A friendly introduction
to Noncommutative Geometry

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Why (NC) Geometry ?

The intention:

- no abstract non-testable theories are harmed
- look at the world and learn what is its geometry
- provide the tools rather than models and solutions

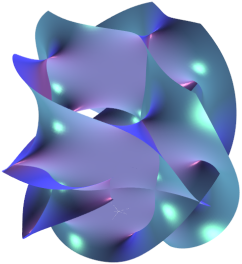
The motivation:

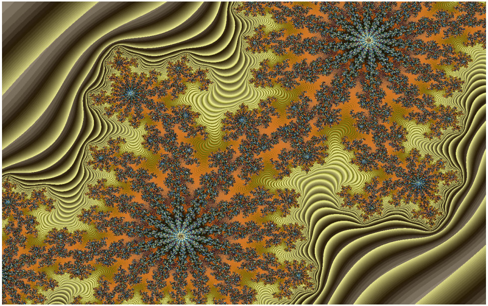
- all geometry is in fact noncommutative
- QM and QFT are in fact NCG
- ... do you really believe that space is a manifold ?

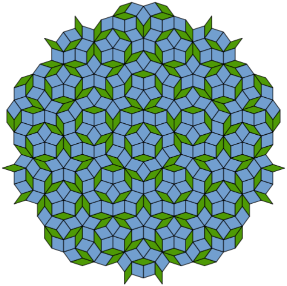
The aim:

- provide a meaningful definition of geometry
- ... tools, which (might) describe the geometry of interactions
- ... clear a path towards quantization of space

Spaces







So let the horror begin...

Space and its C^*-algebra

How do we see a space?

We don't. We see and ,,measure” functions on the space.

Example

If X is a (locally) compact Hausdorff space and $C(X)$ is the algebra of continuous functions on X, then $C(X)$ is a commutative (non) unital C^*-algebra.

What is a C^*-algebra?

Let $\mathcal{A}$ be an involutive Banach algebra (that is a complex normed algebra, which is complete as a topological space in the norm). If:

$$||aa^*|| = ||a||^2,$$

then $\mathcal{A}$ is a C^*-algebra.

C^* algebras are everywhere...

Example (Finite dimensional C^* algebras:)

$$M_{n_1}(\mathbb{C}) \oplus M_{n_2}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C})$$

Example (Bounded operators on $\mathcal{H}$)

Take a separable Hilbert space $\mathcal{H}$ and $B(\mathcal{H})$, the algebra of bounded operators on $\mathcal{H}$ (with the operator norm).

Example (Compact operators on $\mathcal{H}$)

The algebra $\mathcal{K}(\mathcal{H})$ (closure of the algebra of operators of finite rank) is a C^* algebra. It is also a two-sided ideal in $B(\mathcal{H})$.

Example (Group algebras and Lie groups)

For any discrete group Γ there its group C^*-algebra ($C_r\Gamma$), for any Lie group G we have the convolution algebra over G.

Gelfand-Naimark Theorems

The norm:

$$||T|| = \sup_{||v||=1} ||Tv||,$$

Theorem (Gelfand-Naimark (GNS))

Every abstract C^*-algebra $\mathcal{A}$ is isometrically $*$-isomorphic to a concrete C^* algebra of operators on a Hilbert space $\mathcal{H}$. If the algebra $\mathcal{A}$ is separable then we can take $\mathcal{H}$ to be separable.

Theorem (Gelfand-Naimark (GN))

If a C^* algebra is commutative then the Gelfand transform, $\mu : \mathcal{A} \to C_0(X_{\mathcal{A}})$, where $X_{\mathcal{A}}$ is the Gelfand spectrum of $\mathcal{A}$ (space of homomorphisms from $\mathcal{A}$ to $\mathbb{C}$):

$$\mu(a)(\chi) = \chi(a), \quad a \in \mathcal{A}, \chi \in X_{\mathcal{A}},$$

is an isometric $*$-isomorphism.

Gelfand-Naimark: the proof

First, is μ an algebra homomorphism?:

$$\mu(a \cdot b)(\chi) = \chi(a \cdot b) = \chi(a)\,\chi(b) = \mu(a)(\chi)\,\mu(a)(\chi).$$

Is μ a star homomorphism?:

$$\mu(a^*)(\chi) = \chi(a^*) = \chi(a)^* = (\mu(a)(\chi))^* = (\mu(a))^*(\chi).$$

By definition $\mu(a)$ is a continuous function (such is the topology on $X_{\mathcal{A}}$, called weak-$*$ topology on the Banach space dual to $\mathcal{A}$) and $||\mu(a)|| = \sup_{\chi \in X_{\mathcal{A}}} \chi(a)$. But if $\lambda \neq 0$ belongs to the spectrum of a then there exists a character χ of $\mathcal{A}$ such that $\chi(a) = \lambda$.

$$r(a) = \sup\{|\lambda|, \lambda \in \mathbb{C}, (a - \lambda) \text{ is not invertible in } \mathcal{A}\},$$

Finally: $||a|| = r(a) = ||\mu(a)||$.
Exercise: Why there exist $\chi(a) = \lambda$ for $\lambda \in \sigma(a)$?

What is noncommutative topology

TOPOLOGY	vs ALGEBRA
(locally compact) topological space	C^*-algebra
homeomorphism	automorphism
continuous proper map	morphism
compact space	unital C^*-algebra
open (dense) subset	(essential) ideal
compactification	unitization
Stone-Čech compactification	multiplier algebra
cartesian product	tensor product

Is that enough?

There is much more to geometry than topology
There are groups, vector bundles, connections and metric.

Are there interesting C^* algebras ?

Example (Algebras generated by an operator:)

Let T be a bounded operator on a Hilbert space $\mathcal{H}$ and T^* its hermitian conjugate - we consider the algebra $C^*(T)$ as the closure of the algebra of polynomials in T, T^*.

Example (Toeplitz algebra)

Take T to be a unilateral shift $T_s e_n = e_{n+1}, \, n \geq 0$. Then the algebra $C^*(T_s)$ is isomorphic to the algebra $C(S^1) \oplus \mathcal{K}$, in the following sense:

$$0 \to \mathcal{K} \xrightarrow{i} C^*(T_s) \xrightarrow{\pi} C(S^1) \to 0,$$

Example (IRA)

Consider an algebra with unitary operators:

$$UV = e^{2\pi i\theta} VU, \quad (\dagger)$$

Are there interesting C^* algebras ?

Example (Irrational rotation algebra)

Consider the Hilbert space $L^2(S^1)$ and the following operators:

$$(Uf)(z) = zf(z), \quad (Vf)(z) = f(e^{2\pi i\theta}z),$$

We define $\mathbb{T}^2_\theta$ as a C^* algebra generated by the unitary operators U, V, U^*, V^*. We easily check that:

$$UV = e^{2\pi i\theta} VU, \quad (\dagger)$$

Example (Universal C^*-algebra)

For the relation of the type ($\dagger$) - we can consider a universal C^* algebra: that is, an algebra, generated by two unitaries U, V, which satisfy the relation $(\dagger)$ such for any other algebra with the same property (generated by $\tilde{U}, \tilde{V}$) there exists a well-defined homomorphism, which maps U, V to $\tilde{U}, \tilde{V}$.

Smooth functions & vector fields:

- To describe manifolds we need the notion of differentiable functions.
- To go deeper into the differential geometry we need the notion of vector fields and differential forms:

WARNING !

Vector fields are external derivations on the algebra of smooth functions.

$$\delta(f \cdot g) = \delta(f) \cdot g + f \cdot \delta(g),$$

and vector fields are a module over smooth functions:

$$(f \cdot \delta)g = f\,(\delta(g))\,.$$

BOTH CONDITIONS MIGHT NOT BE MET IN NONCOMMUTATIVE CASE

Smooth functions & differential forms:

- Differential forms are a bimodule over smooth functions:

$$df \cdot g = d(fg) - f \cdot (dg).$$

- The external derifative satifies the Leibniz rule (is a derivation) independently of the fact that the algebra is commutative:

$$d(fg) = (df) \cdot g + f \cdot (dg),$$

- The external derivative is a derivation valued in a bimodule over the algebra!

The good news:

We can define differential forms and external derivatives for arbitrary (even noncommutative) algebras!

The bad news:

These construction is by no means unique: there are many too many differential calculi over algebras!

The differential graded algebra

DEFINITION (Graded algebra)

An algebra $\mathcal{A}$ is $\mathbb{Z}(\mathbb{N})$-graded if $\mathcal{A} = \bigoplus_n \mathcal{A}^n$, and $\mathcal{A}^n \cdot \mathcal{A}^m \subset \mathcal{A}^{n+m}$,

DEFINITION (Differential graded algebra)

An algebra Ω is a differential graded algebra if Ω is $\mathbb{N}$-graded, and there exists a linear map $d : \mathcal{A}^n \to \mathcal{A}^{n+1}$ such that:

$$d(\omega \cdot \rho) = (d\omega) \cdot \rho + (-1)^{\mathrm{deg}\omega} \omega \cdot (d\rho),$$

Example: differential forms over a manifold M

$$\Omega^0 = C^\infty(M), \Omega^n = \Omega^n(M),$$

FODC

FODC over an algebra $\mathcal{A}$ is a pair $\Omega^1(\mathcal{A})$, a bimodule over $\mathcal{A}$ and a Ω^1-valued derivation on $\mathcal{A}$: $d(ab) = (da)b + a(db)$.

The differential graded algebras over NC algebras

The universal calculus

Let $\Omega^1_u(\mathcal{A}) = \ker \mu$, where:

$$\mu : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}, \quad \mu(a \otimes a') = aa'.$$

The universal FODC is $\Omega^1_u(\mathcal{A})$ with the universal differential:

$$d_u(a) = a \otimes 1 - 1 \otimes a,$$

Inner FODC

We say that FODC is inner if there exists an element $X \in \Omega^1(\mathcal{A})$ such that:

$$da = X \cdot a - a \cdot X, \quad \forall a \in \mathcal{A}.$$

Example: two points

$$\mathcal{A} = \mathbb{C} \oplus \mathbb{C}, \quad \Omega^1 = \mathbb{C} \oplus \mathbb{C}.$$

The two points example (continued)

The inner derivation:

$$d\begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix} = \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}\right].$$

Is it universal ?

Elements of the algebra: $c_1 e_1 + c_2 e_2$, where $c_1, c_2 \in \mathbb{C}$, $e_i^2 = e_i, e_1 e_2 = e_2 e_1 = 0$. What is Ω_u^1 ?

$$\omega = c_1' e_1 \otimes e_2 + c_2' e_2 \otimes e_1,$$

where $c_1', c_2' \in \mathbb{C}$.

$$\begin{aligned} d(c_1 e_1 + c_2 e_2) =& c_1(e_1 \otimes e_2 - e_2 \otimes e_1) + c_2(e_2 \otimes e_1 - e_1 \otimes e_2) \\ =& (c_1 - c_2)(e_1 \otimes e_2) + (c_2 - c_1)(e_2 \otimes e_1). \end{aligned}$$

What happened to vector fields ?

What are vector fields ?

Vector fields are in fact operators on the Hilbert space!

What is the link between vector fields and forms ?

Imagine the graded differential algebra over $C^\infty(M)$ and take as the Hilbert space the completion (in the L^2 norm) of the module of one-forms. Then the external derivative is a densely defined operator on the Hilbert space. Then:

$$(da)\Psi = d \cdot (a\Psi) - a(d\Psi) = [d,a]\Psi.$$

We can encode the derivatives...

... through commutators with some operators (in this case: first order differential operators). Why not build geometry using this notion ?

Enters: Dirac operator

What is the Dirac operator ?

The usual construction: start with a Riemannian manifold (compact, closed) with a metric g. Find Clifford algebra bundle, identify the spinor bundle, then lift the metric connection (the Levi-Civita one if you want the true Dirac), lift it to the spinor bundle, compose with Clifford map – you get D. Then prove some theorems about D.

Operational definition

Take an algebra represented on a Hilbert space and look for operators, which behave like Dirac operators: first order differential operator such that gives you the metric:

$$d(x,y) = \sup_{||[D,f]\leq 1} |f(x) - f(y)|,$$

is unbounded, discrete spectrum and the eigenvalues have certain growth.

What information carries the Dirac operator ?

1. differential calculus: $da = [D, a]$
2. metric ($d(x,y) = \sup_{||[D,f]|| \leq 1} |f(x) - f(y)|$)
3. additional connection (if spinors twisted by a vector bundle)
4. dimension (growth of eigeinvalues: $N(\Lambda) \sim \Lambda^d$),
5. integral (exotic traces)

$$\mathrm{Tr}(a) = \mathrm{Res}_{z=d} \mathrm{tr}(a|D|^{-z})$$

Extend the notion of vector bundles

Theorem (Serre-Swan)

Let M be a compact manifold, and $E \to M$ a finite dimensional vector bundle. Then the space of continuous sections of E is a finitely generated projective modules over $C(X)$ and every such module is a space of sections of a vector bundle over M.

What are projective modules?

A module $\mathcal{M}$ over an algebra $\mathcal{A}$ is projective if and only if one of the following conditions is satisfied:

- $\mathcal{M}$ is a summand of a free module: $\mathcal{M} \oplus \mathcal{N} = \mathcal{A}^n$,
- Any surjective module morphism $\pi : \mathcal{N} \to \mathcal{M}$ splits, $\exists$ a morphism $\rho : \mathcal{M} \to \mathcal{N}$, such that $\pi \circ \rho = \mathrm{id}_{\mathcal{M}}$,
- Given a surjective module morphism $\pi : \mathcal{N}' \to \mathcal{N}$ any homomorphism $\rho : \mathcal{M} \to \mathcal{N}$ can be lifted to a homomorphism $\rho' : \mathcal{M} \to \mathcal{N}'$ such that $\rho = \pi \circ \rho'$,

Projective modules and projections

How to construct...

...projective modules ? Take an algebra $\mathcal{A}$, take a free module $\mathcal{A}^n$, take $p \in M_n(\mathcal{A})$ such that $p^2 = p = p^*$ (a projection) and define:

$$\mathcal{M}_p = \mathcal{A}^n p$$

It is a finitely generated projective module. WHY?

Connection

Let $\mathcal{M}$ be a left module over $\mathcal{A}$ and Ω^* a DGA. The map $\nabla : \mathcal{M} \to \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{M}$ is a connection if:

$$\nabla(a\xi) = a\nabla(\xi) + da \otimes_{\mathcal{A}} \xi.$$

FGPM are natural for connections.

If the module $\mathcal{M}$ is projective there always exists a connection (actually it is equivalent to $\mathcal{M}$ being projective)

Spectral triples

Define: a spectral triple

Algebra $\mathcal{A}$, its faithful representation π on a Hilbert space $\mathcal{H}$, a selfadjoint operator D, satisfying several conditions:

1. $\forall a \in \mathcal{A}\ [D, \pi(a)] \in B(\mathcal{H})$, D^{-1} is compact
2. even ST: $\exists \gamma \in \mathcal{A}' : \gamma^2 = 1, \gamma = \gamma^\dagger, \gamma D + D\gamma = 0$,
3. $\exists J$, antilinear $J^2 = \pm 1, JJ^\dagger = 1$
   $J\gamma = \pm \gamma J, JD = \pm DJ, [J\pi(a)J, \pi(b)] = 0$,
4. $[[D, a], J\pi(b)J] = 0$ (D: first order differential operator)
5. ...+ conditions of ,,analysis" type

Theorem

If $\mathcal{A} = C^\infty(M)$, M a spin Riemannian compact manifold, $\mathcal{H} = L^2(S)$ (sections of spinor bundle) and D the Dirac operator on M then to $(\mathcal{A}, \mathcal{H}, D)$ is a spectral triple (with a real structure).

The final dictionary

GEOMETRY	ALGEBRA
vector bundle	finitely generated projective module
differential forms	differential forms
differential forms	Hochschild homology
de Rham cohomology	cyclic cohomology
vector fields	operators
group	Hopf algebra
Lie algebra	Hopf algebra
principal fibre bundle	Hopf-Galois extension
measurable functions	von Neumann algebra
infinitesimals	compact operators
metric	Dirac operator
spin^c geometry	spectral triple
spin geometry	real spectral triple
integrals	exotic traces

CHOOSE YOUR WAY !