Remarks on differential structures on quantum spheres.

Tomasz Brzeziński

University of Białystok (Poland)
&
Swansea University (UK)

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Aim:

- One-forms (sections of the cotangent bundle) on the 2-sphere have the interpretation of the direct sum of sections of two line bundles with Chern numbers ± 2.
- These correspond to holomorphic and anti-holomorphic forms.
- This interpretation carries over to the **standard** quantum or Podleś two-spheres.
- We address a question: is such an interpretation possible for **nonstandard** Podleś two-spheres?

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$$\alpha\beta = q\beta\alpha, \quad \alpha\gamma = q\gamma\alpha, \quad \alpha\delta = \delta\alpha + (q - q^{-1})\beta\gamma,$$

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If q is real, the $*$-structure $\delta = \alpha^*$, $\beta = -q\gamma^*$ makes $\mathcal{O}_q(SL(2))$ into $\mathcal{O}_q(SU(2))$ which can be completed to a C^*-algebra if $q \in (0, 1)$ [Woronowicz '86].

Quantum spheres

$\mathcal{O}_q(SL(2))$ is a matrix-type Hopf algebra, with comultiplication and counit

$$\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}, \qquad \varepsilon(u_{ij}) = \delta_{ij},$$

and the antipode:

$$S\alpha = \delta, \quad S\beta = -q^{-1}\beta, \quad S\gamma = -q\gamma, \quad S\delta = \alpha.$$

This is compatible with the $*$-structure of $\mathcal{O}_q(SU(2))$.

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[Podleś '87]: Coordinate algebras of quantum spheres $\mathcal{O}_{q,s}(S^2)$ are generated by

$$\xi = s(q^{-1}\beta^2 - \alpha^2) + (1 - s^2)q^{-1}\alpha\beta, \quad \eta = s(q\gamma^2 - \delta^2) + (s^2 - 1)\gamma\delta,$$

$$\zeta = s(\beta\delta - q\alpha\gamma) + (1 - s^2)q\beta\gamma,$$

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$$\zeta = s(\beta\delta - q\alpha\gamma) + (1-s^2)q\beta\gamma,$$

where $s \in [0,1]$.The derived algebra relations are:

$$\xi\zeta = q^2\zeta\xi, \quad \eta\zeta = q^{-2}\zeta\eta, \quad \xi\eta = (s^2-\zeta)(\zeta+1),$$

$$\eta\xi = (s^2 - q^{-2}\zeta)(q^{-2}\zeta+1).$$

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$\mathcal{O}_{q,s}(S^2)$ are $*$-subalgebras of $\mathcal{O}_q(SU(2))$ with $\zeta^* = \zeta$ and $\eta = \xi^*$.

Quantum line bundles over $\mathcal{O}_{q,s}(S^2)$

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- Line bundles correspond to *invertible* modules, i.e. such $\mathcal{E}$ for which there exists right B-module $\mathcal{F}$ such that $\mathcal{F} \otimes_B \mathcal{E} \cong B$.
- There is a specially nice construction of NCG vector bundles for quantum homogeneous spaces.

Quantum line bundles over $\mathcal{O}_{q,s}(S^2)$

- The right ideal J of $\mathcal{O}_q(SU(2))$ generated by $\zeta, \xi + s, \eta + s$ is a **coideal** in $\mathcal{O}_q(SU(2))$, i.e.

$$\Delta(J) \subseteq \mathcal{O}_q(SU(2)) \otimes J + J \otimes \mathcal{O}_q(SU(2)).$$

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- Crucially,

$$\mathcal{O}_{q,s}(S^2) = \{a \in \mathcal{O}_q(SU(2)) \mid \Delta_R(a) = a \otimes \pi(1)\},$$

[Brzeziński '97].

Quantum line bundles over $\mathcal{O}_{q,s}(S^2)$

Unexpectedly, C has a basis of grouplike elements c_n, $n \in \mathbb{Z}$,

$$c_n = \begin{cases} \pi\left(\prod_{k=0}^{n}(\alpha + q^{k-1}s\beta)\right) = \pi\left(\prod_{k=0}^{n-1}(\alpha + q^{k}s\gamma)\right), & n > 0, \\ \pi(1), & n = 0, \\ \pi\left(\prod_{k=1}^{-n}(\delta - q^{-k-1}s\beta)\right) = \pi\left(\prod_{k=0}^{-n-1}(\delta - q^{-k}s\gamma)\right), & n < 0, \end{cases}$$

[Brzeziński-Majid '00], [Müller-Schneider '99].

Quantum line bundles over $\mathcal{O}_{q,s}(S^2)$

- There is a map $\ell : C \to \mathcal{O}_q(SU(2)) \otimes \mathcal{O}_q(SU(2))$, which ensures that $\mathcal{O}_q(SU(2))$ is a C-equivariantly projective $\mathcal{O}_{q,s}(S^2)$-module and generates line bundles.
- Let $\ell_n := \ell(c_n)$. Then $\ell_0 = 1 \otimes 1$ and, for all $n \in \mathbb{N}$,

$$\ell_{n+1} = \frac{(q\gamma + q^{-n}s\delta)\ell_n(-\beta + q^{-n}s\alpha) + (\alpha + q^{-n-1}s\beta)\ell_n(\delta - q^{-n}s\gamma)}{1 + q^{-2n}s^2}$$

$$\ell_{-n-1} = \frac{(\delta - q^{n+1}s\gamma)\ell_{-n}(\alpha + q^n s\beta) + (\alpha q^n s - q^{-1}\beta)\ell_{-n}(q^n s\delta + \gamma)}{1 + q^{2n}s^2};$$

[Brzeziński-Majid '00]

Quantum line bundles over $\mathcal{O}_{q,s}(S^2)$

- Let $\ell_n = \sum_{i\in I} \ell'_{n,i} \otimes \ell''_{n,i}$.
- For all n,

$$\sum_{i\in I} \ell'_{n,i}\ell''_{n,i} = 1, \qquad \text{[Brzeziński-Majid '00].}$$

- For all n, i, j,

$$\ell''_{n,i}\ell'_{n,j} \in \mathcal{O}_{q,s}(S^2), \qquad \text{[Brzeziński-Hajac '03].}$$

- Hence, $\{\ell''_{n,i}\}_{i\in I}$ generate projective left $\mathcal{O}_{q,s}(S^2)$-modules $\mathcal{E}_n$, while $\{\ell'_{-n,i}\}_{i\in I}$ generate projective right $\mathcal{O}_{q,s}(S^2)$-modules $\tilde{\mathcal{E}}_n$.
- Each of these modules can be interpreted as a line bundle (viewed as a left or right module, correspondingly) of the topological charge $n \in \mathbb{Z}$.

Bundles of charges ±2

- $\mathcal{E}_2$ is generated by:

$$e_1^+ = \beta^2 - s(1+q^{-2})\alpha\beta + q^{-1}s^2\alpha^2, \quad e_3^+ = -q\delta^2 + s(q+q^{-1})\gamma\delta - s^2\gamma^2,$$

$$e_2^+ = q\beta\delta - qs(1+(q+q^{-1})\beta\gamma) + s^2\alpha\gamma.$$

- $\tilde{\mathcal{E}}_{-2}$ is generated by:

$$e_1^- = q^2\gamma^2 + s(q+q^{-1})\gamma\delta + q^{-1}s^2\delta^2,$$

$$e_2^- = -q^2\alpha\gamma - qs(1+(q+q^{-1})\beta\gamma) - q^{-1}s^2\beta\delta,$$
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- Note $e_i^- = e_i^{+*}$.

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- Make C a Hopf algebra with the product $c_n c_m = c_{m+n}$ and antipode $Sc_n = c_{-n}$, i.e. $C = \mathcal{O}(U(1)) = \mathbb{C}[z, z^{-1}]$.

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- This makes $\mathcal{O}_q(SU(2))$ a strongly $\mathbb{Z}$-graded algebra, $\mathcal{E}_n = \tilde{\mathcal{E}}_n$ and

$$\mathcal{O}_q(SU(2)) = \bigoplus_{n \in \mathbb{Z}} \mathcal{E}_n, \quad \mathcal{E}_n \mathcal{E}_m = \mathcal{E}_{m+n}.$$

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- There is a natural covariant 2D-calculus (no such calculus if $s \neq 0$, [Podleś '89]).

2D calculus on $\mathcal{O}_q(S^2)$

- By a 1-st order differential calculus on an algebra B we mean a B-bimodule Ω together with a map $d : B \to \Omega$ such that

$$d(ab) = d(a)b + ad(b), \quad \Omega = \{\sum_i a_i d(b_i) \mid a_i, b_i \in B\}.$$

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- [Woronowicz '89] There is a unique 3D calculus on $\mathcal{O}_q(SU(2))$ compatible with the $\mathbb{Z}$-grading.
- Ω is a free left $\mathcal{O}_q(SU(2))$-module gen. by $\omega_0, \omega_\pm$ with degrees

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$$|\omega_0| = 0, \qquad |\omega_\pm| = \pm 2.$$
- It descends to a calculus Γ on $\mathcal{O}_q(S^2)$ generated by $e_i^+\omega_-, e_i^-\omega_+$, so that $\Gamma \cong \mathcal{E}_2 \oplus \tilde{\mathcal{E}}_{-2}$.
- The calculus splits into antiholomorphic and holomorphic parts
$$\bar{\partial} : \mathcal{O}_q(S^2) \to \mathcal{E}_2, \qquad \partial : \mathcal{O}_q(S^2) \to \tilde{\mathcal{E}}_{-2}.$$

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- Podleś' "no-go" theorem tells us the above method will not work.
- Idea: use the action of the universal enveloping algebra $U_q(sl_2)$ on $\mathcal{O}_q(SU(2))$.
- This will help to construct calculi on $\mathcal{O}_{q,s}(S^2)$

$$\bar{\partial}(\xi^i) = e_i^+, \quad \partial(x) = \bar{\partial}(x^*)^*,$$

where $\xi^i = \xi, \zeta, \eta$ and $x \in \mathcal{O}_{q,s}(S^2)$.

$U_q(sl_2)$ and its action on $\mathcal{O}_q(SU(2))$

$U_q(sl_2)$ is generated by $K^{\pm 1}$, E, F with relations:

$$KE = q^2 EK, \quad KF = q^{-2} FK, \quad [E, F] = \frac{K - K^{-1}}{q - q^{-1}},$$

and Hopf algebra structure, K is grouplike and

$$\Delta E = E \otimes K + 1 \otimes E, \quad \Delta F = F \otimes 1 + K^{-1} \otimes F,$$

$$S(E) = -EK^{-1}, \quad S(F) = -KF.$$

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Hopf algebra dual pairing $\langle -, - \rangle : U_q(sl_2) \times \mathcal{O}_q(SL(2)) \to \mathbb{C}$ is:

$$\langle K, \alpha \rangle = q^{-1}, \quad \langle K, \delta \rangle = q, \quad \langle E, \gamma \rangle = \langle F, \beta \rangle = 1.$$

$U_q(sl_2)$ and its action on $\mathcal{O}_q(SU(2))$

The left action of $U_q(sl_2)$ on $\mathcal{O}_q(SL(2))$, which is given by

$$X \triangleright x = x_{(1)}\langle X, x_{(2)}\rangle, \qquad \text{for all } X \in U_q(sl_2),\, x \in \mathcal{O}_q(SL(2)),$$

comes out as

$$E \triangleright \alpha = \beta, \quad E \triangleright \beta = 0, \quad E \triangleright \gamma = \delta, \quad E \triangleright \delta = 0,$$

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$U_q(su_2)$ is a $*$-Hopf algebra with $K^* = K, E^* = KF$.The pairing is compatible with the $*$-structure in the sense that,

$$\langle K, x^*\rangle = \overline{\langle K^{-1}, x\rangle}, \quad \langle E, x^*\rangle = -q\overline{\langle F, x\rangle}, \quad \langle F, x^*\rangle = -q^{-1}\overline{\langle E, x\rangle}.$$

Differential structures on $\mathcal{O}_{q,s}(S^2)$

- By construction of the left action $U_q(sl_2)$ on $\mathcal{O}_q(SU(2))$, $X \triangleright (xy) = (X_{(1)} \triangleright x)(X_{(2)} \triangleright y)$.

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- Since K is grouplike, $\sigma = K \triangleright -$ is an algebra auto of $\mathcal{O}_q(SU(2))$.
- If we find X such that

$$\Delta(X) = X \otimes K + 1 \otimes X \text{ and } X \triangleright \xi^i = e_i^+,$$

then $X \triangleright - : \mathcal{O}_{q,s}(S^2) \to \mathcal{E}_2 \subseteq \mathcal{O}_q(SU(2))$ will satisfy σ-skew Leibniz rule.

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  then $X \triangleright - : \mathcal{O}_{q,s}(S^2) \to \mathcal{E}_2 \subseteq \mathcal{O}_q(SU(2))$ will satisfy σ-skew Leibniz rule.
- This will give a derivation $\bar{\partial} : \mathcal{O}_{q,s}(S^2) \to \Omega^{(0,1)} = \mathcal{E}_2\sigma(\mathcal{O}_{q,s}(S^2))$.

Differential structures on $\mathcal{O}_{q,s}(S^2)$

- By construction of the left action $U_q(sl_2)$ on $\mathcal{O}_q(SU(2))$, $X \triangleright (xy) = (X_{(1)} \triangleright x)(X_{(2)} \triangleright y)$.
- Since K is grouplike, $\sigma = K \triangleright -$ is an algebra auto of $\mathcal{O}_q(SU(2))$.
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- [Noumi-Mimachi '90] Any such X is necessarily a linear combination of $E, K - 1$ and KF, but do they exist?

Main theorem

Theorem (with R Ó Buachalla)

For any $a, c \in \mathbb{C}$, *let*

$$E^s = aE + \frac{s(a-c)}{q-q^{-1}}(K-1) + s^2 qcKF.$$

Then $\Delta(E^s) = E^s \otimes K + 1 \otimes E^s$, *and*

$$E^s \triangleright \xi = (a-cs^2)e_1^+, \quad E^s \triangleright \zeta = (a-cs^2)e_2^+, \quad E^s \triangleright \eta = (a-cs^2)e_3^+.$$

Hence, if $a \neq s^2 c$, *then* $\Omega^{(0,1)} = \mathcal{E}_2\sigma(\mathcal{O}_{q,s}(S^2))$, *together with*

$$\bar{\partial} : \mathcal{O}_{q,s}(S^2) \to \Omega^{(0,1)}, \quad x \mapsto (a-s^2c)^{-1} E^s \triangleright x,$$

is a first order differential calculus on $\mathcal{O}_{q,s}(S^2)$.

Main theorem (cd)

Theorem (with R Ó Buachalla)

Let $\bar{\sigma} = K^{-1} \triangleright -$ *and*

$$F^s = -q\bar{a}F + \frac{s(\bar{a} - \bar{c})}{q - q^{-1}}(K^{-1} - 1) - s^2\bar{c}K^{-1}E.$$

Then $\Omega^{(1,0)} = \bar{\sigma}(\mathcal{O}_{q,s}(S^2))\tilde{\mathcal{E}}_{-2}$ *together with*

$$\partial : \mathcal{O}_{q,s}(S^2) \to \Omega^{(1,0)}, \quad x \mapsto (\bar{a} - s^2\bar{c})^{-1}F^s \triangleright x,$$

is a first order differential calculus on $\mathcal{O}_{q,s}(S^2)$*.*
Furthermore,$\Omega^{(1,0)^*} = \Omega^{(0,1)}$*, and, for all* $x \in \mathcal{O}_{q,s}(S^2)$*,*

$$\bar{\partial}(x^*) = \partial(x)^*.$$

Maximal prolongations

- To prolong maximally 1st to 2nd order diff. calc. need

  $$d : \Omega^1 \to \Omega^2 = \Omega^1 \otimes_B \Omega^1 / N,$$

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- What are the zero relations in $\Omega^{(0,1)}$ and $\Omega^{(1,0)}$?
- Use the explicit description of $\Omega^{(0,1)}$ and $\Omega^{(1,0)}$ as projective modules provided by the strong connection lifting ℓ.

Toward Ω^2

- Idempotent for $\mathcal{E}_2$ is

$$\mathbf{p} = (\ell''_{2,i}\ell'_{2,j})^{3}_{i,j=1}, \qquad \ell''_{2,i} = e^{+}_{i}, \qquad \ell'_{2,i} = c_i e^{-}_{i},$$

where

$$c_1 = \frac{1}{(1+s^2)(1+q^{-2}s^2)}, \quad c_2 = \frac{(q^{-1}+q^{-2})^2}{(1+s^2)(1+q^{-2}s^2)},$$

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- Note $(1-\mathbf{p})\mathbf{e}^+ = 0$, and so

$$0 = \sum_j (\delta_{ij} - \mathbf{p}_{ij})\bar\partial\xi^j, \quad \text{i.e.} \quad \bar\partial\xi^i = \sum_j p_{ij}\bar\partial\xi^j.$$

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Conclusions

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  - and many more...

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Happy 60th Birthday Andrzej!

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Happy 70th Birthday Ludwik!