A friendly introduction to Noncommutative Geometry part II

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The starting dictionary

GEOMETRY	ALGEBRA
vector bundle	finitely generated projective module
differential forms	differential forms
differential forms	Hochschild homology
de Rham cohomology	cyclic cohomology
vector fields	operators
group	Hopf algebra
Lie algebra	Hopf algebra
principal fibre bundle	Hopf-Galois extension
measurable functions	von Neumann algebra
infinitesimals	compact operators
metric	Dirac operator
spin^c geometry	spectral triple
spin geometry	real spectral triple
integrals	exotic traces

Extend the notion of vector bundles

Theorem (Serre-Swan)

Let M be a compact manifold, and $E \to M$ a finite dimensional vector bundle. Then the space of continuous sections of E is a finitely generated projective modules over $C(X)$ and every such module is a space of sections of a vector bundle over M.

What are projective modules?

A module $\mathcal{M}$ over an algebra $\mathcal{A}$ is projective if and only if one of the following conditions is satisfied:

- $\mathcal{M}$ is a summand of a free module: $\mathcal{M} \oplus \mathcal{N} = \mathcal{A}^n$,
- Any surjective module morphism $\pi : \mathcal{N} \to \mathcal{M}$ splits, $\exists$ a morphism $\rho : \mathcal{M} \to \mathcal{N}$, such that $\pi \circ \rho = \mathrm{id}_{\mathcal{M}}$,
- Given a surjective module morphism $\pi : \mathcal{N}' \to \mathcal{N}$ any homomorphism $\rho : \mathcal{M} \to \mathcal{N}$ can be lifted to a homomorphism $\rho' : \mathcal{M} \to \mathcal{N}'$ such that $\rho = \pi \circ \rho'$,

Projective modules and projections

How to construct...

...projective modules ? Take an algebra $\mathcal{A}$, take a free module $\mathcal{A}^n$, take $p \in M_n(\mathcal{A})$ such that $p^2 = p = p^*$ (a projection) and define:

$$\mathcal{M}_p = \mathcal{A}^n p$$

It is a finitely generated projective module. WHY?

Connection

Let $\mathcal{M}$ be a left module over $\mathcal{A}$ and Ω^* a DGA. The map $\nabla : \mathcal{M} \to \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{M}$ is a connection if:

$$\nabla(a\xi) = a\nabla(\xi) + da \otimes_{\mathcal{A}} \xi.$$

FGPM are natural for connections.

If the module $\mathcal{M}$ is projective there always exists a connection (actually it is equivalent to $\mathcal{M}$ being projective)

Spectral triples

Define: a spectral triple

Algebra $\mathcal{A}$, its faithful representation π on a Hilbert space $\mathcal{H}$, a selfadjoint operator D, satisfying several conditions:

1. $\forall a \in \mathcal{A}\ [D, \pi(a)] \in B(\mathcal{H})$, D^{-1} is compact
2. even ST: $\exists \gamma \in \mathcal{A}' : \gamma^2 = 1, \gamma = \gamma^\dagger, \gamma D + D\gamma = 0$,
3. $\exists J$, antilinear $J^2 = \pm 1, JJ^\dagger = 1$
   $J\gamma = \pm\gamma J, JD = \pm DJ, [J\pi(a)J, \pi(b)] = 0$,
4. $[[D, a], J\pi(b)J] = 0$ (D: first order differential operator)
5. ...+ conditions of ,,analysis" type

Theorem

If $\mathcal{A} = C^\infty(M)$, M a spin Riemannian compact manifold, $\mathcal{H} = L^2(S)$ (sections of spinor bundle) and D the Dirac operator on M then to $(\mathcal{A}, \mathcal{H}, D)$ is a spectral triple (with a real structure).

The Geometer of Particle Physics

Alain Connes's noncommutative geometry offers an alternative to string theory. In fact, being directly testable, it may be better than string theory By ALEXANDER HELLEMANS

If there is a mathematician eagerly waiting for the Large Hadron Collider near Geneva to start up next year, it is Alain Connes of the Collège de France in Paris. Like many physicists, Connes hopes that the Higgs particle will show up in detectors. The Higgs is the still missing crowning piece of the so-called Standard Model—the theoretical framework that describes subatomic particles and their interactions. For Connes, the discovery of the Higgs, which supposedly endows the other particles with mass, is crucial: its existence, and even its mass, emerges from the arcane equations of a new form of mathematics called noncommutative geometry, of which he is the chief inventor.

ALAIN CONNES: REDEFINING SPACE

- Chief inventor of noncommutative geometry, a mathematical space wherein the order of events is more important than the location of objects.
- On its origin: "What is superb is that what has been obtained by a dialogue between experiment and theory turns out to be of an unsuspected mathematical beauty."

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Connes's idea was to extend the relation between geometric space and its commutative algebra of Cartesian coordinates, such as latitude and longitude, to a geometry based on noncommutative algebras. In commutative algebra, the product is independent of the order of the factors: $3 \times 5 = 5 \times 3$. But some operations are noncommutative. Take, for example, a stunt plane that can aggressively roll (rotate over the longitudinal axis) and pitch (rotate over an axis parallel to the wings). Assume a pilot receives radio instructions to roll over 90 degrees and then to pitch over 90 degrees toward the underside of the plane. Everything will be fine if the pilot follows the commands in that order. But if the order is inverted, the plane will take a nosedive. Operations with Cartesian coordinates in space are commutative, but rotations over three dimensions are not.

To gain a clearer vision of what goes on in nature, physicists sometimes resort to "phase space." Such a space is an alternative to Cartesian coordinates—a researcher can plot the position of an electron against its momentum, rather than simply its x and y locations. Because of the Heisenberg uncertainty principle, one cannot measure both quantities simultaneously. As a consequence, position times momentum does not equal momentum times position. Hence, the quantum phase space is noncommutative. Moreover, introducing such noncommutativity into an ordinary space—say, by making the x and the y coordinates noncommutative—produces a space that has noncommutative geometry.

Through such analyses, Connes discovered the peculiar properties of his new geometry, properties that corresponded to the principles of quantum theory. He has spent three decades refining his thinking, and even though he laid down the basics in a 1994 book, researchers beat a path to listen to him. On a day plagued by typical March showers and wind, about 60 of the crème de la crème of French mathematicians fill Salle 5 at the Collège de France. Like a caged lion, the 59-year-old Connes walks quickly back and forth between two overhead projectors, talking rapidly, continually replacing transparencies filled with equations. Outside, police sirens scream amid student protestors trying to occupy the Sorbonne next door in response to the French government's proposed new employment law.

Connes seems oblivious to the commotion—even afterward, while crossing the rue Saint-Jacques past blue police vans and officers in riot gear, he keeps talking about how his research has led him to new insights into physics. As an example, Connes refers to the way particle physics has grown: The concept of spacetime was derived from electrodynamics, but electrodynamics is only a small part of the Standard Model. New particles were added when required, and confirmation came when these predicted particles emerged in accelerators.

But the spacetime used in general relativity, also based on electrodynamics, was left unchanged. Connes proposed something quite different: "Instead of having new particles, we have a geometry that is more subtle, and the refinements of this geometry generate these new particles." In fact, he succeeded in creating a noncommutative space that contains all the abstract algebras (known as symmetry groups) that describe the properties of elementary particles in the Standard Model.

To Connes, physics calculations not only reflect reality but hide mathematical jewels.

The picture that emerges from the Standard Model, then, is that of spacetime as a noncommutative space that can be viewed as consisting of two layers of a continuum, like the two sides of a piece of paper. The space between the two sides of the paper is an extra discrete (noncontinuous), noncommutative space. The discrete part creates the Higgs, whereas the continuum parts generate the gauge bosons, such as the W and Z particles, which mediate the weak force.

Connes has become convinced that physics calculations not only reflect reality but hide mathematical jewels behind their apparent complexity. All that is needed is a tool to peer into the complexity, the way the electron microscope reveals molecular structure, remarks Connes, whose "electron microscope" is noncommutative geometry. "What I'm really interested in are the complicated computations performed by physicists and tested by experiment," he declares. "These calculations are tested at up to nine decimals, so one is certain to have come across a jewel, something to elucidate."

One jewel held infinities. Although the Standard Model proved phenomenally successful, it quickly hit an obstacle: infinite values appeared in many computations. Physicists, including Gerard 't Hooft and Martinus Veltman of the University of Utrecht in the Netherlands, solved this problem by introducing a mathematical technique called renormalization. By tweaking certain values in the models, physicists could avoid these infinities and calculate properties of particles that corresponded to reality.

Although some researchers viewed this technique as a bit like cheating, for Connes renormalization became another opportunity to explore the space in which physics lives. But it wasn't easy. "I spent 20 years trying to understand renormalization. Not that I didn't understand what the physicists were doing, but I didn't understand what the meaning of the mathematics was that was behind it," Connes says. He and physicist Dirk Kreimer of the Institut des Hautes Études Scientifiques near Paris soon realized that the recipe to eliminate infinities is in fact linked to one of the 23 great problems in mathematics formulated by David Hilbert in 1900—one that had been solved. The linkage gave renormalization a mathematically rigorous underpinning—no longer was it "cheating."

The relation between renormalization and noncommutative geometry serves as a starting point to unite relativity and quantum mechanics and thereby fully describe gravity. "We now have to make a next step—we have to try to understand how space with fractional dimensions," which occurs in noncommutative geometry, "couples with gravitation," Connes asserts. He has already shown, with physicist Carlo Rovelli of the University of Marseille, that time can emerge naturally from the noncommutativity of the observable quantities of gravity. Time can be compared with a property such as temperature, which needs atoms to exist, Rovelli explains.

What about string theory? Doesn't that unify gravitation and the quantum world? Connes contends that his approach, looking for the mathematics behind the physical phenomena, is fundamentally different from that of string theorists. Whereas string theory cannot be tested directly—it deals with energies that cannot be created in the laboratory—Connes points out that noncommutative geometry makes testable predictions, such as the Higgs mass (160 billion electron volts), and he argues that even renormalization can be verified.

The Large Hadron Collider will not only test Connes's math but will also give him data to extend his work to smaller scales. "Noncommutative geometry now supplies us with a model of spacetime that reaches down to 10^{-16} centimeter, which still is a long way to go to reach the Planck scale, which is 10^{-33} centimeter," Connes says. That is not quite halfway. But to Connes, the glass undoubtedly appears half full.

Alexander Hellemans is based in Bergerac, France.

Examples of spectral geometries

- The Noncommutative Torus: $UV = e^{2\pi i\theta} VU$
  Dirac operator the same as on the torus [Connes]
- Finite matrix algebras ($M_n(\mathbb{C}) \oplus M_k(\mathbb{C}) \oplus \cdots$
  Dirac operator is a finite hermitian matrix [AS & Paschke, Krajewski]
- Quantum spaces (q-deformations $xy = qyx$)
  Interesting Dirac operators [AS, Dabrowski, et al
- Moyal deformation $[x^\mu, x^\nu] = \theta^{\mu\nu}$
  The usual Dirac [Gracia-Bondia et al]
- κ-deformation $[x^0, x^i] = \frac{1}{\kappa} x^i$
  Doubly Special Relativity

How to construct them?

There is so far no general method. Mathematically, spectral triple is an unbounded Fredholm module, a representative of the K-homology class.

How to get some numbers?

Use the Dirac and exotic (regularized) traces

The Dixmier trace is a good example, take T a positive operator:

$$\text{Tr}_{\omega}(T) = \lim_{N\to\infty} \frac{1}{\log N} \sum_{i=1}^{N} \mu_i(T).$$

Residues...

Take an unbounded operator D such that D^{-1} is compact. For a sufficiently large $r > 0$ $|D|^{-r}$ is trace class, consider:

$$\zeta_D(z) = \text{Tr}\, |D|^{-z}$$

as a complex function - then it is analytic for the real part of z sufficiently big. Then take the analytic continuation to $\mathbb{C}$ and just compute the residue:

$$\text{Res}_{z=d}\zeta_D(z)$$

Getting the numbers...

Take S^1:

The Dirac operator is $De_n = ne_n$, the ζ function is

$$\zeta_D(z) = 2\sum_{n=1}^{\infty} \frac{1}{n^z} = 2\zeta(z),$$

The dimension is not a number

It is a discrete set in the complex plane

The dimension spectrum of a manifold

id $d, d-1, d-2, \ldots$

For fractals...

it could be a set of real numbers...

Family of Dirac operators

Dirac operators as dynamical variables

One we have one Dirac operator we have many of them, just take the space of all inner fluctuactions of Dirac operators:

$$\mathsf{D}_A = \{D' = D + A\},$$

where A is a self-adjoint one-form $A = \sum_i a_i [D, b_i]$. Classically This corresponds to twisting of the Dirac operator by a (trivial) complex line bundle, so adding $U(1)$ gauge field.

Inner fluctuations...

Inner fluctuation of inner fluctuation are inner fluctuations!

Warning:

The space of all possible Dirac operators is much bigger in the classical case than the space of metrics (but for real spectral geometries it excludes $U(1)$ gauge connections). The Dirac operators with torsion are, for instance, allowed.

A proposition for the action...

Let the action be spectral

For a fixed function f (cutoff) consider the following functional on the space of D_A:

$$S(D_A) = \mathrm{Tr}\, f\left(\frac{D_A}{\Lambda}\right),$$

Then use the asymptotic expansion

$$S(D_A, f, \Lambda) = \sum_{k \in Sd^+} f_k \Lambda^k \left(\mathrm{Res}_{z=k} \zeta_{D_A}(z)\right) + f(0)\, \zeta_{D_{\mathcal{A}}}(0) + \mathcal{O}(\Lambda^{-2})$$

where Sd^+ is the positive part of the dimension spectrum of $(\mathcal{A}, \mathcal{H}, D)$, $f_k = \frac{1}{2}\int_0^\infty f(t)\, t^{k/2-1}\, dt$. Moreover:

$$\zeta_{D_A}(0) - \zeta_D(0) = \sum_{q=1}^{n} \frac{(-1)^q}{q} \mathrm{Res}_{z=d}\left(\mathrm{Tr}(AD^{-1})^q |D|^{-z}\right),$$

What is the aim?

- We aim to reconstruct the classical Lagrangian
- The fermionic fields: elements of the Hilbert space
- All remaining (bosonic) fields come from geometry
- Freedom is in the choice of the Dirac operator
- **This includes gravity (metric)**
- **This includes (possibly) torsion, dilaton...**
- **This includes gauge fields $D+A$!**
- **This includes Higgs doublet!**

How can it be possible ?

For just a simple particular model...

.. of the two-point space.

$$d\begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix} = \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}\right].$$

So, what is the space of all Dirac operators ?

$$D(\Phi) = \begin{pmatrix} 0 & \Phi \\ \Phi^* & 0 \end{pmatrix}$$

The action for two-point space alone makes little sense...

BUT if you consider $M^4 \times \mathbb{Z}_2$: then you obtain (take M^4 to be flat):

$$S(D,\Phi,\Lambda) = \Lambda^4 \mathrm{Vol}(M) + c(\Lambda^2)\int_M (\Phi\Phi^* - 1)^2 + c_1 \int_M (\nabla\Phi)(\nabla\Phi)^*.$$

A more refined model...

For the geometries of the type $M \times F$, where M is a Riemannian manifold and F is a discrete geometry we obtain

$$
\begin{aligned}
S &= \frac{1}{\pi^2}(48 f_4 \Lambda^4 - f_2 \Lambda^2 c + \frac{f_0}{4} d) \int \sqrt{g}\, d^4x \\
&+ \frac{96 f_2 \Lambda^2 - f_0 c}{24\pi^2} \int R \sqrt{g}\, d^4x \\
&+ \frac{f_0}{10\pi^2} \int (\frac{11}{6} R^* R^* - 3 C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma}) \sqrt{g}\, d^4x + \cdots
\end{aligned}
$$

Spectral action:

$$
\begin{aligned}
\cdots \quad + \quad & \frac{(-2\,a\,f_2\,\Lambda^2 + e\,f_0)}{\pi^2} \int |\varphi|^2 \sqrt{g}\, d^4x \\
+ \quad & \frac{f_0}{2\,\pi^2} \int a\,|D_\mu \varphi|^2 \sqrt{g}\, d^4x \\
- \quad & \frac{f_0}{12\,\pi^2} \int a\,R\,|\varphi|^2 \sqrt{g}\, d^4x \\
+ \quad & \frac{f_0}{2\,\pi^2} \int (g_3^2\, G^i_{\mu\nu}\, G^{\mu\nu i} + g_2^2\, F^\alpha_{\mu\nu}\, F^{\mu\nu\alpha} + \frac{5}{3}\, g_1^2\, B_{\mu\nu}\, B^{\mu\nu}) \sqrt{g}\, d^4x \\
+ \quad & \frac{f_0}{2\,\pi^2} \int b\,|\varphi|^4 \sqrt{g}\, d^4x
\end{aligned}
$$

How to get that?

- Choose a suitable finite geometry:

$$F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$$

- Take a product geometry:

$$\mathcal{A} = C^\infty(M) \otimes F$$

- Choose the Hilbert space (representation):

$$\mathcal{H} = \ldots$$

- Construct all possible NCG on this algebra:

$$D = D \otimes 1 + \gamma \otimes D_F,$$

How to choose the Hilbert space?

- Start from a bigger algebra:

$$F \ni (\lambda, q, m) \to \big((\lambda, \bar{\lambda}), q, (\lambda, m)\big) \in \mathbb{H} \oplus \mathbb{H} \oplus M_4(\mathbb{C})$$

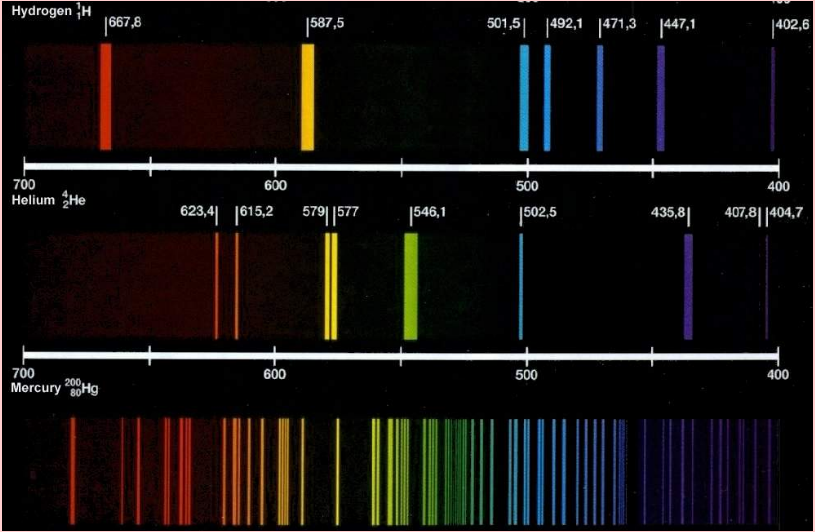
  Then the Hilbert space is a unique fundamental representation of dimension 32.
- Look for all possible Dirac operators, which satisfy order one connection and connect all disconnected components. Take $\mathcal{A}_F$ as the maximal subalgebra such that $[D, \mathcal{A}_F] \neq 0$.
- Or: take the input from physics !
- Once we are finished we can identify the elements of the Hilbert space with fermions: leptons and quarks.
- The finite part of the Dirac operator is an arbitrary self-adjoint matrix, which satisfies order one condition.

CONCLUSIONS

- NCG is a way to describe geometry more generally.
- NCG allows for much broader scope than differential geometry.
- NCG is a sound mathematics !
- NCG is not a claim that the world looks like Moyal or κ-Minkowski !
- NCG allows for freedom yet restricts quite a lot !

CONCLUSIONS:

The last time I checked the world was commutative...



The notes: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3½Lectures on Noncommutative Geometry

A. Sitarz

1 What is noncommutative geometry?

The slides and the some earlier notes (a bit more mathematical but with many more examples) will be available soon on my web page. Notes for these lectures will appear later.

Advanced literature: