* is a torsion-free metric compatible bimodule connection.

One-cross bundles

Definition: Let A and U be a dual pair of Hopf *-algebras, and let $W \subseteq U$ be a Hopf *-subalgebra such that the space of invariants $B := {}^{W}\!A$ is a quantum homogeneous space. A *one-cross bundle* for B is a weak one-cross bundle $X \in W$ such that W admits a central element Z, such that for any second or third root of unity θ,

$$X \triangleleft Z \neq \theta X\,.$$

Theorem: *The unique covariant connection* $\nabla : \Omega^1 \to \Omega^1 \otimes_B \Omega^1$ *is a Levi-Civita connection with respect to* g_λ.

Corollary: *For any one-cross bundle, there is a unique covariant connection*

$$\nabla : \mathscr{F} \to \Omega^1 \otimes_B \mathscr{F}, \quad \textit{for all simple} \quad \mathscr{F} \cong A \,\square_W V.$$

Corollary: *For any one-cross bundle* (W, X)*, the maximal prolongation* $\Omega^\bullet$ *of the associated first-order calculus* Ω^1 *has relations that are generated, as a* B*-bimodule, by the elements*

$$\omega \otimes \nu + \sigma(\omega \otimes \nu), \ \textit{for } \omega, \nu \in \Omega^1, \quad \textit{and} \quad \nabla(db), \textit{ for } b \in \mathrm{Gen}(B),$$

where $\mathrm{Gen}(B)$ *is a generating set of* B.

Complex structures

Theorem: *For a one-cross bundle of complex type we have an $\mathbb{N}_0^2$-algebra grading $\Omega^{(\bullet,\bullet)}$ on $\Omega^\bullet$ satisfying*

1. $\Omega^k \simeq \bigoplus_{a+b=k} \Omega^{(a,b)}$,

2. $(\Omega^{(a,b)})^* = \Omega^{(b,a)}$,

3. *the decomposition is integrable, that is, the decomposition of d with respect to $\Omega^{(\bullet,\bullet)}$ gives a double complex.*

Theorem: *If the degree two relations of the maximal prolongation are all of the form*

$$\omega \otimes \nu + \sigma(\omega \otimes \nu), \qquad \text{for all } \omega, \nu \in \Omega^1(B),$$

(that is, the torsion relations $\nabla(\mathrm{d})$ *can be forgotten) then the decomposition of the space of* 1*-forms* $\Omega^1(B) \cong \Omega^{(1,0)} \oplus \Omega^{(0,1)}$ *extends to a covariant complex structure on* $\Omega^\bullet(B)$.

Corollary: *For every simple relative Hopf module* $\mathcal{F} \in {}^{A}_{B}\mathrm{mod}$ *, the evident projection to a* $(0,1)$*-connection*

$$\overline{\partial}_{\mathcal{F}} : \mathcal{F} \to \Omega^{(0,1)} \otimes_B \mathcal{F}$$

is flat, which is to say , it is a holomorphic structure.

Theorem: *The complex structure is* ***factorisable****, which is to say*

$$\Omega^{(a,b)} \cong \Omega^{(a,0)} \oplus \Omega^{(0,b)}$$

if and only if

$$\sigma^2(\omega \otimes \nu) = \omega \otimes \nu, \qquad \forall \;\; \omega \otimes \nu \in \Omega^{(1,0)} \otimes \Omega^{(0,1)}, \text{ and } \Omega^{(0,1)} \otimes \Omega^{(1,0)}.$$

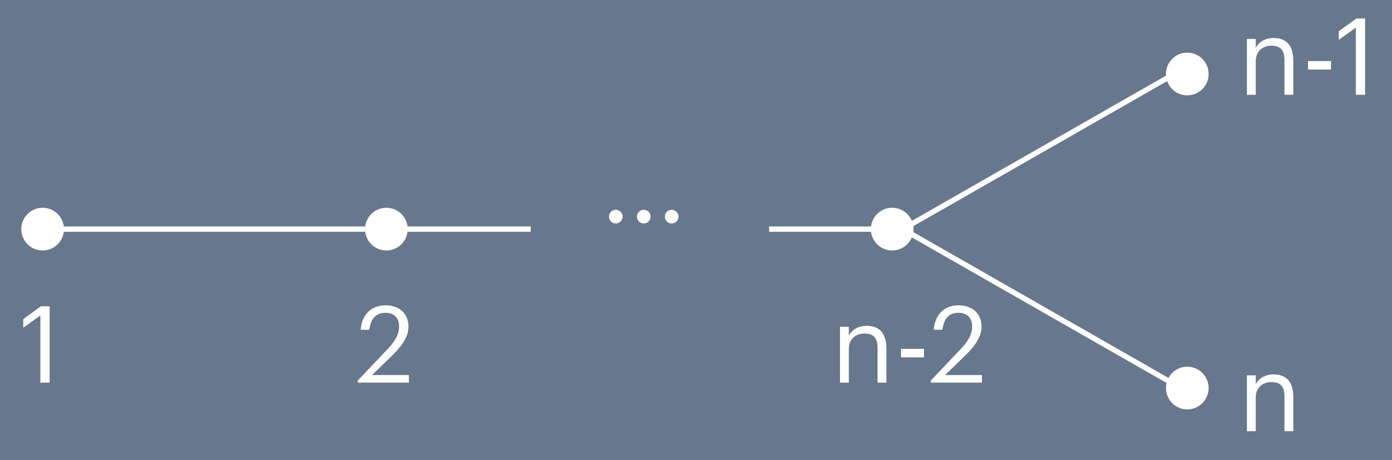
Examples

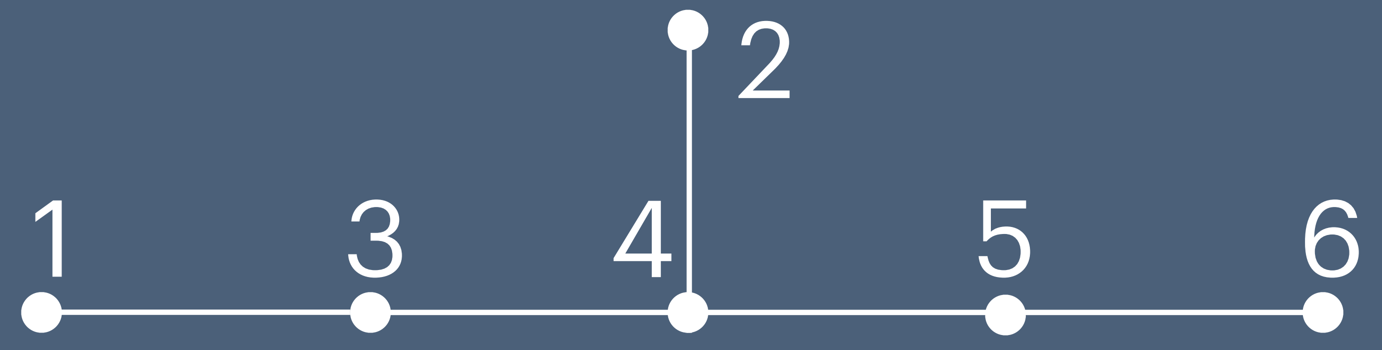
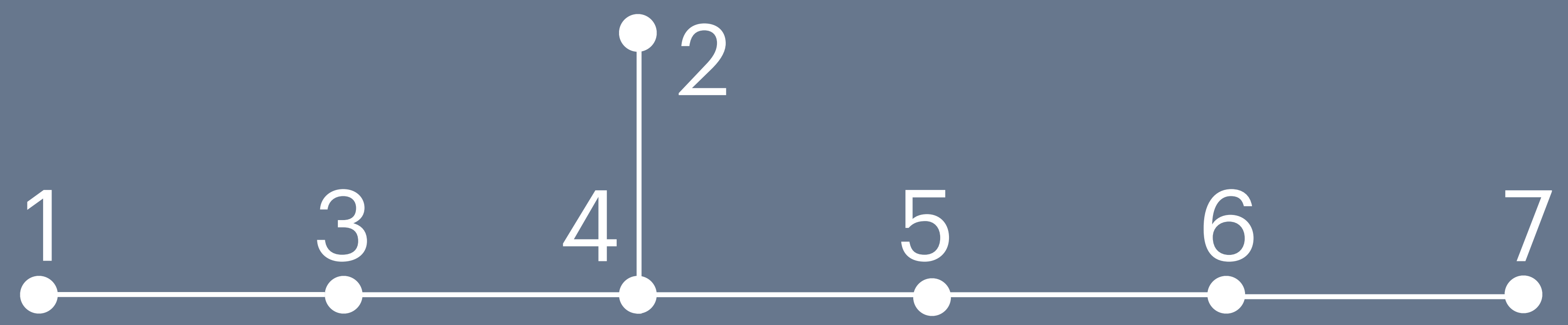
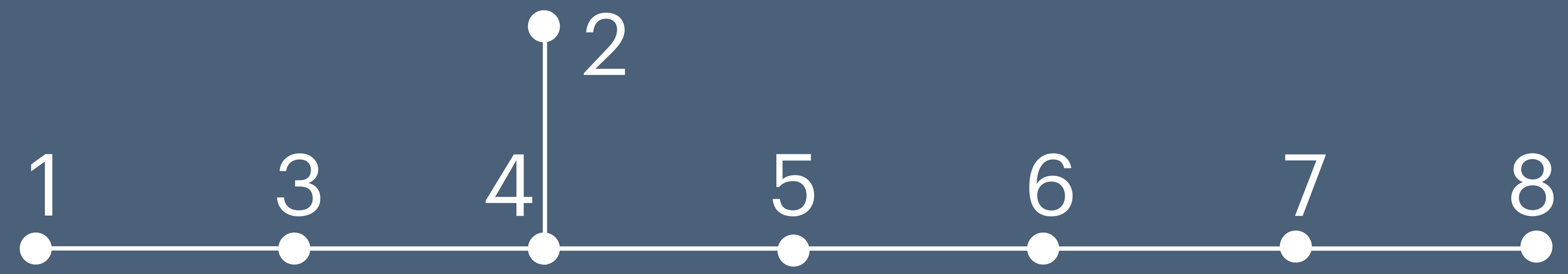
Example: If we take $A = \mathcal{O}_q(SU_2)$, $U = U_q(\mathfrak{sl}_2)$, and $W = \langle K, K^{-1} \rangle$, then we get the Podleś sphere. Choosing the element $X = E$ we get a one-cross bundle and recover the Podles calculus with the well-known holomorphic structures on its line modules $\mathcal{E}_k$.

Example: More generally, if we take $A = \mathcal{O}_q(SU_{n+1})$, $U = U_q(\mathfrak{sl}_{n+1})$ and

$$W = \langle E_i, F_i, K_i^{\pm 1} \mid i \neq 1 \rangle,$$

then we get quantum projective space $\mathcal{O}_q(\mathbb{C}P^n)$. If we set $X = E_1$, then we also get a one-cross bundle.

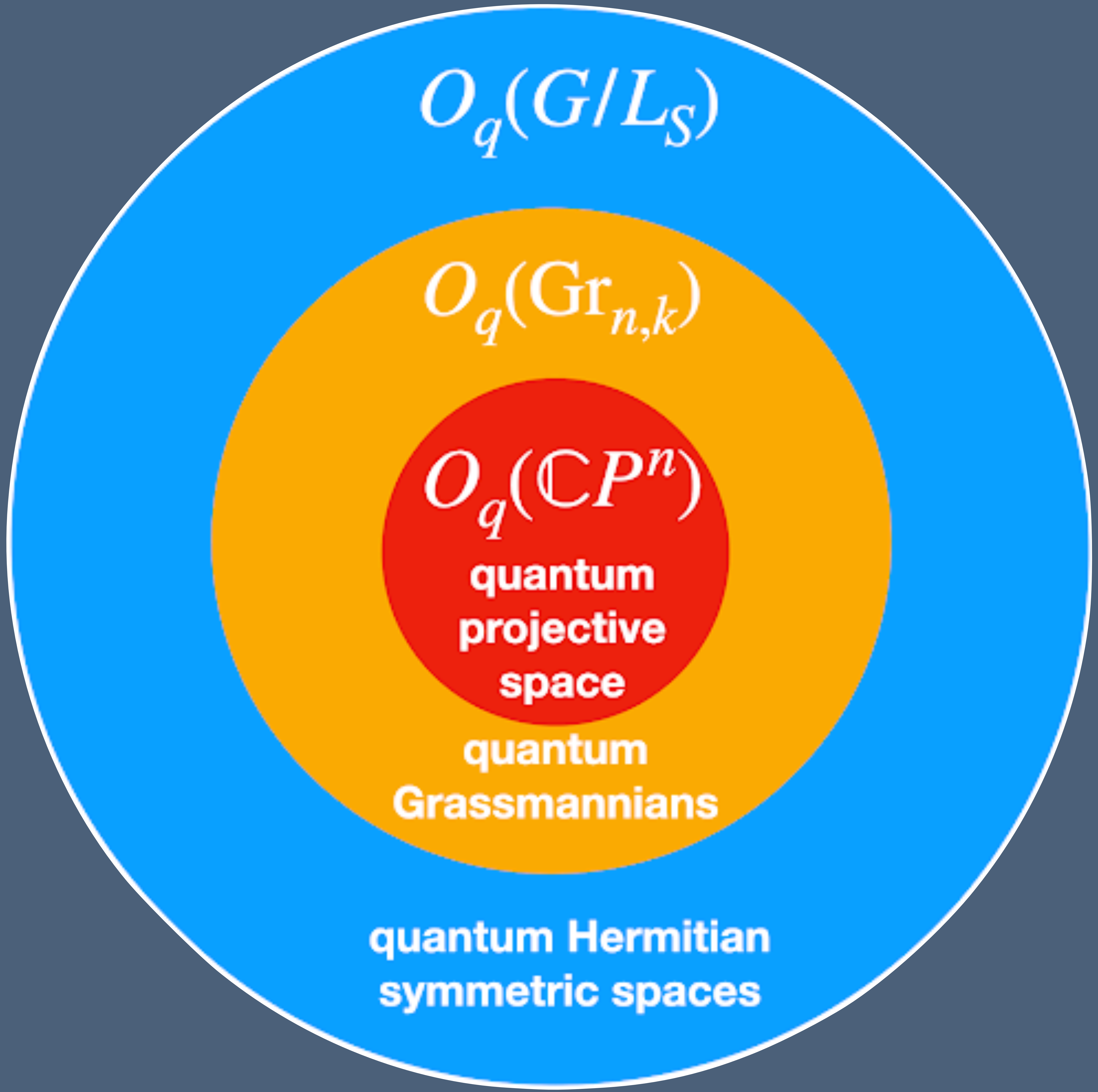
Root system	Dynkin diagram	Highest weight
A_n		$\alpha_1 + \cdots + \alpha_n$
B_n		$\alpha_1 + 2(\alpha_2 + \cdots + \alpha_n)$
C_n		$2(\alpha_1 + \cdots + \alpha_{n-1}) + \alpha_n$
D_n		$\alpha_1 + 2(\alpha_2 + \cdots + \alpha_{n-2}) + \alpha_{n-1} + \alpha_n$

Root system	Dynkin diagram	Highest weight
E_6		$\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$
E_7		$\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4$ $+3\alpha_5 + 2\alpha_6 + \alpha_7$
E_8		$2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4$ $+5\alpha_1 + 4\alpha_2 + 3\alpha_3 + 2\alpha_4$
F_4		$2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$
G_2		$3\alpha_1 + 2\alpha_2$

Example: Let $\mathfrak{g}$ be a complex simple Lie algebra, and G the associated connected simple-connected compact Lie group. If we take $A = \mathcal{O}_q(G)$, $U = U_q(\mathfrak{g})$ and

$$W = U_q(\mathfrak{l}_{\alpha_i}) := \langle E_i, F_i, K^{\pm 1} i \mid \alpha_i \text{ is not cominiscule} \rangle,$$

then we get the irreducible quantum flag manifolds $\mathcal{O}_q(G/L_S)$. If we set $X = E_x$, for α_x the cominiscule root, then we recover the celebrated **Heckenberger–Kolb differential calculi** and the holomorphic structures of their covariant noncommutative associated vector bundles.


$\mathcal{O}_q(G/L_S)$
$\mathcal{O}_q(\mathrm{Gr}_{n,k})$
$\mathcal{O}_q(\mathbb{C}P^n)$
quantum projective space
quantum Grassmannians
quantum Hermitian symmetric spaces

Example: Consider now the non-cominiscule case, which is to say, when

$$W = U_q(\mathfrak{l}_S) = \langle E_i, F_i, K_j \mid i \in \Pi \backslash \{\alpha_i\}, j \in \Pi \rangle.$$

For a non-cominiscule root α_i. Taking $X = E_i$ we again get a one-cross bundle, but this time we get **something totally new...**

We conjecture that the associated quantum exterior algebras will extend **Berenstein and Zwicknagl's** constructions beyond the cominiscule case.

★Now for something completely different

Example: The **free unitary quantum group** of Wang is the Hopf *-algebra generated by the generators u_{ij}, for $i, j = 1, \ldots, N$, subject to the following relations:

The matrix $u = (u_{ij})$ is **unitary,** that is to say

$$u^* u = u u^* = I_n,$$

or in components,

$$\sum_{k=1}^{n} u_{ki}^* u_{kj} = \delta_{ij}, \quad \sum_{k=1}^{n} u_{ik} u_{jk}^* = \delta_{ij} .$$

Example (continued): We have a surjective map

$$\pi : U_N^+ \to \mathcal{O}(U_N)$$

given by including the commutativity relations.

Dually, we have an inclusion map

$$\iota : \mathfrak{u}_N \hookrightarrow (U_N^+)^\circ .$$

Consider the **Levi subalgebra** $\mathfrak{l}_S \subseteq \mathfrak{u}_N$, for some $S = \Pi \setminus \{\alpha_x\}$. The **free flag manifold** is the quantum homogeneous space

$$F_{N,S}^+ := {}^{\iota(U(\mathfrak{l}_S))}U_N^+ .$$

The element $X = E_x$ gives a one-cross bundle for the free flag $F_{N,S}^+$.

Spectral Geometry?

DIRAC OPERATOR ON THE STANDARD PODLEŚ QUANTUM SPHERE

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Communications in Mathematical Physics

Noncommutative Riemannian and Spin Geometry of the Standard q-Sphere

S. Majid

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THE NONCOMMUTATIVE GEOMETRY OF THE QUANTUM PROJECTIVE PLANE

FRANCESCO D'ANDREA, LUDWIK DABROWSKI, and GIOVANNI LANDI

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Communications in Mathematical Physics

Dirac Operators on Quantum Projective Spaces

Francesco D'Andrea[1], Ludwik Dąbrowski[2]

We want a Dolbeault-Dirac operator

$$D_{\overline{\partial}} = \overline{\partial} + \overline{\partial}^\dagger$$

So we need to make sense of the adjoint $\overline{\partial}^\dagger$.

For an adjoint we need a metric!

On 1-forms $\Omega^1(B)$ we know how to do this, but how do we extend to higher forms?

- This is very much work in progress!
- So let's restrict to the addressing this problem for the irreducible quantum flags and their Heckenberger-Kolb calculi.

Warsaw, 2013


Why don’t you define the Hodge first?


Why don’t you define the Hodge first?
I’m allowed to do that!?


Why don’t you define the Hodge first?
I’m allowed to do that!?
Why not?


Now hurry up, you know those vegan burgers take hours to make.


I don’t think the people who work there get enough protein.

ELSEVIER

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Noncommutative Kähler structures on quantum homogeneous spaces

Réamonn Ó Buachalla [1]

At the level of a one-cross bundle, the form (1,1)-form

$$\kappa := \wedge \circ (\mathrm{id} \otimes I)(g)$$

is closed, that is $\mathrm{d}\kappa = 0$, moreover, central, and real (up to rescaling), making it a natural candidate for a Kahler form...

We have a bounded representation $\lambda : \mathcal{O}_q(G/L_S) \to \mathbb{B}(L^2(\Omega^\bullet))$ given by left multiplication.

The Dolbeault–Dirac operator $D_{\bar\partial} := \bar\partial + \bar\partial^\dagger$ is by construction a densely defined unbounded operator. It is in fact closable, and moreover essentially self-adjoint.

The Dirac $D_{\bar\partial}$ and hence the Laplacian $\Delta_{\bar\partial}$ are diagonalisable.

The commutators $[D_{\bar\partial}, \lambda(b)]$ are bounded, for all $b \in \mathcal{O}_q(G/L_S)$.

We even know that the domain of the closure of $D_{\bar\partial}$ is closed under the action λ of $\mathcal{O}_q(G/L_S)$.

As for all one-cross quantum flag manifolds, their line bundles $\mathcal{E}_k$ are labelled by the integers $k \in \mathbb{Z}$, and come endowed with a unique covariant holomorphic structure.

We can tensor the anti-holomorphic subcomplex $\Omega^{(0,\bullet)}$ by any $\mathcal{E}_k$ to get a twisted Dirac operator $D_{\bar{\partial},\mathcal{E}_k}$.

Theorem: For $k \in \mathbb{Z}_{>0}$, and $q \in I$, the $\mathcal{E}_k$-twisted Dolbeault–Dirac operator is a Fredholm operator, has a spectral gap around zero.

Theorem: For $k \in \mathbb{Z}_{>0}$, and $q \in I$, the $\mathscr{E}_k$-twisted Dolbeault–Dirac operator is a Fredholm operator. Moreover, the index of the operator is given by

$$\operatorname{index}\left(D_{\bar{\partial}_{\mathscr{E}_{-k}}}\right) = \begin{cases} \dim\left(V_{(k-C_S)\varpi_x}\right), \text{ if } k \geq C_S, \\ 0, \text{ otherwise} \end{cases}$$

where $2M$ is the total dimension of the dc, Δ^+ is the set of positive roots of $\mathfrak{g}$, and ρ is the half-sum of positive roots.

In particular, if $k \geq C_S$, then

$$\operatorname{index}\left(D_{\bar{\partial}_{\mathscr{E}_{-k}}}\right) = (-1)^M \frac{\prod_{\alpha \in \Delta^+} \left(\rho + (k - C_S)\varpi_x, \alpha\right)}{\prod_{\alpha \in \Delta^+} (\rho, \alpha)}.$$

TABLE 2. Irreducible quantum flag manifolds: notation for the CQGA-homogeneous space, the Heckenberger–Kolb calculus complex dimension, and the identification of the top holomorphic forms with a line module.

$\mathcal{O}_q(G/L_S)$	$M := \dim\big(\Omega^{(1,0)}\big)$	Canonical line module $\Omega^{(M,0)}$
$\mathcal{O}_q(\mathrm{Gr}_{n+1,s})$	$s(n+1-s)$	$\mathcal{E}_{-(n+1)}$
$\mathcal{O}_q(\mathbf{Q}_{2n+1})$	$2n-1$	$\mathcal{E}_{-2n+1}$
$\mathcal{O}_q(\mathbf{L}_n)$	$\frac{n(n+1)}{2}$	$\mathcal{E}_{-(n+1)}$
$\mathcal{O}_q(\mathbf{Q}_{2n})$	$2(n-1)$	$\mathcal{E}_{-2(n-1)}$
$\mathcal{O}_q(\mathbf{S}_n)$	$\frac{n(n-1)}{2}$	$\mathcal{E}_{-2(n-1)}$
$\mathcal{O}_q(\mathbb{OP}^2)$	16	$\mathcal{E}_{-12}$
$\mathcal{O}_q(\mathbf{F})$	27	$\mathcal{E}_{-18}$


HAPPY BIRTHDAY
Sto lat!

Dziękuję za wysłuchanie