Geometry of Quantum Graphs: Unknowns and Why They Matter

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with $\Phi_{\mathrm{cl}}(b|a)$ denoting the *transition matrix*, i.e., the conditional probability of b given a.

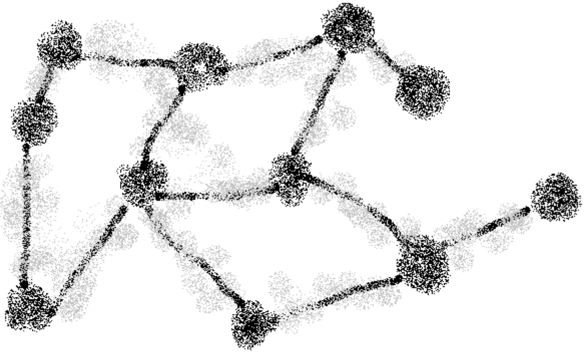
- One can define the *confusability graph*

$$G_{\Phi_{\mathrm{cl}}} = (V_{\Phi_{\mathrm{cl}}}, E_{\Phi_{\mathrm{cl}}})$$

by taking $V_{\Phi_{\mathrm{cl}}} = A$ and

$$(a_1, a_2) \in E_{\Phi_{\mathrm{cl}}} \;\Leftrightarrow\; \exists b \in B \text{ such that } \Phi_{\mathrm{cl}}(b|a_1)\,\Phi_{\mathrm{cl}}(b|a_2) \neq 0.$$

A
B
????



Graphs as operator spaces

- With a classical graph $G = (V, E)$ one can associate operator space $\mathcal{S}_G = \mathrm{span}\{|v_i\rangle\langle v_j| : (v_i v_j) \in E\}$ in von Neumann algebra $B(\ell^2(V))$.

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CPTP map $\mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_B)$

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Idea:

Quantum graph = an operator system $\mathcal{S}$ satisfying some extra properties.

Another perspective

- Instead of specifying the set of edges E for a classical graph $G = (V, E)$, we can work with the adjacency matrix A.
- One can consider $\mathrm{C}(V)$ and think of the adjacency matrix in terms of the operator $A : \mathrm{C}(V) \to \mathrm{C}(V)$.
- Intuitively: $A_{i,j}$ *corresponds* to $|v_i\rangle\langle v_j| \in \mathcal{S}_G$, for the classical graphs.

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Under certain technical assumptions, operator systems defining quantum graphs correspond to some endomorphisms A of the C^*-algebra $\mathrm{C}(\mathbb{V})$.

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- $mm^* = \delta^2 \mathrm{id}_{L^2(\mathcal{G})}$ for some $\delta > 0$
- $A : L^2(\mathcal{G}) \to L^2(\mathcal{G})$ s.t.

$$A = \delta^{-2} m(A \otimes A)m^*, \quad A = (\mathrm{id} \otimes \eta^* m)(1 \otimes A \otimes 1)(m^* \eta \otimes \mathrm{id}).$$

Characteristics of (quantum) graphs

- clique number
- colorings and chromatic numbers
- connectivity
- ...

Clique numbers

$$\omega(\mathcal{G}) = \max\Big\{ |\{\psi_k \in L^2(\mathcal{G}) : k \in K\}| : \ \psi_k \neq 0 \,, |\psi_i\rangle\langle\psi_j| \in \mathcal{S}_{\mathcal{G}} \text{ for } i,j,k \in K \text{ and } i \neq j \Big\}.$$

$$\omega(\mathcal{G}) = \max\{|K_n| : \ K_n \to \mathcal{G}\}$$

(We say that there exists a homomorphism between quantum graphs $\mathcal{G}$ and $\mathcal{F}$, and write $\mathcal{G} \to \mathcal{F}$, if there exists a Hilbert space H and an isometry $\mathcal{J} : L^2(\mathcal{G}) \to L^2(\mathcal{F}) \otimes \mathrm{H}$ s.t. $\mathcal{J} S_{\mathcal{G}} \mathcal{J}^* \subseteq S_{\mathcal{F}} \otimes B(\mathrm{H})$.)

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One can also define the *quantum* clique number $\omega_q(\mathcal{G})$ by replacing in the above definition the complete graph by its quantum analog.

Colorings

The quantum graph $\mathcal{G}$ possesses a quantum c-coloring if there exists a finite von Neumann algebra $\mathcal{N}$ and a partition of unity $\{P_a\}_{a=1}^{c} \subset \mathrm{C}(\mathcal{G}) \otimes \mathcal{N}$, i.e. a set of orthogonal projections satisfying $\sum_{a=1}^{c} P_a = \mathbb{1}_{\mathrm{C}(\mathcal{G})\otimes\mathcal{N}}$, and such that

$$\forall_{1\leq a\leq c}\,\forall_{X\in S_{\mathcal{G}}} : P_a(X\otimes\mathbb{1}_{\mathcal{N}})P_a = 0.$$

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- The quantum chromatic number:

$$\chi_q(\mathcal{G}) = \min\{c \in \mathbb{N} | \exists\, c\text{-colouring with } \dim(\mathcal{N}) < \infty\}.$$

- The classical chromatic number

$$\chi_{\mathrm{loc}}(\mathcal{G}) = \min\{c \in \mathbb{N} | \exists\, c\text{-colouring with } \dim(\mathcal{N}) = 1\}.$$

Colorings

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- Chromatic numbers can be reformulated in terms of non-local games.
- Additional conditions on *the second leg of the tensor product* lead to different scenarios (winning strategies) in terms of quantum (non-local) games. [This is closely related to the MIP*=RE problem].

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Some classical problems can be made quantum

- Does there exist a triangle-free graph of arbitrarily large chromatic number?
- Mycielski construction: from a given graph G, we can construct a larger one $\mu(G)$ s.t. $\omega(\mu(G)) = \omega(G)$ and $\chi(\mu(G)) = \chi(G) + 1$.
- We found a quantum version of this construction and studied how it impacts (quantum) characteristics of quantum graphs.

Symmetries of quantum graphs

The *quantum symmetry group* of $\mathcal{G}$ is the compact quantum group $\mathrm{QAut}(\mathcal{G})$ equipped with a $\psi_{\mathcal{G}}$-preserving action $\rho_{\mathcal{G}}\colon \mathsf{C}(\mathcal{G}) \to \mathsf{C}(\mathcal{G}) \otimes \mathsf{C}(\mathrm{QAut}(\mathcal{G}))$ such that $\rho_{\mathcal{G}} \circ A_{\mathcal{G}} = (A_{\mathcal{G}} \otimes \mathrm{id}_{\mathsf{C}(\mathrm{QAut}(\mathcal{G}))}) \circ \rho_{\mathcal{G}}$ and which is universal among all $\psi_{\mathcal{G}}$-preserving compact quantum group actions $\alpha\colon \mathsf{C}(\mathcal{G}) \to \mathsf{C}(\mathcal{G}) \otimes \mathsf{C}(\mathbb{G})$ such that $\alpha \circ A_{\mathcal{G}} = (A_{\mathcal{G}} \otimes \mathrm{id}_{\mathsf{C}(\mathbb{G})}) \circ \alpha$.

Symmetries preserving partitions of unity

- Let A be a unital C*-algebra and let $\rho\colon \mathsf{A} \to \mathsf{A} \otimes \mathrm{C}(\mathbb{G})$ be an action of a compact quantum group $\mathbb{G}$ on the quantum space underlying A.

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- Let $\mathfrak{P} = \{P_1, \ldots, P_n\}$ be a partition of unity in A, and define the ideal $\mathsf{J} \subset \mathsf{C}(\mathbb{G})$ generated by

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- Set

$$\mathsf{C}(\mathbb{G}_{\mathfrak{P}}) := \mathsf{C}(\mathbb{G})/\mathsf{J}, \qquad \pi\colon\ \mathsf{C}(\mathbb{G}) \to \mathsf{C}(\mathbb{G}_{\mathfrak{P}}),$$

$$\rho_{\mathfrak{P}} := (\mathrm{id} \otimes \pi) \circ \rho\colon \mathsf{A} \to \mathsf{A} \otimes \mathsf{C}(\mathbb{G}_{\mathfrak{P}}).$$

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- Let $\mathbb{Y}$ be a compact quantum space and $\varphi\colon \mathrm{C}(\mathbb{G}) \to \mathrm{C}(\mathbb{Y})$ a unital $*$-homomorphism with $(\mathrm{id} \otimes \varphi)\rho(P_a) = P_a \otimes \mathbb{1}$. Then $\mathrm{J} \subset \ker \varphi$.

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- Applying this to $\mathbb{Y} = \mathbb{G}_{\mathfrak{P}} \times \mathbb{G}_{\mathfrak{P}}$ and $\varphi = (\pi \otimes \pi) \circ \Delta_{\mathbb{G}}$, we get a unique $\Delta_{\mathbb{G}_{\mathfrak{P}}}\colon \mathrm{C}(\mathbb{G}_{\mathfrak{P}}) \to \mathrm{C}(\mathbb{G}_{\mathfrak{P}}) \otimes \mathrm{C}(\mathbb{G}_{\mathfrak{P}})$ s.t. $(\pi \otimes \pi) \circ \Delta_{\mathbb{G}} = \Delta_{\mathbb{G}_{\mathfrak{P}}} \circ \pi$.

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- Thus there exists a unique $\Lambda\colon \mathrm{C}(\mathbb{G}_{\mathfrak{P}}) \to \mathrm{C}(\mathbb{Y})$ such that $\varphi = \Lambda \circ \pi$.
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- It satisfies $(\rho_{\mathfrak{P}} \otimes \mathrm{id}) \circ \rho_{\mathfrak{P}} = (\mathrm{id} \otimes \varphi) \circ \rho = (\mathrm{id} \otimes \Delta_{\mathbb{G}_{\mathfrak{P}}}) \circ \rho_{\mathfrak{P}}$ and $\Delta_{\mathbb{G}_{\mathfrak{P}}} \otimes \mathrm{id}) \circ \Delta_{\mathbb{G}_{\mathfrak{P}}} \circ \pi = (\mathrm{id} \otimes \Delta_{\mathbb{G}_{\mathfrak{P}}}) \circ \Delta_{\mathbb{G}_{\mathfrak{P}}} \circ \pi$.
- Furthermore, the density conditions hold, i.e., $(\mathrm{C}(\mathbb{G}_{\mathfrak{P}}), \Delta_{\mathbb{G}_{\mathfrak{P}}})$ is a compact quantum group.

Distinguishing labeling

- A c-distinguishing labeling of G is a function $\phi\colon V \to \{1, \ldots, c\}$ such that for every $\sigma \in \mathrm{Aut}(G) \setminus \{\mathrm{id}\}$, there exists $v \in V$ such that $\phi(v) \neq \phi(\sigma(v))$.
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- The distinguishing number $D(G)$ of G is the smallest c such that G has a labeling that is c-distinguishing.
- Quantum version: Let $\mathfrak{P} = \{P_1, \dots, P_c\} \subset \mathrm{C}(\mathcal{G}) \otimes \mathrm{B}(\mathscr{H})$ be a quantum c-labeling of $\mathcal{G}$. We say that $\mathfrak{P}$ is a *quantum c-distinguishing labeling* of $\mathcal{G}$ if the quantum group $\mathrm{QAut}(\mathcal{G})_{\mathfrak{P}}$ is trivial.

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- E.g., $\lambda_2(L(G)) = 0$ iff G is disconnected.
- For quantum graphs: A quantum graph $S_{\mathcal{G}} \subseteq \mathrm{Mat}_n$ is connected if there exists p such that $\mathcal{S}_{\mathcal{G}}^p = \mathrm{Mat}_n$.
- There exist (several) notions of Laplacian for quantum graphs, e.g., $L = D - A$, where $Dx = x(A\mathbb{1})$ is the quantum degree operator.
- Question: Explore connectivity properties in terms of quantum Laplacians and quantum incidence operators

Insight into representation theory

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Insight into representation theory

- Let $\mathcal{G}$ be quantum Cayley graph associated with some quantum group $\mathbb{G}$. What information about $\mathbb{G}$ is contained in the spectrum of $L(\mathcal{G})$?
- Classically: the eigenvalues of the Laplacian of a Cayley graph are related to the irreducible representations of the underlying group.
- The spectral gap relates to the expander properties of the graph.

Laplacian is important also because...

- The Colin de Veriére invariant defined in terms of the spectrum of the (generalized) Laplacian tells us about planarity of the graph.
- Spectrum of Laplacian characterize spanning trees.
- . . . and many others!

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- Question: Which characteristics of quantum graphs we can read from spectral triples?
- Question: What are the *physical* implications on noisy quantum channel?

Spectral Geometry for Quantum Information?

Thank you for your attention!