SPECTRAL TRIPLES WITH NON-PRODUCT DIRAC OPERATORS.

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Motivation: what is metric?

Metrics for NC spaces

- Algebraic approach: metric in $\Omega^1(A) \otimes_A \Omega^1(A)$
  (or a map $\Omega^1(A) \otimes_A \Omega^1(A) \to A$)
- Spectral triples approach: metric is **the Dirac operator**

Metric for the products of NC spaces

- Algebraic approach: $A \otimes B$
  construct the differential calculus over the the product and use the algebraic procedure
- construct a spectral triple over the product:

$$(A \otimes B, H_A \otimes H_B, D_{A\otimes B} = D_A \otimes \mathrm{id} + \gamma_A \otimes D_B)$$

Problem: how to go beyond product metric?

Fluctuations

- Starting with the product-type Dirac operator one can **fluctuate** - find a family of Dirac operators that differ from $D_{A \otimes B}$ by a one-form.
- Of course, such operators **are not** product-type Dirac operators of the type we have started with,
- Yet: this does not change the metric (at least in the classical sense).

Is it possible to go beyond it?

Yes! One of the first examples of non-product Dirac operators are conformally-rescaled Dirac operators:

$$D \mapsto kDk,$$

where k is a positive (bounded) element of A' (commutant of A)

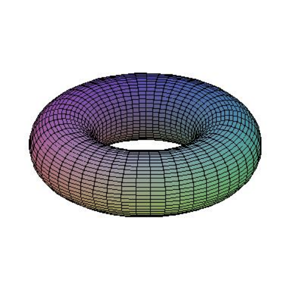
Problem: how to go beyond product metric?

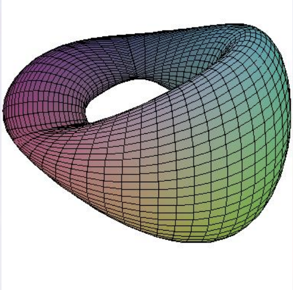
An asymmetric noncommutative torus

Consider a noncommutative torus with the Dirac operator:

$$D = \sigma^1 \delta_1 + k \sigma^2 \delta_2 k,$$

where $k \in C(\mathbb{T}^2_\theta)'$ (again bounded, positive).





L.Dąbrowski & AS, SIGMA 11 (2015), 075;
T. Brzezinski, L. Dabrowski, AS, Lett Math Phys (2019) 109: 643;
L. Dąbrowski, AS, arXiv:1911.12873 (to appear in JNCG)

Almost commutative geometries & SM

Spectral Triples and Particle Physics

The applications of spectral triples to describe particle physics use the product geometry of a 4-dimensional manifold M with a finite algebra:

$$A = C^\infty(M) \otimes (\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}))\,.$$

and the product Dirac operator.

Is this true physics?

No! To construct the product geometry one has to double the Hilbert space of fermions and to have the Dirac operator **real** one has to double it again.

Is there a solution?

Yes! In the end one has to restrict the Hilbert space to physical degrees of freedom.

ALMOST COMMUTATIVE GEOMETRIES & SM

Some more problems.

- Passing from Euclidean to Lorentzian formulation: how to *rotate* and how to *compute* Lorentzian spectral action?
- Possibility of $SU(3)$-symmetry breaking: *usual* minimal axioms do not fully eliminate unphysical models.
- Some *predictions* do not agree with experiments, e.g. the value of the Higgs mass.
- Gravity is not modified (apart from quadratic terms in the spectral action).

Why gravity is important?

The problem with *dark matter* and *dark energy* still is one of the biggest puzzles of theoretical physics.

THE QUEST CONTINUES...

- **Lorentzian formulation** [Paschke–Sitarz, 2006] [Barret, 2007] [Eckstein–Franco, 2014] [van den Dungen, 2015] [Brouder–Bizzi–Besnard, 2015] [Devastato–Farnsworth–Lizzi--Martinetti, 2018] [Bochniak–Sitarz, 2018] [Martinetti–Singh, 2019], [Dang– Wrochna, 2020]. . .
- **Fermion doubling problem** [Lizzi–Mangano–Miele–Sparano, 1997] [Gracia-Bondia– Iochum–Schucker, 1998] [D'Andrea–Kurkov–Lizzi, 2016],. . .
- **Classification of finite Dirac operators and leptoquarks** [Krajewski, 1998] [Paschke–Sitarz, 1998] [Paschke–Scheck–Sitarz, 1999] [Farnsworth– Boyle, 2014] [Dąbrowski–D'Andrea– Sitarz, 2018] [Bochniak–Sitarz, 2018],. . .
- σ **field** [Stephan, 2009] [Chamseddine–Connes, 2012], . . .
- **Twisted Spectral Triples** [Landi–Martinetti, 2016] [Devastato–Martinetti, 2017], . . .

Gravity in almost commutative geometries.

Questions and answers

- What is gravity for **non-product geometries** in the simplest almost-commutative model?

$$C^\infty(M) \otimes F, \qquad F = \mathbb{C} \oplus \mathbb{C},$$

  (the **toy** Connes-Lott model)
- Is the model physically feasible?
- How the modification influences the cosmological models?
- Does it relate to *modified gravity* models

Based on:

AS, Class.Quant.Grav. **36**, 19 (2019)
A.Bochniak and AS, Phys. Rev. D **103**, 044041 (2021)
A.Bochniak, AS, *in preparation*

NONPRODUCT GEOMETRIES FOR TWO-POINT GRAVITY

The first possibility to modify *almost-commutative geometries* is to assume dependence of the Dirac operator on the discrete coordinate.
Assuming the Connes-Lott discrete geometry over a manifold and a Dirac operator:

$$\mathcal{D} = \begin{pmatrix} D_1 & \gamma\Phi \\ \gamma\Phi^* & D_2 \end{pmatrix}$$

where D_1 and D_2 are independent Dirac operators.

Problems

- compute the spectral action (at least the leading part)
- check if the model physically viable (stability)
- is there a mechanism leading to $D_1 = D_2$?
- is it **bimetric** gravity ?

NONPRODUCT GEOMETRIES FOR „BIMETRIC" GRAVITY

Assume FLRW type geometry (Euclidean):

$$ds^2 = b(t)^2 dt^2 + a(t)^2 \left(d\chi^2 + S_k^2(\chi) \left(d\theta^2 + \sin^2(\theta) d\phi^2 \right) \right),$$

where $S_k(\chi) = r\,\mathrm{sinc}(r\sqrt{k})$ for $k = 0, \pm 1$, so

$$D = \gamma^a dx^\mu(\theta_a) \frac{\partial}{\partial x^\mu} + \frac{1}{4} \gamma^c \omega_{cab} \gamma^a \gamma^b,$$

with $\{\gamma^a, \gamma^b\} = -2\delta^{ab} I$.
Apply conformal rescaling:

$$D_h = h^{-1} D h \quad \text{with } h(t) = a(t)^{-3/2} b(t)^{-1/2}$$

and take the doubled geometry with D_1, D_2 as earlier, Φ a constant field ($\Phi \neq 0$), $\gamma^2 = \kappa = \pm 1$ and $a_i(t), b_i(t)$ positive smooth functions ($i = 1, 2$).

THE ACTION:

$$\begin{aligned}\mathcal{S}(\mathcal{D}) &= \Lambda^4 \,\mathrm{Wres}(\mathcal{D}^{-4}) + c\Lambda^2 \,\mathrm{Wres}(\mathcal{D}^{-2}) \\ &= \int_M \int_{\|\xi\|=1} \left(\Lambda^4 \,\mathrm{Tr}\,\mathrm{Tr}_{Cl}\, \mathfrak{b}_0^2 + c\Lambda^2 \,\mathrm{Tr}\,\mathrm{Tr}_{Cl} \mathfrak{b}_2 \right)\end{aligned}$$

$$\begin{aligned}\mathcal{S}(\mathcal{D}) \sim \int dt \Big\{ &\left(\Lambda^4 - c\kappa\Lambda^2 |\Phi|^2 \right) (a_1^3 b_1 + a_2^3 b_2) \\ &- \frac{c\Lambda^2}{12} \left(a_1^3 b_1 R(a_1, b_1) + a_2^3 b_2 R(a_2, b_2) \right) \\ &+ c\kappa\Lambda^2 |\Phi|^2 b_1 b_2 \frac{(a_1 - a_2)^2}{(a_1 b_2 + a_2 b_1)^2} \Big[a_1^2 (2 a_2 b_1 + a_1 b_2) \\ &+ a_2^2 (2 a_1 b_2 + a_2 b_1) \Big] + \\ &+ c\kappa\Lambda^2 |\Phi|^2 \frac{(b_1 - b_2)^2}{(a_2 b_1 + a_1 b_2)^2} a_1^2 a_2^2 (a_1 b_1 + a_2 b_2) \Big\}\end{aligned}$$

where $R(a, b)$ is the usual scalar curvature density.

The potential and bimetric gravity models

The symmetries

- The interaction potential between the metrics satisfies:

$$V\left(\sqrt{g_2^{-1}g_1}\right)\sqrt{g_2} = V\left(\sqrt{g_1^{-1}g_2}\right)\sqrt{g_1}, \qquad (*)$$

- Introduce $x = \frac{b_1}{b_2}$, $y = \frac{a_1}{a_2}$, (functions of $X^a_c = g_2^{ab} g_{1\,bc}$), then:

$$V\left(\sqrt{g_2^{-1}g_1}\right) = \frac{1}{(x+y)^2}\Big(x^2 + 2xy - 2x^2y + y^2 - 6xy^2$$
$$+4x^2y^2 + 4xy^3 - 6x^2y^3 + x^3y^3 - 2xy^4 + 2x^2y^4 + xy^5\Big)$$

- V is a rational function of symmetric polynomials in $\sqrt{X}$.

Modified gravity ?

For **bimetric gravity models** [Hassan– Rosen, 2012]: polynomial function of symmetric polynomials in $\sqrt{X}$ satisfying $(*)$.

ARE COSMOLOGICAL MODELS STABLE?

- Wick rotate into $(-,+,+,+)$
- Introduce parameters:

$$\Lambda_e = \frac{12}{c}(\Lambda^2 - c\kappa|\Phi|^2), \quad \alpha = 12|\Phi|^2\kappa.$$

- Introduce interactions with matter and/or radiation.
- We are free to fix one of b_1, b_2. Choose $b_{1,2}(t) = 1 \pm b(t)$.
- Study linearized **equations of motions**:

$$\dot{r}(t) = \frac{3\lambda^2 a(t)^2(1+w) - (\dot{a}(t)^2 + k)(1+3w)}{2a(t)\dot{a}(t)} r(t) + \left(\dot{a}(t) + \alpha\frac{a(t)^2}{6\dot{a}(t)}\right) s(t),$$

$$\dot{s}(t) = 3\frac{\dot{a}(t)}{a(t)}\left(\frac{r(t)}{a(t)} - s(t)\right),$$

where $\Lambda_e = 6\lambda^2$ and

$$a_1(t) = a(t) + \epsilon r_1(t), \quad a_2(t) = a(t) + \epsilon r_2(t), \quad b(t) = \epsilon s(t)$$

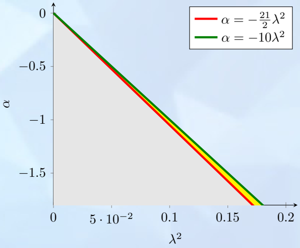
Empty de Sitter universe

$$a(t) = a_0 e^{\sqrt{\lambda} t}$$

$$s(t) = C_1 e^{-\frac{3}{2}\lambda t + \frac{1}{2}\sqrt{21\lambda^2 + 2\alpha}\, t} + C_2 e^{-\frac{3}{2}\lambda t - \frac{1}{2}\sqrt{21\lambda^2 + 2\alpha}\, t},$$

$$r(t) = C_3 e^{-\frac{1}{2}\lambda t + \frac{1}{2}\sqrt{21\lambda^2 + 2\alpha}\, t} + C_4 e^{-\frac{1}{2}\lambda t - \frac{1}{2}\sqrt{21\lambda^2 + 2\alpha}\, t},$$

$$C_3 = C_1 \frac{a_0}{6\lambda} \left(3\lambda + \sqrt{21\lambda^2 + 2\alpha}\right), \qquad C_4 = C_2 \frac{a_0}{6\lambda} \left(3\lambda - \sqrt{21\lambda^2 + 2\alpha}\right).$$

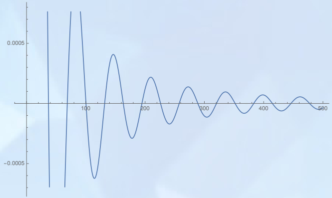


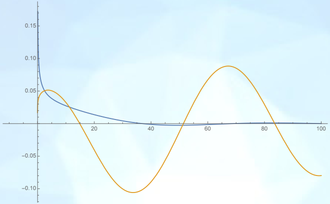
- yellow region: exponential damping,
- gray region: exponentially growth
- red line: degenerate solution

Radiation dominated universe

The solutions for $s(t)$ is,

$$s(t) = c_1 t^{-\frac{5}{4}} J_{\sqrt{\frac{13}{16}}}\left(\sqrt{-\frac{\alpha}{2}}\, t\right) + c_2 t^{-\frac{5}{4}} Y_{\sqrt{\frac{13}{16}}}\left(\sqrt{-\frac{\alpha}{2}}\, t\right),$$





THE POTENTIAL FOR INFINITESIMAL PERTURBATIONS

Assume here that the metric differs infinitesimally from the flat one: $g_{ij} = \delta_{ij} + \epsilon h_{ij}$. What is the potential for the interaction between two perturbations h_1 and h_2 ? The Diracs become:

$$D = \gamma^i \left(\delta^j{}_i - \frac{1}{2}\epsilon h^j{}_i + \frac{3}{8}\epsilon^2 h^{jk} h_{ki} - \frac{5}{16}\epsilon^3 h^j{}_k h^k{}_l h^l{}_i \right) \partial_j,$$

and the interaction is:

$$\begin{aligned} S(h_1, h_2) \sim \quad & \epsilon^2 \frac{1}{2} \mathrm{Tr}\,(h_1 - h_2)^2 \\ & + \epsilon^3 \frac{1}{4} \mathrm{Tr}\left((h_1 - h_2)^2 (h_1 + h_2) \right) \\ & + \epsilon^3 \frac{1}{8} \mathrm{Tr}\,(h_1 - h_2)^2 \,\mathrm{Tr}\,(h_1 + h_2) \\ & - \epsilon^3 \frac{3}{4} \mathrm{Tr}\left((h_1 - h_2)(h_1^2 - h_2^2) \right) + \cdots \end{aligned}$$

Further results

Diagonal geometries

$$ds^2 = a^2(dx_1^2 + dx_2^2) + b^2(dx_3^2 + dx_4^2),$$

With $x = \frac{b_1}{b_2}$ and $y = \frac{a_1}{a_2}$ the potential is

$$\hat{V}(g_1, g_2) \sim V\left(\sqrt{g_2^{-1} g_1}\right)\sqrt{g_2}, \qquad \text{where:}$$

$$V\left(\sqrt{g_2^{-1} g_1}\right) = \frac{4x^2y^2(x-1)(y-1)}{(x-y)(x+y)^2}\log\left(\frac{y}{x}\right) + (x^2y^2+1)(x-y) - 2xy\frac{x-y}{x+y}(xy+1).$$

Conjecture

$$V\left(\sqrt{g_2^{-1} g_1}\right)\sqrt{g_2} = V\left(\sqrt{g_1^{-1} g_2}\right)\sqrt{g_1},$$

Back to particle physics

How does the SM really look like?

- Do not assume **almost-commutativity**: allow general (nonproduct) spectral triples: $(\mathcal{D})$ over the same algebra.
- Our approach: look at the **physical** Standard Model – and **then** try to explore what is **the geometry** (in spectral triple language) used to describe it.
- Keep the information about the **Lorentzian structure**.

The Krein-shifted Dirac

- Taking the Dirac for $(1,3)$-Minkowski space: $\mathcal{D} = i\gamma^\mu \partial_\mu$ we introduce the **Krein shift** of $\mathcal{D}$: $\widetilde{\mathcal{D}} = \gamma^0 \mathcal{D}$.
- $\widetilde{\mathcal{D}}$ - symmetric $\Longleftrightarrow$ $\mathcal{D}$ - Krein-self-adjoint: $\mathcal{D}^\dagger = \gamma^0 \mathcal{D} \gamma^0$
- $\mathcal{D}\gamma = -\gamma\mathcal{D}$, $\mathcal{D}\mathcal{J} = \mathcal{J}\mathcal{D}$ $\quad\Leftrightarrow\quad$ $\widetilde{\mathcal{D}}\gamma = \gamma\widetilde{\mathcal{D}}$, $\widetilde{\mathcal{D}}\mathcal{J} = -\mathcal{J}\widetilde{\mathcal{D}}$.

The structure of the Standard Model

What is the finite spectral triple ?

Finite (Riemannian) spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \pi_L, \pi_R)$ has the following components:

- $\mathcal{A}$ - finite dimensional algebra
- π_L - representation of $\mathcal{A}$ on $\mathcal{H}$
- π_R - representation of $\mathcal{A}^{\rm op}$ on $\mathcal{H}$
- $[\pi_L(a), \pi_R(b)] = 0$ - (0th order condition)
- $[[\mathcal{D}, \pi_L(a)], \pi_R(b)] = 0$ - (1st order condition)

Observe:

- bimodule structure
- no explicit reality structure
- 1st order condition required

Explicit construction of the Standard Model

- We start with the particle content (one generation):

$$\Psi = \begin{pmatrix} \nu_R & u_R^1 & u_R^2 & u_R^3 \\ e_R & d_R^1 & d_R^2 & d_R^3 \\ \nu_L & u_L^1 & u_L^2 & u_L^3 \\ e_L & d_L^1 & d_L^2 & d_L^3 \end{pmatrix} \in M_4(H_W)$$

  where each entry is **the Weyl spinor** over the manifold M !
- Algebra $\mathcal{A}$ of $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$-valued smooth functions on the spacetime, with representations:

$$\pi_L(\lambda, q, m)\,\Psi = \begin{pmatrix} \lambda & & \\ & \overline{\lambda} & \\ & & q \end{pmatrix} \Psi, \quad \pi_R(\lambda, q, m)\,\Psi = \Psi \begin{pmatrix} \overline{\lambda} & \\ & m^\dagger \end{pmatrix}.$$

DIRAC OPERATOR FOR THE STANDARD MODEL

- At every point of the Minkowski space, linear operators on the space of particles can be encoded as a matrix from $M_4(\mathbb{C}) \otimes M_2(\mathbb{C}) \otimes M_4(\mathbb{C})$.
- Dirac operator: $\mathcal{D}_{SM}\Psi = \mathcal{D}\Psi + \mathcal{D}_F\Psi$, where

$$\mathcal{D} = \begin{pmatrix} & & i\widetilde{\sigma}^\mu\partial_\mu & \\ & & & i\widetilde{\sigma}^\mu\partial_\mu \\ i\sigma^\mu\partial_\mu & & & \\ & i\sigma^\mu\partial_\mu & & \end{pmatrix},$$

  and $\mathcal{D}_F$ is a finite endomorphism of $M_4(H_W)$. Here $\widetilde{\sigma}^0 = \sigma^0 = 1_2$ and $\widetilde{\sigma}^j = -\sigma^j$ for $j = 1, 2, 3$.

Some conditions:

Lorentz invariance

- $\mathcal{D}_F$ has to be in $M_4(\mathbb{C}) \otimes 1_2 \otimes M_4(\mathbb{C})$ in order to have the Lorentz invariance of the full Dirac operator.
- As a consequence $\mathcal{D}_F$ commutes with the chirality operator $\Gamma = \pi_L(1, -1, 1)$.
- Therefore, $\mathcal{D}_{SM} = \mathcal{D} + \mathcal{D}_F$ with $\{\mathcal{D}, \Gamma\} = 0$ and $[\mathcal{D}_F, \Gamma] = 0$.
- Krein-shifted operators behave in the opposite way.

Spin-c or Hodge condition

We reformulate the spin-c or Hodge type condition [Dąbrowski– D'Andrea, 2016], [Dąbrowski– D'Andrea– Sitarz, 2018], [Dąbrowski– Sitarz, 2019]

- spin$_c$ type geometry: $(Cl_{\mathcal{D}}(\pi_L(\mathcal{A})))' = \pi_R(\mathcal{A})$.
- Hodge condition: $(Cl_{\mathcal{D}}(\pi_L(\mathcal{A})))' = Cl_{\mathcal{D}}(\pi_R(\mathcal{A}))$,

The main result

Theorem (Bochniak-S-Zalecki)

With the above assumptions,

- *Requiring $\widetilde{\mathcal{D}_{SM}}$ to satisfy the first order condition implies*

$$\widetilde{\mathcal{D}_F} = \begin{pmatrix} & M_l \\ M_l^\dagger & \end{pmatrix} \otimes 1_2 \otimes e_{11} + \begin{pmatrix} & M_q \\ M_q^\dagger & \end{pmatrix} \otimes 1_2 \otimes (1_4 - e_{11}),$$

  where $M_l, M_q \in M_2(\mathbb{C})$.

- *if M_l, M_q nondegenerate then $\widetilde{\mathcal{D}_{SM}}$ satisfies the spin$_c$ condition.*

The Standard Model with three generations

- Hilbert space: $M_4(H_W) \otimes \mathbb{C}^3$.
- Representation enlarged diagonally.
- $M_l, M_q \in M_2(\mathbb{C}) \otimes M_3(\mathbb{C})$:

$$M_l = \begin{pmatrix} \Upsilon_\nu & \\ & \Upsilon_e \end{pmatrix}, \qquad M_q = \begin{pmatrix} \Upsilon_u & \\ & \Upsilon_d \end{pmatrix},$$

with Υ_e, Υ_u - diagonal, $\Upsilon_\nu = U\widetilde{\Upsilon_\nu}U^\dagger$, $\Upsilon_d = V\widetilde{\Upsilon_d}V^\dagger$,

U– Pontecorvo–Maki–Nakagawa–Sakata matrix,
V– Cabibbo–Kobayashi–Maskawa matrix.

THE STANDARD MODEL WITH THREE GENERATIONS

THEOREM

The spin-c condition holds provided that for both pairs of matrices $(\Upsilon_\nu, \Upsilon_e)$ *and* (Υ_u, Υ_d) *their eigenvalues are pairwise different.*

- This is the same condition as for Hodge duality [Dąbrowski–Sitarz, 2019]
- This condition is satisfied for physical Standard Model provided that there is no massless neutrino [Dąbrowski–Sitarz, 2019]
- The model can be doubled: the resulting spectral triple satisfies the Hodge duality and is the finite part of the one studied in the almost-commutative framework.

More physics: CP violation

Real structure

- Although the "spatial" spectral triple is **real** the usual 0th and 1*st* order condition is not implemented by $\mathcal{J}$
- A real full Dirac operator implies the reality of M_l and M_q.
- One generation: fermion masses are real.
- Three generations: both Wolfenstein parameter $\overline{\eta}$ and CP-violating phase δ^{ν}_{CP} have to vanish.
- **CP-violation** $\Leftrightarrow$ shadow of the $\mathcal{J}$-symmetry violation in the non-doubled spectral triple.

Twisting

Note that $\widetilde{\mathcal{D}_{SM}}$ satisfies the order one condition, while $\mathcal{D}_{SM}$ satisfies its twisted version: $[[\mathcal{D}_{SM}, \pi_L(a)]_\beta, \pi_R(b)]_\beta = 0$, where $[x, y]_\beta = xy - \beta y \beta^{-1} x$.

NEXT STEP: GAUGE TRANSFORMATIONS

To make connection with physics we shall consider fluctuated Dirac operators:

- $U_{LR} := \pi_L(U)\pi_R(U)$ for $U = (u_1, u_2, u_3) \in \mathcal{U}(\mathcal{A})$. They form a group $(U(1) \times SU(2) \times U(3))/(\mathbb{Z}/2\mathbb{Z})$.
- To have $SU(3)$ rather than $U(3)$ one could impose unimodularity condition.
- The left action is already unimodular, while for the right one, the unimodularity can be imposed either on each fundamental component or in the full representation.
- In the first case: $u_1 \det u_3 = 1$ and the gauge group of the Standard Model $(U(1) \times SU(2) \times SU(3))/(\mathbb{Z}/6\mathbb{Z})$, while in the second one: $(u_1 \det u_3)^{12} = 1$ and the group differs by a finite factor.

Fluctuated Dirac operator

$$\widetilde{\mathcal{D}_{SM}}^{\omega} = \widetilde{\mathcal{D}_{SM}} + A_\mu e_{11} \otimes \sigma^\mu \otimes (1_4 - e_{11}) - 2A_\mu e_{22} \otimes \sigma^\mu \otimes e_{11}$$
$$- A_\mu e_{22} \otimes \sigma^\mu \otimes (1_4 - e_{11}) - A_\mu (e_{33} + e_{44}) \otimes \tilde{\sigma}^\mu \otimes e_{11}$$
$$+ \begin{pmatrix} 0_2 & \\ & W_\mu \end{pmatrix} \otimes \widetilde{\sigma}^\mu \otimes 1_4 + \begin{pmatrix} 1_2 & \\ & 0_2 \end{pmatrix} \otimes \sigma^\mu \otimes \begin{pmatrix} 0_1 & \\ & G_\mu \end{pmatrix}$$
$$+ \begin{pmatrix} 0_2 & \\ & 1_2 \end{pmatrix} \otimes \widetilde{\sigma}^\mu \otimes \begin{pmatrix} 0_1 & \\ & G_\mu \end{pmatrix} + \begin{pmatrix} & M_l \Phi \\ \Phi^\dagger M_l^\dagger & \end{pmatrix} \otimes 1_2 \otimes e_{11}$$
$$+ \begin{pmatrix} & M_q \Phi \\ \Phi^\dagger M_q^\dagger & \end{pmatrix} \otimes 1_2 \otimes (1_4 - e_{11}).$$

Physical parametrization

- Since $\Phi \in \mathbb{H}$ we can write $\Phi = \begin{pmatrix} \phi_1 & \phi_2 \\ -\overline{\phi_2} & \overline{\phi_1} \end{pmatrix}$.
- Define $\Phi_x := M_x(1_2 + \Phi)$, for $x = l, q$.
- Define the Higgs doublet $H = \begin{pmatrix} 1 + \phi_1 \\ \phi_2 \end{pmatrix}$

Why this model is slightly different

Nonproduct geometry

- It is not chiral: finite part is chiral (Krein-shifted) and spatial part is chiral (Lorentzian)
- We have: $(D_s + D_F)^2 \neq D_s^2 + D_F^2$

Lorentzian geometry

- The model is Lorentzian: is there a (good) Euclidean version?
- No easy way to spectral action – use Wick rotation?
- Next step: use Dang-Wrochna for spectral action?

WICK ROTATED MODEL - LEPTONIC SECTOR

We take the Lorentzian Dirac operator:

$$\mathcal{D}_L = i\begin{pmatrix} & 1_2 \otimes \widetilde{\sigma}^\mu \\ 1_2 \otimes \sigma^\mu & \end{pmatrix}\partial_\mu + A_\mu \begin{pmatrix} & -1_2 \otimes \widetilde{\sigma}^\mu \\ (\sigma^3 - 1_2) \otimes \sigma^\mu & \end{pmatrix}$$

$$+ \begin{pmatrix} & W_\mu \otimes \widetilde{\sigma}^\mu \\ 0_4 & \end{pmatrix} + \begin{pmatrix} \Phi_l^\dagger & \\ & \Phi_l \end{pmatrix} \otimes 1_2.$$

and Wick rotate it: $\sigma^j \to i\sigma^j$:

$$\mathcal{D}_{L,w} = i\begin{pmatrix} & 1_2 \\ 1_2 & \end{pmatrix} \otimes 1_2 \partial_0 + i\begin{pmatrix} & -i1_2 \\ i1_2 & \end{pmatrix} \otimes \sigma^j \partial_j$$

$$+ A_0 \begin{pmatrix} & -1_2 \\ (\sigma^3 - 1_2) & \end{pmatrix} \otimes 1_2 + A_j \begin{pmatrix} & i1_2 \\ i(\sigma^3 - 1_2) & \end{pmatrix} \otimes \sigma^j$$

$$+ \begin{pmatrix} & W_0 \\ 0_2 & \end{pmatrix} \otimes 1_2 - \begin{pmatrix} & iW_j \\ 0_2 & \end{pmatrix} \otimes \sigma^j + \begin{pmatrix} \Phi_l^\dagger & \\ & \Phi_l \end{pmatrix} \otimes 1_2$$

Wick rotated model: results

Doing the same for the quark sector we compute $\mathcal{D}^\dagger\mathcal{D}$ and the Gilkey-Seeley-DeWitt coefficients:

$$a_2 = \frac{3}{4\pi^2} a \int d^4x |H|^2,$$

$$a_4 = \frac{1}{8\pi^2} \int d^4x \Big[b|H|^4 - a\mathrm{Tr}|D_\mu|^2 + \frac{20}{3}F^2 + 2\mathrm{Tr}(W^2) + 2\mathrm{Tr}(G^2) + 12\varepsilon^{jkl}F_{jk}F_{0l} - 6\varepsilon^{jkl}\mathrm{Tr}(W_{jk}W_{0l}) \Big]$$

which leads to the Euclidean action:

$$\mathcal{L}_{\mathrm{gauge}} = \frac{f(0)}{\pi^2}\left(\frac{10}{3}F^2 + \mathrm{Tr}(W^2) + \mathrm{Tr}(G^2) + 6\varepsilon^{jkl}F_{jk}F_{0l} - 3\varepsilon^{jkl}\mathrm{Tr}(W_{jk}W_{0l}) \right),$$

$$\mathcal{L}_H = \frac{bf(0)}{2\pi^2}|H|^4 + \frac{6f_2\Lambda^2}{\pi^2}a|H|^2 - \frac{af(0)}{2\pi^2}\mathrm{Tr}|D_\mu H|^2$$

SM WITH NON-PRODUCT GEOMETRY

The good side

- No fermion **doubling**.
- **No** $SU(3)$ **breaking**.
- Order-one condition holds.
- Lack of real structure $\rightarrow$ **CP violation.**
- The Morita condition of spin$_c$ geometry holds.
- **Spectral action** reproduces the action of the Lorentzian Standard Model with an additional electroweak "θ-term".

The problems and outlook

- Still Higgs mass needs to be explained (within this model or extensions)
- Need Lorentzian spectral action!
- Are electroweak θ-terms significant?

THE LAST PAGE

Summary:

Non-product spectral triples: interesting new features **motivated** by physics.

Outlook:

Understanding gravity in the full Standard Model.
Computing spectral action for Lorentzian Dirac operator.

Questions:

- Computing explicit interaction term between the metrics is **a challenge.**
- Are there **observable** physical consequences of a non-product metric ?

THANK YOU !