Deformation quantization for actions of non-Abelian Lie groups.

Pierre Bieliavsky (UCLouvain, Belgium)

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Weights on Lie groups

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Example:

- $\mathfrak{d}(x) := \sqrt{1 + |\mathtt{Ad}_x|^2 + |\mathtt{Ad}_{x^{-1}}|^2}$
- modular weight Δ_G

Symbol spaces

$(\mathcal{E}\,,\,\{|\;|_j\}_{j\in\mathbb{N}})$ Fréchet space, $\underline{\mu}\;:=\;\{\mu_j\}_{j\in\mathbb{N}}$

$$\mathcal{B}^{\underline{\mu}}(G\,,\,\mathcal{E})\;:=\;\{F\in C^\infty(G\,,\,\mathcal{E})\mid\forall j\,,\,A\in\mathcal{U}(\mathfrak{g})\;:\;\left|\widetilde{A}(F)\right|_j\;<\;C\,\mu_j\}$$

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$|F|_{j,A}\ :=\ \sup_{x\in G}\left\{\frac{1}{\mu_j(x)}\left|\widetilde{A}(F)\right|_j\right\}$ Fréchet

Lemma: If $\underline{\mu'}\ \succ\ \underline{\mu}$, then

$$\mathrm{closure}_{\mathcal{B}^{\underline{\mu'}}(G\,,\,\mathcal{E})}(\mathcal{D}(G\,,\,\mathcal{E}))\ \supset\ \mathcal{B}^{\underline{\mu}}(G\,,\,\mathcal{E})$$

Tempered Lie groups

G group with

$$\mathbb{R}^d \to G$$

bijection such that multiplication and inverse are tempered functions.

Examples: 1. $(\mathbb{R}, +)$

2. $\mathbb{R} \to \mathbb{R} : x \mapsto \sinh(x) \quad \rightsquigarrow \quad (\mathbb{R}, +^{(\sinh)})$

Tempered pairs

G Lie group

$$S : G \to \mathbb{R} \quad \text{smooth}$$

such that

$$\widetilde{\mathrm{d}S} \ : \ G \to \mathfrak{g}^\star \ : \ x \mapsto \widetilde{\mathrm{d}S}_x \ := \ [\mathfrak{g} \to \mathbb{R} : X \mapsto \widetilde{X}_x(S)]$$

equips G with a structure of tempered Lie group

(G, S) is called a **tempered pair**

Kaehlerian Lie groups: symplectic symmetric spaces

Proposition. Let $\mathfrak{P} := SO_o(1,1) \ltimes \mathbb{R}^2$ denote the Poincaré group. Then

1. Every generic coadjoint (massive) orbit $\mathbb{M}$ of $\mathfrak{P}$ carries a unique Poincaré-invariant linear connection ∇.
2. The connection ∇ is torsionfree and symplectic.
3. Every geodesic symmetry centered at a point $x \in \mathbb{M}$:
$$s_x \ : \ \mathbb{M} \to \mathbb{M} : y \mapsto s_x(y)$$
is affine: $s_x(\nabla) = \nabla$.
4. The **kaleidoscope map**
$$\Phi : \mathbb{M}\times\mathbb{M}\times\mathbb{M} \to \mathbb{M}\times\mathbb{M}\times\mathbb{M} : (x,y,z) \mapsto (\xi\ ,\ s_z(\xi)\ ,\ s_y\,(s_z(\xi)))$$
where
$$\xi \ = \ s_x\,(s_y\,(s_z(\xi)))$$
is well-defined as a global diffeomorphism.

Tempered pairs for Kahlerian Lie groups

$$\mathbb{B} \;:=\; SO_o(1,1) \ltimes \mathbb{R} \;\subset\; \mathfrak{P} \;=\; SO_o(1,1) \ltimes \mathbb{R}^2$$

$$\mathbb{B} \;\longrightarrow\; \mathbb{M} \subset \mathrm{Lie}(\mathfrak{P})^\star : \; x \mapsto \mathtt{Ad}^\flat_x(\xi_0) \quad \text{diffeomorphism}$$

$$S_{\mathbb{M}}(x,y,z) \;:=\; \int_{\Phi(x,y,z)} \omega^{KKS}$$

$$S \;:\; G \;:=\; \mathbb{B} \times \mathbb{B} \to \mathbb{R} : (x,y) \mapsto S_{\mathbb{M}}(x,y,e) \;.$$

Then: (G, S) is a tempered pair.

Admissible phases

(G, S) tempered pair

$$\mathfrak{g} = \bigoplus_{n=0}^{N} V_n \tag{1}$$

For every $n = 0, \ldots, N$ $\rightsquigarrow$ ordered basis of V_n

$$\{e^n_{j_n}\}_{j_n=1,\ldots,\dim(V_n)}$$

$\rightsquigarrow$ global coordinates on G:

$$x_n^{j_n} := \left(\widetilde{e^n_{j_n}} S\right)(x) \tag{2}$$

Definition

Set $\mathbf{E} := \exp(iS)$ $\quad (G,S)$ *is called* **admissible**, *if*
$\forall\ n = 0, \ldots, N \quad \exists\, X_n \in \mathfrak{S}(V_n) \subset \mathcal{U}(\mathfrak{g})$

$$\widetilde{X}_n \mathbf{E} \; =: \; \alpha_n \mathbf{E} :$$

(i) $\exists\ C_n\,,\ \rho_n > 0$: $|\alpha_n| \;\geq\; C_n \big(1 + |x_n|_n^{\rho_n}\big)$
(ii) $\forall\ n = 0, \ldots, N\ \exists$ **tempered function** $0 < \mu_n \in C^\infty(G)$:
(ii.1) For every $A \in \prod_{k=0}^{n} \mathfrak{S}(V_k) \subset \mathcal{U}(\mathfrak{g})$ *there exists* $C_A > 0$ *such that:*

$$\big|\widetilde{A}\,\alpha_n\big| \;\leq\; C_A\ |\alpha_n|\ \mu_n\ .$$

(ii.2) The function μ_n *is* **independent of the variables** $\{x_r^{j_r}\}_{j_r=1,\ldots,\dim(V_r)}$, *for all* $r \leq n$:

$$\frac{\partial \mu_n}{\partial x_r^{j_r}} \;=\; 0\,, \quad \forall r \leq n\,,\ \ \forall j_r = 1, \ldots, \dim(V_r)\ .$$

Kahlerian Lie groups

Theorem. Let $\mathbb{B}$ be a Kahlerian Lie group. Then: the associated tempered pair

$$\left(G \;:=\; \mathbb{B}\times\mathbb{B}\,,\; S \;:=\; S_{\mathbb{M}}(.\,,\,.\,,\,e)\right)$$

is admissible.

Oscillatory integrals: the fundamental operators

Theorem. (G, S) admissible tempered pair, $\underline{\mu} := \{\mu_j\}_{j\in\mathbb{N}}$ tempered weights.

Then: there exists $\mathbf{D} \in DO(G)$ such that

$$\mathbf{D}^* \,\mathbf{E} = \mathbf{E}$$

and

$$\mathbf{D}(F) \in L^1(G\,,\,\mathcal{E}) \qquad \forall\, F \in \mathcal{B}^{\underline{\mu}}(G,\mathcal{E})$$

Oscillatory integrals: Integrations by parts

(G, S) admissible tempered pair, $\mathbf{m} \in \mathcal{B}^{\mu}(G)$, then

$$\mathcal{D}(G, \mathcal{E}) \to \mathcal{E} : F \mapsto \int_G \mathbf{m}\mathbf{E}\, F = \int_G \mathbf{E}\,\mathbf{D}(\mathbf{m}\, F)$$

extends to **continuous map**:

$$\widetilde{\int_G \mathbf{m}\,\mathbf{E}} : \mathcal{B}^{\underline{\mu}}(G, \mathcal{E}) \to \mathcal{E} \ ,$$

that we refer to as an **oscillatory integral**.

Oscillatory integrals: Fréchet symbol spaces

Lemma: Let $\mathcal{A}$ be a Fréchet algebra and $\mathbb{B}$ a Lie group. Then

$$R^\star \times R^\star : \mathcal{B}^{\underline{\mu}}(\mathbb{B},\mathcal{A}) \times \mathcal{B}^{\underline{\mu'}}(\mathbb{B},\mathcal{A}) \to \mathcal{B}^{\underline{\mu}\otimes\underline{\mu'}}(\mathbb{B}\times\mathbb{B}, \mathcal{B}^{\underline{\mu'\mu}}(\mathbb{B},\mathcal{A}))$$

$$(F_1\,,\,F_2) \mapsto \big[(x,y) \mapsto R^\star_x F_1\, R^\star_y F_2 \;:=\; [\mathbb{B}\to\mathcal{A} : z \mapsto F_1(zx)\,F_2(zy)]\big]$$

Theorem: Assume $(G := \mathbb{B}\times\mathbb{B}, S)$ is admissible pair. Then the bilinear map

$$\star^S \;:=\; \widetilde{\int_G \mathbf{m}\,\mathbf{E}} \circ (R^\star \times R^\star) \;:\mathcal{B}^{\underline{\mu}}(\mathbb{B},\mathcal{A}) \times \mathcal{B}^{\underline{\mu'}}(\mathbb{B},\mathcal{A}) \to \mathcal{B}^{\underline{\mu'\mu}}(\mathbb{B},\mathcal{A})$$

is separately continuous.

Kahlerian Lie groups

Theorem. Let $\mathbb{B}$ be a Kahlerian Lie group with kaleidoscope map $\Phi : \mathbb{B}^3 \to \mathbb{B}^3$. Let $\theta \in \mathbb{R}_0$. Then:
(i) the formula

$$u \star_\theta^\Phi v(x) \;:=\; \frac{1}{\theta^m} \int_{\mathbb{B}^2} \left|\mathrm{Jac}_\Phi(x,y,z)\right|^{1/2} \; e^{\frac{i}{\theta} S_{\mathbb{M}}(x,y,z)} \, u(y)\, v(z) \,\mathrm{d}y\,\mathrm{d}z$$

extends from $(u, v) \in \mathcal{D}(\mathbb{B}) \times \mathcal{D}(\mathbb{B})$ to an associative Hilbert algebra structure:

$$\star_\theta^\Phi : L^2(\mathbb{B}) \times L^2(\mathbb{B}) \;\to\; L^2(\mathbb{B})\ .$$

(ii) For all $u, v \in \mathcal{D}(L)$, one has the asymptotic expansion:

$$u \,\star_\theta^\Phi\, v \;\sim\; u\,v \;+\; \frac{\theta}{2i}\,\{u, v\}^{KKS} \;+\; 0(\theta^2)$$

Tempered Fréchet G-module-algebras

$(\mathbb{A}\,,\,\{|\;|_j\}_{j\in\mathbb{N}})$ Fréchet algebra

$$\alpha : G \times \mathbb{A} \to \mathbb{A} : (x, f) \mapsto \alpha_x(f)$$

linear action by automorphisms.

1. Strongly continuous: $\forall\, f \in \mathbb{A}$

$$\alpha(f)\;\; : G \to \mathbb{A} : x \mapsto \alpha_x(f) \quad \text{is continuous}$$

2. Tempered: $\forall j \;\exists\; \mu_j$ weight such that

$$|\alpha_x(f)|_j \;<\; C\,\mu_j(x)\;|f|_{k(j)}\;.$$

The space of C^∞-vectors:

$$\mathbb{A}^\infty \;:=\; \{a \in \mathbb{A} \;:\; \alpha(f) \;\in\; C^\infty(G, \mathbb{A})\}$$

is Fréchet for

$$|f|_{j,A} \;:=\; \left|\widetilde{A}(\alpha(f))(e)\right|_j \quad (A \in \mathcal{U}(\mathfrak{g}))$$

Fundamental property of tempered actions

Proposition. Let $(\alpha\,,\,\{\mu_j\})$ be a tempered action of G on Fréchet algebra $\mathbb{A}$. Then

1. The action α on $\mathbb{A}^\infty$ is tempered: $\nu_{j,A} \;:=\; \mathfrak{d}^{|A|}\,\mu_j$
2. $\alpha : \mathbb{A}^\infty \hookrightarrow \mathcal{B}^{\underline{\nu}}(G, \mathbb{A}^\infty)$ continuous injection.
3. The evalution map $\delta_e : \mathcal{B}^{\underline{\nu}}(G, \mathbb{A}^\infty) \to \mathbb{A}^\infty : F \mapsto F(e)$ is continuous.

Corollary. Let $(\mathbb{B} \times \mathbb{B} \;=:\; G, S)$ admissible tempered pair. Let $(\alpha, \overline{\mu})$ tempered action of $\mathbb{B}$ on Fréchet algebra $\mathbb{A}$. Then, the formula

$$a \,\star^S_{\mathbb{A}^\infty}\, b \;:=\; \delta_e \left(\alpha(a) \,\star^S\, \alpha(b)\right)$$

defines a jointly continuous bilinear map:

$$\star^S_{\mathbb{A}^\infty} \;:\; \mathbb{A}_\infty \times \mathbb{A}_\infty \;\to\; \mathbb{A}_\infty \,.$$

UDF for tempered actions of Kahlerian Lie groups

Lemma. Let $\mathbb{B}$ be a Kahlerian Lie group with kaleidoscope map Φ and associated admissible tempered pair $(G := \mathbb{B} \times \mathbb{B}\,,\, S)$. Then

$$\mathbf{m} \;:=\; \left|\mathrm{Jac}_\Phi\right|^{1/2} \;\in\; \mathcal{B}^1(G)\,.$$

Corollary. Let $(\alpha, \overline{\mu})$ tempered action of $\mathbb{B}$ on Fréchet algebra $\mathbb{A}$. Then, the formula

$$a \;\star^S_{\mathbb{A}^\infty}\; b \;:=\; \delta_e \left(\alpha(a) \;\star^S\; \alpha(b)\right)$$

defines an associative Fréchet algebra $(\mathbb{A}^\infty\,,\, \star^S_{\mathbb{A}^\infty})$.

The Spectral: $C^\star$-algebras $\Leftarrow$ NC-Calderon-Vaillancourt

A : $C^\star$-algebra

$$(\mathcal{H}_{\theta,\mathbf{m}}\,,\,\pi_{\theta\mathbf{m}})\ \in\ \widehat{\mathbb{B}}$$

$\mathcal{S}^{(G,S)}(\mathbb{B}\,,\,A)\ =\ \text{“}\mathcal{B}^{-\infty}(\mathbb{B}\,,\,A)\text{”}$ “non-Abelian Schwartz space”

“Cheap” : Pseudo-differential (Weyl-type) calculus (quantization)

$$\Omega_{\theta,\mathbf{m}}\ :\ \mathcal{S}^{(G,S)}(\mathbb{B}\,,\,A)\ \longrightarrow\ \mathcal{K}(CH_{\theta,\mathbf{m}})\otimes A$$

extends to (use UDF for Fréchet)

$$\rightsquigarrow\ \Omega_{\theta,\mathbf{m}}\ :\ \mathcal{S}^{(G,S)}(\mathbb{B}\,,\,A)\ \longrightarrow\ \mathcal{L}(CH_{\theta,\mathbf{m}})\otimes A\,.$$

“Harder” : Calderon-Vaillancourt :

$$\Omega_{\theta,\mathbf{m}}\ :\ \mathcal{S}^{(G,S)}(\mathbb{B}\,,\,A)\ \longrightarrow\ \mathcal{B}(CH_{\theta,\mathbf{m}})\otimes A\,.$$

Theorem. $(\mathbb{A}^\infty\,,\,\star^S_{\mathbb{A}^\infty})$ carries a pre-$C^\star$ structure.

(Quantum) Symmetry ?

- "Oscillating quantum groups" (Smooth) OK !
- But those are *not* Locally compact quantum groups !
- Kohn-Nirenberg quantization $\rightsquigarrow$ Locally compact quantum groups (deforming $\mathbb{B}$)
- But ... non smooth !!