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Pointless Geometry[*]

We present a brief overview of the mathematical and physical motivations for a change of paradigm: to develop a geometry without points. Based on that we sketch the basics ideas of noncommutative geometry, focusing on the notion of measurements. Finally, we extend the idea of measuring the distance between states towards the notion of distances between representations.

1. Introduction

The fundamental physical theories are a big success: we have fantastic knowledge of gravity and the forces acting between macroscopic objects on one hand while at the same time we can use and test quantum mechanics and quantum field theory with impressive precision. With the recent discovery [ATLAS Collaboration 2012] of the Higgs boson, all pieces of the Standard Model are put together and give a working and correct description of the elementary constituents of matter. However, on the other hand, the arguments that the universe is filled with dark matter and dark energy [Capozziello & Lambiase 2013], which we do not understand give a hint that we are still far from reaching the point of knowing everything. Even further lies an understanding of mathematical language and fine details of Quantum Field Theory or the penultimate quest of theoretical physics, which hopes to unify General Relativity with Quantum Mechanics.

Having outlined the questions and problems we might start to think whether the current knowledge we have requires a step forward towards the change of paradigm in our description of the world. To understand what needs to be changed we need to understand what is the basis of the description and what suggestions we might have.

Let us observe two facts - the first that the physics is in fact *geometry* as it is justified by the purely geometric description of gravity through Riemannian)

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geometry and Yang-Mills theory (gauge theories), which is the basis for fundamental particle interactions. The second known fact is that the world is *quantum* (as could be seen from the atomic spectra) and Quantum Mechanics together with *Quantum Field Theory* aims to describe particle interactions.

Let us stress that both the fundamentals of modern physics are quite distinct and there appears no clear, evident and certain passage from the world of geometry into the world of quanta.

In practice, we use these tools separately, hoping that we understand sufficiently well the link between them or that we know why they both appear (at different levels) in our description of physical phenomena.

There is, however, a tempting question – what if these are just appearances of just one single mathematical description? What if we see the same object but only from two different sides, what if the world is *simultaneously* **quantum** and **geometric**?

If this is the case, we need to reunite these descriptions into one solid, viable, mathematical framework. *Noncommutative Geometry* is a proposition, which goes into this direction. Beware that is not a finished chapter yet, being developed, which has a lot of mathematical applications (starting from the theory of C^*-algebras, K-theory, index theorem, Hopf algebras etc) as well as many interesting physical ones (Standard Model, Quantum Hall Effect, quantum computing, effective quantum field theories, integrable systems).

The aim of this exposition is to demonstrate that Noncommutative Geometry provides tools which could be effectively used to construct physical models. Then, in turn, one can invert the problem and by looking at the world (in particular at particle physics, described by the Standard Model) try to recover the *geometry* behind that reality – expressed in noncommutative terms.

The long term goal is certainly to provide a meaningful definition of geometry, which would allow to incorporate both the fundamental interactions as we know them with the notion of quantised space (for some arguments and models see [Doplicher, Fredenhagen & Roberts 1995]).

2. The illusion of points

Let us focus first on some mathematical and physical motivations. Geometry is based on the principle of describing spaces which are sets of points equipped with an additional structure. Yet, physically a point is an elusive theoretical (and experimental) concept and, moreover, even though we describe the world as a space (a collections of points), effectively, we always use coordinates, which are functions on the space. Therefore, the notion of a function (in particular a continuous function) appears to be more fundamental and practical than that of a point. In physics this is clearly visible when we take into account quantum effects, in particular, the Heisenberg uncertainty principle.

We shall therefore ask the question – what are points if seen from the point of view of functions and, later, if there are some physical limits on the *measurement* of points.

2.1. Points from an algebraic point of view

As we noted earlier one cannot see a point though we can easily visualise a space, in particular, finite and discrete spaces (lattices) as a collection of what we call points. Now, to describe functions on points we need some more structure, in particular we need some way of telling the points apart. This is provided by a collection of well-chosen functions, which in the language of spaces is topology and translates to continuous functions. A good collection of functions is therefore a collection of complex-valued continuous functions, which separate the points. Such a collection has a special structure, which is so evident that we usually forget about it: functions form an involutive algebra, or, in easy words, we may multiply the continuous functions and the result is a continuous functions and conjugate them. What makes the structure interesting is the following important remark:

Remark 2.1. *If X is a (locally) compact Hausdorff space and $C(X)$ is the algebra of complex-valued continuous functions on X, then $C(X)$ is a commutative (non) unital C^*-algebra.*

A C^*-algebra is a complex normed algebra, that is, an algebra, equipped with a submultiplicative norm, which is complete as a topological space in the given norm (means any sequence which converges to something, converges actually to an element in the algebra), and, additionally, for every element $a \in A$:

$$||aa^*|| = ||a||^2.$$

We need to learn two important theorems, which set up the framework of what we want to achieve.

Theorem 2.2 (Gelfand-Naimark-Segal [Segal 1947]). *Every abstract C^*-algebra $\mathcal{A}$ is isometrically $*$-isomorphic to a concrete C^* algebra of operators on a Hilbert space $\mathcal{H}$. If the algebra $\mathcal{A}$ is separable then we can take $\mathcal{H}$ to be separable.*

Theorem 2.3 (Gelfand-Naimark [1943]). *If a C^* algebra is commutative then it is an algebra of continuous functions on some (locally compact, Hausdorff) topological space.*

The first theorem tells us that every C^* algebra is an algebra of bounded operators on a Hilbert space thus showing a link between the geometry (points) and quantum theory (operator algebras) whereas the second one assures that we do not lose anything when passing from one description to the other.

Example 2.4 (Irrational Rotation Algebra *aka* Noncommutative Torus). *An interesting example is a family of C^* algebras defined as follows. Consider the Hilbert space $L^2(S^1)$ and the following operators:*

$$(Uf)(z) = zf(z), \quad (Vf)(z) = f(e^{2\pi i\theta}z),$$

where $0 < \theta < 1$ is an irrational real number. We define $\mathbb{T}^2_\theta$ as a C^* algebra generated by the unitary operators U, V, U^*, V^*. We easily check that:

$$UV = e^{2\pi i\theta} VU.$$

Depending on the value of θ the algebras have dramatically different properties!

2.2. What are points in mathematics?

Usually one starts with a point and then defines functions but let us look at it the other way round. Imagine we have a commutative algebra of functions and we want to *define* points. How can we do it?

There are at least three ways to do it:

Remark 2.5. *Let $\mathcal{A}$ be a commutative C^* algebra, which corresponds to $C(X)$ for some topological space X. Then the following are equivalent:*

- $x \in X$ *is a point*
- $\chi_x : \mathcal{A} \to \mathbb{C}$, $\chi_x(f) = f(x)$ *is a character of the algebra* $\mathcal{A}$
- χ_x *defined as above is a pure state on* $\mathcal{A}$
- χ_x *is an irreducible representation of* $\mathcal{A}$
- $I_x \subset \mathcal{A}$, $I_x = \{f \in \mathcal{A} : f(x) = 0\}$ *is a maximal ideal*

Note that all these notions are, of course, related to each other and could be defined for any algebra, not necessarily for a commutative one. However, only for the commutative one they all coincide. Of course, for a nonabelian algebra we might easily see that they are distinct.

Example 2.6. *Let $A = M_2(\mathbb{C})$, the algebra of 2×2 complex matrices. It is the smallest nontrivial nonabelian C^*-algebra, and we have:*

- *there are no nontrivial characters of the algebra,*
- *the space of pure states is S^2 (a sphere)*
- *there exists exactly one irreducible representation (on $\mathbb{C}^2$),*
- *there are no nontrivial two-sided ideals in $M_2(\mathbb{C})$.*

Even this simple example signifies that one cannot easily generalise the notion of a point.

2.3. Points in physics

According to quantum mechanics it follows from the uncertainty principle that it is impossible (both in theory and in practice) to *measure* simultaneously position and momenta with an arbitrarily small accuracy. This means that that the classical phase space (the space of possible positions and momenta) of a physical object is no longer a space in quantum theory. Instead of using points we already know that what we see as coordinates are, in reality, operators on a Hilbert space. Moreover, what we usually describe as the state of a physical object (by giving the coordinates describing position and momenta) corresponds to the expectations values of these operators for a given state (a normalised vector) in the Hilbert space.

The main question is, however, does it follow that we cannot measure small distances at all? First of all, note that currently our limit of measurement is 10^{-15}m (size of proton) or, if we consider time measurements, it is an attosecond 10^{-18}s [Corkum & Krausz 2007].

Of course, there exists a universal constant, the Plank length, which is built of fundamental constants:

$$\ell_{\mathrm{P}} = \sqrt{\frac{\hbar G}{c^3}} \approx 1.6162 \times 10^{-35}\mathrm{m},$$

which is somehow 18 orders or magnitude smaller. Its origin, apart from purely dimensional considerations is linked with though experiments and arguments linking uncertainty relations of quantum mechanics with the Schwarzschild radius of general relativity (for a review see [Garay 1995]).

There are numerous propositions for the limits of measurements and uncertainty relations for the Cartesian coordinates, for instance Salecker and Wigner

[Salecker & Wigner 1958] proposed that the limit on the accuracy of measurement of a distance l is:

$$\delta l \geq (l \ell_{\mathrm{P}}^2)^{\frac{1}{3}},$$

which roughly suggests that for the typical distances (1m) it would be 10^{-23}m.

3. What is Noncommutative Geometry?

3.1. The first dictionary

As we have shown there exists an equivalence between commutative C^*-algebras and spaces, which could be extended to other topological constructions, like continuous maps between spaces, Cartesian products etc. The following dictionary is just a short list of corresponding operations:

TOPOLOGY	ALGEBRA
locally compact Hausdorff topological space	nonunital C^*-algebra
homeomorphism	automorphism
continuous proper map	morphism
compact Hausdorff topological space	unital C^*-algebra
open (dense) subset	(essential) ideal
compactification	unitization
Stone-Čech compactification	multiplier algebra
Cartesian product	tensor product

This, however, does not define a geometry: to have it we need to learn how to *differentiate* and *measure* the objects we have.

3.2. How to differentiate in algebras?

This appears to be an easy task, as differentiation is defined as an operation in the algebra of functions. To be more precise, differentiation is a derivation in the algebra. However, classically, to define a derivative of a function we use points and limits, then prove that functions, which are differentiable still form an algebra.

From the point of view of the algebraic approach, the derivative is a *derivation* of an algebra: a linear map, which obeys the Leibniz rule. In noncommutative algebras there are some derivations, which do not exist for abelian ones – those that come from commutators with an element of an algebra – such derivations are called inner. The interesting derivations are outer (simply those that are not inner). All derivations of commutative algebras are, of course, outer, whereas for the noncommutative ones we might have plenty (too many) of inner ones and very few (or even none) outer derivations. A good example is again the simplest case of $M_2(\mathbb{C})$, which has no outer derivations at all.

For this reason we cannot repeat the classical scheme but we might pass directly to the construction of differential forms. Again, classically this is almost the final step of all constructions, here we shall just start with it. Let us recall the notion of a module. If $\mathcal{A}$ is an algebra, then we say that $\mathcal{N}$ is a left module, if it is a linear space with a bilinear map: $\mathcal{A} \times \mathcal{N} \ni (a, n) \to a \cdot n \in \mathcal{N}$ such that:

$$a \cdot (b \cdot n) = (ab) \cdot n, \quad \forall a, b \in \mathcal{A}, n \in \mathcal{N}.$$

Similarly we define a right-module, the $\mathcal{M}$ is a bimodule if it is simultaneously a left and a right module with mutually commuting left and right multiplications:

$$a \cdot (m \cdot b) = (a \cdot m) \cdot b, \quad \forall a, b \in \mathcal{A}, m \in \mathcal{M}.$$

The motivation comes from the following fact: all differential one-forms $\Omega^1(M)$ over a manifold M form a bimodule over the algebra of smooth functions $C^\infty(M)$ and the external derivative $d : C^\infty(M) \to \Omega^1(M)$ is a bimodule-valued derivation on the algebra $\mathbb{C}^\infty(M)$:

$$d(fg) = (df) \cdot g + f \cdot (dg), \quad \forall f, g \in C^\infty(M).$$

Observe that the external derivative satisfies the Leibniz rule (so it is a derivation) independently of the fact that the algebra is commutative. Therefore, we can generalise a bimodule-valued derivation over any algebra, even if it is noncommutative:

Definition 3.1. *Let $\mathcal{A}$ be an algebra. The first order differential calculus over $\mathcal{A}$ is a pair $(\Omega^1(\mathcal{A}), d)$ where $\Omega^1(\mathcal{A})$ is a bimodule over $\mathcal{A}$ and d is an $\Omega^1(\mathcal{A})$-valued derivation of $\mathcal{A}$:*

$$d(ab) = (da) \cdot b + a \cdot (db), \quad \forall a, b \in \mathcal{A}.$$

This construction has one major drawback: it is by no means unique and there are many (even too many) differential calculi over algebras. There exists (as usually) a universal object

Example 3.2 (Universal differential calculus)**.** *Let $\mathcal{A}$ be a unital algebra and*

$\Omega^1_u(\mathcal{A}) = ker\ \mu$, where:

$$\mu : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}, \quad \mu(a \otimes a') = aa'.$$

The universal first order differential calculus is then $\Omega^1_u(\mathcal{A})$ with the universal differential:

$$d_u(a) = a \otimes 1 - 1 \otimes a.$$

Example 3.3 (Inner Differential Calculi)**.** *For any noncommutative algebras we might have a huge family of differential calculi, which are* inner*, that is if there exists an element $X \in \Omega^1(\mathcal{A})$ such that:*

$$da = X \cdot a - a \cdot X, \quad \forall a \in \mathcal{A}.$$

3.3. How to measure in a geometry without points?

To measure a distance between two points without using any specialised equipment (apart from the clock) the best way is to travel from one point to another with a constant speed and then to find the minimal time needed to reach the destination. To mimic this procedure we measure the length of a curve connecting two points or, even better, we can find a function such that the norm of its gradient is less or equal 1 and just take the absolute value of difference its values at the two points.

The last procedure does not use any notion of a curve or path and therefore is best suited for the approach without points. The necessary element is, however, the norm of the gradient (or equivalently of a differential form).

4. The Spectral Triples

4.1. The Dirac operator

Instead of introducing the metric as an abstract construction (related with distances, norms or metric tensor) we shall rather connect it with the natural operators that are closely bound with metric: the Laplace operator and the Dirac operator. One may wonder why the external derivative is not a natural candidate, however, as it is not a selfadjoint operator, one must look for its natural extension, which is the signature operator. The latter then falls into the class of generalised Dirac operators.

The recipe to construct the Dirac operator (or Dirac-type) over a spin manifold is quite easy. You start with a Riemannian manifold (compact, closed) with a fixed metric g. Find the Clifford algebra bundle, identify the spinor bundle, then lift the metric connection (the Levi-Civita one if you want the *true* Dirac, then lift it to the spinor bundle, compose with the Clifford map and you get D: the first order differential operator on the sections of the spinor bundle. Then finally you are ready to prove some theorems about D.

We shall take a different approach and follow the operational definition, which is as follows: just take a suitable algebra of functions ($C^\infty(M)$, for example) represented on a Hilbert space of some sections of a suitable vector bundle over M and look for operators, which behave more or less like the Dirac operators. That is, they are first order differential operators and the commutator with them gives the natural structure of the first order differential calculus, which should be isomorphic to the *standard* bimodule of differential forms:

$$da := [D, a],$$

understood as an operator on the Hilbert space. What follows, is a simple (in principle) formula for the distances:

$$d(x,y) = \sup_{||[D,f]|| \leq 1, f \in C^\infty(M)} |f(x) - f(y)|, \quad \forall x, y \in M.$$

The Dirac operator (and that includes the signature operator as well) carries a lot of information about the additional structures on the manifold. We have already mentioned the differential calculus and the metric. What is also interesting is that having the Dirac operator we may directly recover the dimension of the manifold as well as perform the integration – using the spectral properties of the Dirac. Here is a short table presenting what the Dirac operator gives us:

tool/object	method
differential forms	$da = [D, a]$
distance between points	$d(x,y) = \sup_{\lvert\lvert[D,f]\rvert\rvert \leq 1} \lvert f(x) - f(y)\rvert$
connections (twisted Dirac operators)	$D \to D + \sum a[D, b]$
metric dimension	eigenvalues asymptotics: $N(\Lambda) \sim \Lambda^d$),
integration	Dixmier trace : $f \to \mathrm{Tr}_\omega(f\lvert D\rvert^{-d})$
trace(s) on pseudodifferential operators	$\mathrm{Tr}(P) = \mathrm{Res}_{z=0}\mathrm{tr}(P\lvert D\rvert^{-z})$

4.2. Dirac operators for noncommutative algebras

We are ready now to present a proposition for a generalisation of Riemannian spin geometry to the realm of noncommutative algebras, which is based on the properties of Dirac operators and constructions we discussed earlier.

Definition 4.1. *A* spectral triple *is a collection of the following data: an algebra $\mathcal{A}$ (which is a dense subalgebra of a C^* algebra) together with its faithful representation π on a Hilbert space $\mathcal{H}$ and a densely defined selfadjoint operator D (called later Dirac operator), which satisfies several conditions:*

- $\forall a \in \mathcal{A}\ [D, \pi(a)] \in B(\mathcal{H})$,
- *even spectral triples:* $\exists \gamma \in \mathcal{A}' : \gamma^2 = 1, \gamma = \gamma^\dagger, \gamma D + D\gamma = 0$,
- $\exists J$, *antilinear* $J^2 = \pm 1, JJ^\dagger = 1$
  $J\gamma = \pm\gamma J, JD = \pm DJ, [J\pi(a)J, \pi(b)] = 0$,
- $[[D, a], J\pi(b)J] = 0$ *(D: first order differential operator),*
- *D has a compact resolvent (so the spectrum of* $|D|$ *is discrete and its eigenvalues have no other accumulation point than* $+\infty$*),*
- *…more technical conditions, which guarantee that we indeed work with differentiable "functions" and the "noncommutative space" is not singular (for simplicity we skip details and refer to any review, like [Connes 2000*b*]).*

Remark 4.2. *If* $\mathcal{A} = C^\infty(M)$, M *a spin Riemannian compact manifold,* $\mathcal{H} = L^2(S)$ *is the Hilbert space of summable sections of the spinor bundle and* D *the Dirac operator on* M *then to* $(\mathcal{A}, \mathcal{H}, D)$ *is a spectral triple (with a real structure).*

The definition (more or less in this form) was proposed by Connes in [Connes 1995] then developed later by many authors. Details of the proof of the above theorem could be found in [Gracia-Bondía, Várilly & Figueroa 2001].

The examples of nontrivial and interesting spectral triples over noncommutative algebras exist although they are not abundant. The first and best known is the spectral triple for the noncommutative torus (see example 2.4) [Connes 1995, Paschke & Sitarz 2006], for theta deformations of manifolds (NC Torus is a special case), [Connes & Landi 2001], finite matrix algebras: $(M_{n_1}(\mathbb{C}) \oplus M_{n_2}(\mathbb{C}) \oplus \cdots)$ [Krajewski 1998, Paschke & Sitarz 1998] and some quantum spaces (q-deformations of spheres, $SU_q(2)$), [Dąbrowski & Sitarz 2003, Dąbrowski, Landi, Sitarz, Van Suijlekom & Várilly 2005] as well as for the Moyal deformation ($[x^\mu, x^\nu] = \theta^{\mu\nu}$) [Gayral, Gracia-Bondía, Iochum, Schücker & Várilly 2004].

5. On distances between representations

In this section we shall propose how to extend the notion of distances not only on states (as shown above) but also on the parallel notions of *points* – that is on maximal ideals and irreducible representations.

Definition 5.1. *Let π_1 and π_2 be two representations. For any $a \in \mathcal{A}$, $\pi_i(a)$ is a bounded operator on a Hilbert space. Let $\sigma_i(a)$ denote the spectrum of $\pi_i(a)$. We set:*

$$d_a(\pi_1, \pi_2) = dist(\sigma_1(a) - \sigma_2(a)),$$

and

$$d(\pi_1, \pi_2) = sup_{||[D,a]|| \leq 1} d_a(\pi_1, \pi_2).$$

Let us observe that the definition has at least some of the desired properties. First of all, for commutative algebras if π_i are irreducible representations, which are then characters of the algebra, the spectrum of $\sigma_i(a)$ consists of a single point in $\mathbb{C}$, which is just the value of the character on a. Thus we recover the usual formula for the distance between points, as presented earlier. Moreover, since the distance on $\mathbb{C}$ satisfies the triangle inequality, so does the above formula. Furthermore, the spectrum of an element does not depend on the unitary transformations, the distance depends only on the class of the representation with respect to unitary equivalence.

5.1. Open problems

Before we finish this short overview and presentation of some new ideas let us indicate a series of open problems, which are natural in this context.

- If the distance is zero, does it imply that the representations are unitarily equivalent?
- Does the defined distance give a metric on the space of all irreducible representations?
- Is there a relation between the distance on states and the distance on representations?
- Does the existence of the distance between representations implies existence of distance between states?
- Primitive ideals are kernels of irreducible representations - is there a natural distance on ideals (maximal ideals)?

6. Conclusions

There is, of course, more to noncommutative geometry than spectral triples alone – and to give an idea we present a short extension of the dictionary of classical terms. The mathematical foundations of Noncommutative Geometry could be found in the original paper by Connes [1985] and the book [1994]. A more recent book by Connes and Marcolli [2008] includes also details on the noncommutative description of the Standard Model. A good basic and thorough textbook (again, mathematically oriented) is the one by Gracia-Bondia, Várilly and Figureoa [2001], some other aspects are presented in a textbook by Khalkhali [2009]. There are numerous old and new reviews and lecture notes like [Connes 2000*a*, 2000*b*], a collection of short expositions [Scheck, Upmeier & Werner 2002] and some expository articles [Kastler 2000]. A slightly different approach can be found in [Dubois-Violette 2002]. A specific explanation on the Standard Model (more on the physics side) can be found in [Jureit, Krajewski, Schuecker & Stephan 2007], a similar (more mathematical review in [Sitarz 2008]. A concise and excellent review of spectral geometry, spectral action and exotic traces is in [Iochum 2011].

GEOMETRY	ALGEBRA
sections of a vector bundle	finitely generated projective module
differential forms	differential forms
differential forms	Hochschild homology
de Rham cohomology	cyclic cohomology
vector fields	operators
group	Hopf algebra
Lie algebra	Hopf algebra
principal fibre bundle	Hopf-Galois extension
measurable functions	von Neumann algebra
infinitesimals	compact operators
metric	Dirac operator
spin^c geometry	spectral triple
spin geometry	real spectral triple
integrals	exotic traces

The extension of *distances* for the irreducible representations which we have proposed here, adds two points to the above list: metric can be then understood as distances not only on states but also as metric on the set of irreducible representations.

We see it just as a next step towards a better understanding of what noncommutative geometry really allows us to measure.

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