RIEMANNIAN GEOMETRY OF A DISCRETIZED CIRCLE AND TORUS

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MOTIVATION: GRAVITY

- General Relativity is based on the theory of metric-compatible linear connections = (pseudo)Riemannian Geometry.
- Modifications of GR exist but differ only mildly from GR.
- Quantization of gravity is a mystery (despite many attempts).
- Some successful approaches are based on discretization.
- Commonly used argument requires "quantum spacetime" to be the basis for a quantum theory of gravity.

AFTER MANY YEARS...

we are far from understanding the "Riemannian geometry" of noncommutative spaces.

NC RIEMANNIAN GEOMETRY: THE APPROACHES

- Naive approach.
- Moyal approach.
- Algebraic approach.
- Spectral approach.

CLASSICAL GEOMETRY...

is the only intersection offering a computable and reasonable (that is consistent between all methods) answer.

EXAMPLES?

classical geometries, *q*-deformations, Moyal deformation, fuzzy spaces, twists (NC Tori), finite spaces.

RIEMANNIAN GEOMETRY: THE ALGEBRAIC WAY

THE BASICS:

The basic tool of NC Riemannian geometry is the differential calculus (differential graded algebra $\Omega(A)$ over an algebra A):

$$d(\omega\rho) = (d\omega)\rho + (-1)^{|\omega|}\omega(d\rho).$$

THE NEXT STEP:

A linear connection on the left module $\Omega^1(A)$ is a linear map

$$\nabla : \Omega^1(A) \to \Omega^1(A) \otimes_A \Omega^1(A),$$

such that

$$\nabla(a\omega) = da \otimes_A \omega + a\nabla(\omega),$$

WHAT IS MISSING?

Classically, the Levi-Civita connection requires more conditions.

RIEMANNIAN GEOMETRY: THE ALGEBRAIC WAY

LINEAR AND LEVI-CIVITA CONNECTIONS

- Bimodule compatibility.
- Torsion freeness.
- Compatibility with the metric.

But what is the metric?

THE METRIC: A DEFINITION

We define the metric as a pair,

$$\mathbf{g} = \mathbf{g}^{(1)} \otimes_A \mathbf{g}^{(2)} \in \Omega^1(A) \otimes_A \Omega^1(A),$$

$$(\cdot, \cdot) : \Omega^1(A) \otimes_A \Omega^1(A) \to A,$$

such that the pairing between them is nondegenerate:

$$(\omega, \mathbf{g}^{(1)})\mathbf{g}^{(2)} = \omega = \mathbf{g}^{(1)}(\mathbf{g}^{(2)}, \omega)$$

TOWARDS NC LEVI-CIVITA CONNECTIONS

THE TORSION

$$\wedge \circ \nabla - d \equiv 0.$$

THE BIMODULE COMPATIBILITY

A linear connection on the bimodule $\Omega^1(A)$ is a pair (∇, σ), where ∇ is a connection, σ a bimodule map,

$$\sigma : \Omega^1(A) \otimes_A \Omega^1(A) \to \Omega^1(A) \otimes_A \Omega^1(A),$$

$$\nabla(a\omega) = da \otimes_A \omega + a\nabla\omega,$$
$$\nabla(\omega a) = (\nabla\omega)a + \sigma(\omega \otimes_A da),$$

TOWARDS NC LEVI-CIVITA CONNECTIONS

THE METRIC COMPATIBILITY

A linear connection (∇, σ) is said to be compatible with the metric **g** if

$$(\nabla \otimes \mathrm{id})\mathbf{g} + (\sigma \otimes \mathrm{id})(\mathrm{id} \otimes \nabla)\mathbf{g} = 0.$$

THE STAR STRUCTURE

We say that the metric **g** is compatible with $*$, if $\mathbf{g}^* = \mathbf{g}$, that is:

$$\mathbf{g}^* = (\mathbf{g}^{(1)} \otimes_A \mathbf{g}^{(2)})^* = \left(\mathbf{g}^{(2)}\right)^* \otimes_A \left(\mathbf{g}^{(1)}\right)^* = \mathbf{g},$$

$$(\omega^*, \rho^*) = (\rho, \omega)^*, \quad \forall \omega, \rho \in \Omega^1(A).$$

TOWARDS NC LEVI-CIVITA CONNECTIONS

THE EXISTENCE OF THE METRIC(S)?

It is not clear whether such metrics exist and, if so, under what conditions.
Moreover, the space of the metrics is much smaller than in the classical case, for example, the noncommutative torus admits only the constant (flat) metric.

THE EXISTENCE OF METRIC-COMPATIBLE LINEAR CONNECTIONS ?

The existence and in (some cases) the uniqueness of metric-compatible, torsion-free linear connections can be proven only in some very special cases [Goswami-Landi].

TOWARDS THE RIEMANNIAN GEOMETRY OF $\mathbb{Z}_N$.

WHY FINITE GROUPS?

- Toy examples with a physical background.
- Group structure.
- Bicovariant differential calculi.

WHY CYCLIC GROUPS?

The easiest example of a fuzzy space with possible extension to products (fuzzy torus) ?

TOWARDS THE RIEMANNIAN GEOMETRY OF $\mathbb{Z}_N$.

THE MINIMAL DIFFERENTIAL CALCULUS.

Consider the cyclic group $\mathbb{Z}_N, N > 2$ with the generator p. Denote by $\tilde{p}$ its inverse in $\mathbb{Z}_N$. Then there exists a minimal bicovariant, star-compatible connected differential calculus, generated by $\theta_p, \theta_{\tilde{p}}$ with the following structure:

$$\begin{aligned} f\theta_p &= \theta_p R_p(f), & f\theta_{\tilde{p}} &= \theta_{\tilde{p}} R_{\tilde{p}}(f), \\ \theta_p^* &= -\theta_{\tilde{p}}, & df &= -\left[\theta_p + \theta_{\tilde{p}}, f\right], \\ d\theta_p &= 0, & d\theta_{\tilde{p}} &= 0, \\ d\omega &= \theta \wedge \omega + \omega \wedge \theta. \end{aligned}$$

TOWARDS THE RIEMANNIAN GEOMETRY OF $\mathbb{Z}_N$.

THE LINEAR CONNECTION ON $\mathbb{Z}_N$.

Since the differential calculus is **inner**, every bimodule connection is of the form, for some bimodule maps σ, α:

$$\nabla\omega = \theta \otimes \omega - \sigma(\omega \otimes \theta) + \alpha(\omega),$$

An immediate consequence of the above:

PROPOSITION

For a minimal bicovariant calculus over $C(\mathbb{Z}_N)$ with $N \neq 3$, a bimodule linear connection is determined by a bimodule map σ.

PROOF.

There are no bimodule maps apart from the zero map between $\Omega^1(A)$ and $\Omega^1(A) \otimes_A \Omega^1(A)$ unless p is of order 3. Therefore, $\alpha \equiv 0$. Hence, the bimodule connection ∇ and σ are mutually determined. ☐

TOWARDS THE RIEMANNIAN GEOMETRY OF $\mathbb{Z}_N$.

PROPOSITION.

For a minimal bicovariant calculus over $C(\mathbb{Z}_N)$ the torsion-free connection that is determined by σ, the map σ must satisfy,

$$\omega \wedge \theta = - \wedge \circ \sigma(\omega \otimes_A \theta).$$

PROOF.

Comparing

$$\begin{aligned} \nabla(\omega f) &= \nabla(\omega) f + \sigma(\omega \otimes_A df), \\ d(\omega f) &= (d\omega) f - \omega \wedge df, \end{aligned}$$

we immediately get $\omega \wedge df = - \wedge \circ \sigma(\omega \otimes_A df)$, which gives us the claimed formula. ☐

TOWARDS THE RIEMANNIAN GEOMETRY OF $\mathbb{Z}_N$.

PROPOSITION.

The torsion-free bimodule connections over the minimal bicovariant calculi over $C(\mathbb{Z}_N)$ with $N \neq 4$ are determined by a family of functions $A_p, A_{\tilde{p}}, B_p, B_{\tilde{p}}$, so that σ is,

$$\begin{aligned}
\sigma(\theta_p \otimes_A \theta_p) &= A_p\, \theta_p \otimes_A \theta_p, \\
\sigma(\theta_{\tilde{p}} \otimes_A \theta_{\tilde{p}}) &= A_{\tilde{p}}\, \theta_{\tilde{p}} \otimes_A \theta_{\tilde{p}}, \\
\sigma(\theta_p \otimes_A \theta_{\tilde{p}}) &= B_p \left(\theta_p \otimes_A \theta_{\tilde{p}} + \theta_{\tilde{p}} \otimes_A \theta_p\right) - \theta_p \otimes_A \theta_{\tilde{p}}, \\
\sigma(\theta_{\tilde{p}} \otimes_A \theta_p) &= B_{\tilde{p}} \left(\theta_p \otimes_A \theta_{\tilde{p}} + \theta_{\tilde{p}} \otimes_A \theta_p\right) - \theta_{\tilde{p}} \otimes_A \theta_p.
\end{aligned}$$

PROOF.

It follows directly from the fact that σ is a bimodule map, $p^2 \neq \tilde{p}^2$ for $N \neq 4$, and the compatibility condition of σ with the product in the differential algebra $\wedge$. ☐

TOWARDS THE RIEMANNIAN GEOMETRY OF $\mathbb{Z}_N$.

PROPOSITION.

The connections are star-compatible if and only if the relations below are fulfilled:

$$(R_g\overline{A_g})(R_{g^{-1}}A_{g^{-1}}) = 1,$$
$$|B_g - 1|^2 + \overline{B_g}B_{g^{-1}} = 1,$$

for $g \in \{p, \tilde{p}\}$.

FINALLY:

The last step: we need to introduce the **metric**.

THE METRIC OVER $\mathbb{Z}_N$

POSSIBLE METRICS

A nondegenerate metric over the minimal bicovariant calculus over $C(\mathbb{Z}_N)$ is given by functions G_p, $G_{\tilde{p}}$, which are everywhere different from 0,

$$\begin{aligned} \mathbf{g} &= G_p\,\theta_p \otimes_A \theta_{\tilde{p}} + G_{\tilde{p}}\,\theta_{\tilde{p}} \otimes_A \theta_p, \\ (\theta_a, \theta_b) &= \frac{1}{R_{a^{-1}}\,G_{a^{-1}}}\delta_{a^{-1},b}, \qquad a, b = \{p, \tilde{p}\}. \end{aligned}$$

REMARK

If the metric is also compatible with the higher-order differential calculus (i.e. $\wedge\mathbf{g} = 0$), then it can be described by the only one function $G := G_p = G_{\tilde{p}}$.

THE METRIC OVER $\mathbb{Z}_N$

METRICS AND NORMS

If we define,

$$\langle \cdot , \cdot \rangle : \Omega^1(A) \otimes_A \Omega^1(A) \to A; \qquad \langle \omega_1 , \omega_2 \rangle := (\omega_1^* , \omega_2),$$

then if all G_g are real and negative we conclude that $\Omega^1(A)$ equipped with $\langle \cdot , \cdot \rangle$ is a Hilbert C^*-module over A.

RIGHT (LEFT)-INVARIANT METRICS

The metric is right-invariant if $R_h(\mathbf{g}) = \mathbf{g}$ (resp. left-invariant if $L_h(\mathbf{g}) = \mathbf{g}$), for every $h \in G$, where we have used the unique extension of right (resp. left) translations to the whole differential algebra, so that

$$R_g(df) = d(R_g f), \qquad \text{(resp.} \quad L_g(df) = d(L_g f)).$$

WARM-UP: $\mathbb{Z}_2$ CASE.

LEVI-CIVITA BIMODULE CONNECTIONS FOR $\mathbb{Z}_2$

In the case of $\mathbb{Z}_2$, we have $p = \tilde{p}$ and,

$$\nabla(\theta_p) = (S-1)\theta_p \otimes_A \theta_p, \qquad \sigma(\theta_p \otimes_A \theta_p) = S\,\theta_p \otimes_A \theta_p,$$

the metric is given by $G\,\theta_p \otimes_A \theta_p$, the metric compatibility:

$$(G - R_pG) + G(S-1) + GS(R_p(S) - 1) = 0.$$

Using notation $G_0 = G(e)$, $G_1 = G(p)$ and S_0, S_1 for the respective values of S we have

$$G_1 = G_0 S_0 S_1, \qquad G_0 = G_1 S_1 S_0,$$

which lead to $G_1 = \pm G_0$ and $S_0 S_1 = \pm 1$, so even in the case of constant metric: metric compatible connections given by:

$$S_0 = z, \qquad S_1 = \frac{1}{z}.$$

METRIC COMPATIBILITY FOR $\mathbb{Z}_N$

THEOREM

For the torsion-free bimodule connection for the minimal bicovariant calculus over $\mathbb{Z}_N$ with $N > 4$ the metric compatibility conditions take the following form:

$$\begin{aligned}
&G_g(R_{g^{-1}}B_{g^{-1}})A_g = R_{g^{-1}}G_g,\\
&G_{g^{-1}}(R_gB_g - 1)B_{g^{-1}} + G_g(B_g - 1)(R_{g^{-1}}A_{g^{-1}}) = 0,\\
&R_gG_g = G_{g^{-1}}(R_gB_g - 1)(B_{g^{-1}} - 1) + G_gB_g(R_{g^{-1}}A_{g^{-1}}),
\end{aligned}$$

PROBLEM

Can these equations be solved ?
Is the solution unique ??
Can we compute the curvature ???

SOLVING THE EQUATIONS

Introducing $A_g = a_g + 1$ and $B_g = b_g + 1$, $X_g = \frac{R_g G_g}{G_{g^{-1}}}$ and combining the first and the third equation we obtain

$$b_{g^{-1}}\left(R_g b_g\right) = X_g - R_{g^{-1}} X_g.$$

As the left-hand side is unchanged when we replace g by g^{-1} and act on the result with R_g, we obtain

$$X_g - R_{g^{-1}} X_g = R_g\left(X_{g^{-1}} - R_g X_{g^{-1}}\right),$$

Since X_g satisfies $X_g R_g X_{g^{-1}} = 1$, we obtain:

$$X_g - R_{g^{-1}} X_g = \frac{1}{X_g} - R_g \frac{1}{X_g},$$

and as a result

$$X_g + \frac{1}{R_g X_g} = c = \mathrm{const}.$$

SOLVING THE EQUATIONS

Writing explicitly $X_p, X_{\tilde{p}}$ as functions over $\mathbb{Z}_N$, the relation can be reformulated, as

$$\begin{cases} (c - f(n))f(n+1) = 1, \\ f(0) = f(N), \end{cases}$$

for a function $f : \mathbb{N} \to \mathbb{C}$. Note that we can equivalently choose the equation for $X_{\tilde{p}}$ (denote this function as F) but this corresponds to the choice of -1 as the generator of $\mathbb{Z}_N$ and therefore give the equations,

$$\begin{cases} (c - F(n))F(n-1) = 1, \\ F(0) = F(N), \end{cases}$$

which is equivalent to the one above since

$$F(n) = \frac{1}{f(n-1)}.$$

THEOREM

For the minimal calculus on $\mathbb{Z}_N$, $N > 4$ with $H = \{p, \tilde{p}\}$ the only allowed torsion-free connections compatible with the metric **g** are determined by the bimodule map σ, where

$$A_g = \frac{R_{g^{-1}} G_g}{G_g \left(1 + R_{g^{-1}} b_{g^{-1}}\right)},$$

and for $B_g = 1 + b_g$ we have the following possibilities depending on X_g,
Case I. If $X_g \neq \mathrm{const}$ the only following functions X_p are allowed,

$$X_p(n) = \cos\left(\frac{l\pi}{N}\right) + \sin\left(\frac{l\pi}{N}\right) \cot\left(\phi - \frac{(n+1)l\pi}{N}\right),$$

for $l = 1, ..., N-1$, and an arbitrary constant ϕ such that $e^{2i\phi} \neq e^{\frac{2l\pi}{N}(n+1)}$. Then $c = 2\cos\left(\frac{l\pi}{N}\right)$ and there exist three possible solutions for b:

THEOREM (CONTINUED)

$$(a) \qquad b_p = -1 - R_{\tilde{p}} X_p, \qquad b_{\tilde{p}} = \frac{c+2}{X_p+1} - \frac{1+X_p}{X_p}.$$

$$(b) \qquad b_{\tilde{p}} = -1 - R_p X_{\tilde{p}}, \qquad b_p = \frac{c+2}{X_{\tilde{p}}+1} - \frac{1+X_{\tilde{p}}}{X_{\tilde{p}}}.$$

and, provided that $\prod_{k\in\mathbb{Z}_N} X_g(k) = (-1)^N$,

$$(c) \qquad b_g = \frac{(c+2)R_{g^{-1}}X_g}{1+R_{g^{-1}}X_g+(c+2)y_g^{\mathrm{hom}}} - R_{g^{-1}}X_g - 1, \qquad g = \{p, \tilde{p}\},$$

$$y_p^{\mathrm{hom}}(n) = \kappa_p\,(-1)^n \prod_{k=0}^{n-1} X_p(k), \qquad y_{\tilde{p}}^{\mathrm{hom}}(n) = \kappa_{p^{-1}}\,(-1)^n \prod_{k=0}^{n-1} X_p(k).$$

with $\kappa_p \kappa_{\tilde{p}} = \frac{1}{(c+2)^2}\left(\frac{X_p(N-1)}{X_p(0)} - 1\right), \qquad y_p^{\mathrm{hom}} + y_p^0 \neq 0.$

Case II.

If $X_g = \gamma \equiv \mathrm{const}$:

- $b_g = b_{g-1} = 0$ is always a solution (independently of γ),
- if $\gamma^N \neq (-1)^N$, then there are two more independent solutions:
  - (A) $b_p = 0$ and $b_{\tilde{p}} = -1 - \frac{1}{\gamma}$,
  - (B) $b_p = -1 - \gamma$ and $b_{\tilde{p}} = 0$.
- if $\gamma^N = (-1)^N$ and $\gamma \neq -1$ then $X \neq$ const from previous case is also a solution.

For $\gamma = -1$ the cases (a) and (b) reduce to the first bullet point.

THE ONLY UNIQUE SOLUTION

exists when $X_p = -1$.

THE METRICS

LEMMA

For the real metric satisfying $\mathbf{g} = \mathbf{g}^*$, the constant solutions above are restricted to real constant X_g, whereas the non-constant solutions are restricted by an additional demand that ϕ is a real parameter. Only the solutions with $X_g = \text{const} > 0$ give the real metric $\mathbf{g}$ that equips the module of one-forms with a Hilbert C^*-module structure.

LEMMA

If we further assume that the metric is compatible with the differential calculus, $\wedge \mathbf{g} = 0$, the solution for X_g provides the solution for $G_p = G_{\tilde{p}}$ given by:

$$G(n) = G_0 \prod_{k=0}^{n-1} f_{l,\phi}(k).$$

UNIQUENESS

PROPOSITION

Suppose that we have a Hilbert C^*-module structure on $\Omega^1(A)$ given by the metric **g**. Then there exists a unique torsion free, metric compatible and star-compatible linear connection.

NEXT: CURVATURE

The Riemannian curvature for a given connection ∇ is a map:

$$\mathbf{R}_\nabla : \Omega^1 \to \Omega^2 \otimes_A \Omega^1,$$

defined by the following prescription

$$\mathbf{R}_\nabla = (d \otimes_A \mathrm{id} - \mathrm{id} \wedge \nabla)\, \nabla.$$

THE CURVATURE OF THE CIRCLE

THE RIEMNANNIAN CURVATURE

The Riemannian curvature for the connection ∇ is:

$$\mathbf{R}_\nabla(\theta_g) = \theta_g \wedge \theta_{g^{-1}} \otimes_A \rho_g, \quad g = p, \tilde{p} \tag{1}$$

where

$$\begin{aligned}\rho_g = &\Big[B_g(R_g A_g) - A_g(R_{g^{-1}} B_g) - \Big(R_{g^{-1}} B_{g^{-1}} - 1\Big)(B_g - 1)\Big]\,\theta_g \\ &+ \Big[(R_{g^{-1}} A_{g^{-1}})(1 - B_g) + B_g(R_g B_g - 1)\Big]\,\theta_{g^{-1}}.\end{aligned}$$

MORE CURVATURE

RICCI AND SCALAR

To define the objects corresponding to Ricci and scalar curvature we need, however, some more structure. Let ι be a bimodule map representing two-forms in $\Omega^1(A) \otimes_A \Omega^1(A)$,

$$\iota : \Omega^2 \to \Omega^1 \otimes_A \Omega^1, \qquad \wedge \circ \iota = \mathrm{id},$$

Then, we define:

$$\widetilde{\mathbf{R}}_\nabla \equiv (\iota \otimes \mathrm{id})\mathbf{R}_\nabla : \Omega^1 \to \Omega^1 \otimes_A \Omega^1 \otimes_A \Omega^1,$$

and the Ricci tensor is defined as

$$\mathrm{Ricci} = \left(\mathbf{g}^{(1)}, \widetilde{\mathbf{R}}_\nabla\left(\mathbf{g}^{(2)}\right)_{(1)}\right) \widetilde{\mathbf{R}}_\nabla\left(\mathbf{g}^{(2)}\right)_{(2)} \otimes_A \widetilde{\mathbf{R}}_\nabla\left(\mathbf{g}^{(2)}\right)_{(3)},$$

where the Sweedler's notation on $\Omega^1 \otimes_A \Omega^1 \otimes_A \Omega^1$ is used.

MORE CURVATURE

SCALAR

We define:

$$R = \left(\mathbf{g}^{(1)}, \widetilde{\mathbf{R}}_\nabla\left(\mathbf{g}^{(2)}\right)_{(1)}\right) \cdot \left(\widetilde{\mathbf{R}}_\nabla\left(\mathbf{g}^{(2)}\right)_{(2)}, \widetilde{\mathbf{R}}_\nabla\left(\mathbf{g}^{(2)}\right)_{(3)}\right).$$

REMARK:

In our case of $A = C(\mathbb{Z}_N)$ with $N > 4$ we observe that the most general form of the lifting map ι is

$$\iota(\theta_p \wedge \theta_{\tilde{p}}) = \theta_p \otimes_A \theta_{\tilde{p}} + \beta(\theta_p \otimes_A \theta_{\tilde{p}} + \theta_{\tilde{p}} \otimes_A \theta_p),$$

where $\beta \in C(\mathbb{Z}_N)$. As an immediate consequence we finally obtain for the Ricci tensor,

$$\mathrm{Ricci} = -\frac{R_{\tilde{p}}\beta}{X_{\tilde{p}}}\theta_p \otimes_A \rho_{\tilde{p}} + \frac{1 + R_p\beta}{X_p}\theta_{\tilde{p}} \otimes_A \rho_p.$$

RESULTS

CONNECTIONS AND CURVATURES

For a positive parameter $X_p = \gamma > 0$ and a metric $\mathbf{g}$ with $G := G_p < 0$, for odd N there exist three possible torsion-free and metric compatible linear connections given by the functions $B_p, B_{\tilde{p}}$ with the corresponding Ricci tensor and the scalar curvature (for an arbitrary lift of Ω^2 given by the function β):

case (a) $\quad B_p = 1, \; B_{\tilde{p}} = 1,$

$$\mathrm{Ricci}(n) = \gamma\beta(n-1)Z_+(n)\,\theta_p \otimes_A \theta_{\tilde{p}} + \frac{1+\beta(n+1)}{\gamma} Z_-(n+1)\,\theta_{\tilde{p}} \otimes_A \theta_p,$$

$$R(n) = \gamma^2\beta(n-1)W_+(n) + \frac{1+\beta(n+1)}{\gamma} W_-(n+1).$$

case (b) $\quad B_p = 1, \; B_{\tilde{p}} = -\frac{1}{\gamma},$

$$\begin{aligned}\mathrm{Ricci}(n) = &-\beta(n-1)Z_+(n)\theta_p \otimes_A \theta_{\tilde{p}} - (1+\beta(n+1))Z_-(n+1)\theta_{\tilde{p}} \otimes \theta_p + \\ &+ \beta(n-1)S_-(n)\theta_p \otimes_A \theta_p,\end{aligned}$$

$$R(n) = -\gamma\beta(n-1)W_+(n) - (1+\beta(n+1))W_-(n+1).$$

RESULTS

CONNECTIONS AND CURVATURES

case (c) $B_p = -\gamma,\ B_{\tilde{p}} = 1,$

$$\mathrm{Ricci}(n) = -\beta(n-1)Z_+(n)\theta_p \otimes_A \theta_{\tilde{p}} - (1+\beta(n+1))Z_-(n+1)\theta_{\tilde{p}} \otimes_A \theta_p - \\ - (1+\beta(n+1))S_+(n+1)\theta_{\tilde{p}} \otimes \theta_{\tilde{p}},$$

$$R(n) = -\gamma\beta(n-1)W_+(n) - (1+\beta(n+1))W_-(n+1).$$

WHERE:

$$Z_+(n) = \frac{G(n+1)}{G(n)} - \frac{G(n)}{G(n-1)}, \qquad Z_-(n) = \frac{G(n)}{G(n+1)} - \frac{G(n-1)}{G(n)},$$

$$S_+(n) = \frac{\gamma+1}{\gamma^2}\left(\frac{G(n+1)}{G(n)} - \gamma^2\right), \qquad S_-(n) = \gamma(\gamma+1)\left(\frac{G(n-1)}{G(n)} - \frac{1}{\gamma^2}\right).$$

and

$$W_\pm(n) = \frac{Z_\pm(n)}{G(n)}.$$

SPECIAL SYMMETRIC CASE

THE CURVATURE

In the case of left-right symmetric metric $\gamma = 1$ and the standard choice of the lift $\beta = -\frac{1}{2}$ the scalar curvature is $R(n) = \pm\frac{1}{2}(W_+(n) - W_-(n+1))$, i.e.

$$R(n) = \pm\frac{1}{2}\left[\frac{G(n+1)^3 + G(n)^3}{G(n+1)^2 G(n)^2} - \left(\frac{1}{G(n-1)} + \frac{1}{G(n+2)}\right)\right],$$

with the sign $-$ for the case (a) and $+$ for cases (b) and (c). On the other hand, for the special cases discussed at the end of the previous theorem, the scalar curvature is:

$$R(n) = \frac{1}{2}\left[\frac{B_{\tilde{p}}(n)}{B_p(n)}\left(\frac{1}{G(n+2)} - \frac{G(n+1)}{G(n)^2}\right) + \frac{B_p(n)}{B_{\tilde{p}}(n)}\left(\frac{1}{G(n-1)} - \frac{G(n)}{G(n+1)^2}\right)\right].$$

In particular for the constant metric G, this curvature vanishes in all these cases.

APPROACHING THE CONTINUUM

THE LIMIT

It is interesting to see the continuous limit of the expression for the scalar curvature. A simple computation gives that if we denote by $g(t)$ the limit of the $G(n)$ function, for the parametrisation of the curve with t, then the curvature $R(t)$ becomes:

$$R(t) = \pm \frac{g''(t)g(t) - g'(t)^2}{g(t)^3} = \pm \frac{1}{g(t)} \frac{d}{dt} \left(\frac{\frac{d}{dt} g(t)}{g(t)} \right).$$

EXAMPLES

ELLIPSE

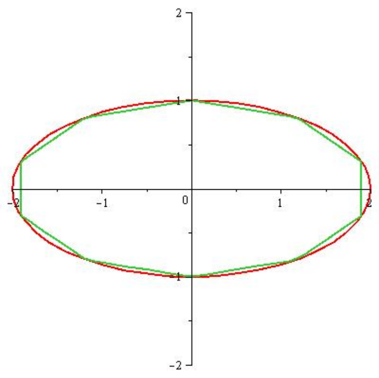
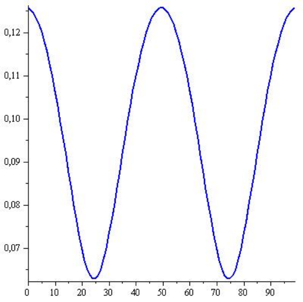


FIGURE: Ellipse and ellipse-like polygon ($N = 10$).

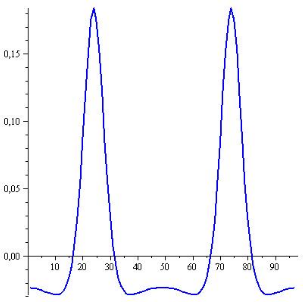
We can then compute the scalar curvature for the assumed form of the metric and approximate its continuous limit.

EXAMPLES

ELLIPSE



the metric



the scalar curvature

THE INVERSE PROBLEM

CONSTANT SCALAR CURVATURE

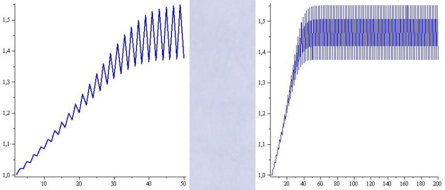


FIGURE: The metric for constant positive scalar curvature.

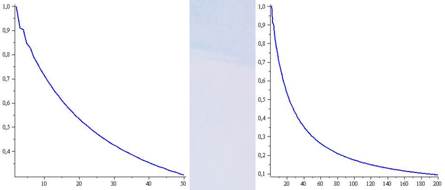


FIGURE: The metric for constant negative scalar curvature.

THE TORUS ($\mathbb{Z}_N \otimes \mathbb{Z}_M$)

THE METRIC

The only bimodule metric over $C(\mathbb{Z}_N) \otimes C(\mathbb{Z}_M)$ is of the form:

$$\mathbf{g} = G_p\,\theta_p \otimes \theta_{\tilde{p}} + G_{\tilde{p}}\,\theta_p \otimes \theta_{\tilde{p}} + G_s\,\theta_s \otimes \theta_{\tilde{s}} + G_{\tilde{s}}\,\theta_{\tilde{s}} \otimes \theta_s,$$

THE TORUS

THE CONNECTION

$$
\begin{aligned}
\sigma(\theta_p \otimes_A \theta_p) &= A_p\, \theta_p \otimes_A \theta_p,\\
\sigma(\theta_{\tilde{p}} \otimes_A \theta_{\tilde{p}}) &= A_{\tilde{p}}\, \theta_{\tilde{p}} \otimes_A \theta_{\tilde{p}},\\
\sigma(\theta_s \otimes_A \theta_s) &= A_s\, \theta_s \otimes_A \theta_s,\\
\sigma(\theta_{\tilde{s}} \otimes_A \theta_{\tilde{s}}) &= A_{\tilde{s}}\, \theta_{\tilde{s}} \otimes_A \theta_{\tilde{s}},\\
\sigma(\theta_p \otimes_A \theta_{\tilde{p}}) &= B_p\, \left(\theta_p \otimes_A \theta_{\tilde{p}} + \theta_{\tilde{p}} \otimes_A \theta_p\right) - \theta_p \otimes_A \theta_{\tilde{p}} + W_p\, (\theta_s \otimes_A \theta_{\tilde{s}} + \theta_{\tilde{s}} \otimes_A \theta_s),\\
\sigma(\theta_{\tilde{p}} \otimes_A \theta_p) &= B_{\tilde{p}}\, \left(\theta_p \otimes_A \theta_{\tilde{p}} + \theta_{\tilde{p}} \otimes_A \theta_p\right) - \theta_{\tilde{p}} \otimes_A \theta_p + W_{\tilde{p}}\, (\theta_s \otimes_A \theta_{\tilde{s}} + \theta_{\tilde{s}} \otimes_A \theta_s),\\
\sigma(\theta_s \otimes_A \theta_{\tilde{s}}) &= B_s\, (\theta_s \otimes_A \theta_{\tilde{s}} + \theta_{\tilde{s}} \otimes_A \theta_s) - \theta_s \otimes_A \theta_{\tilde{s}} + W_s\, (\theta_p \otimes_A \theta_{\tilde{p}} + \theta_{\tilde{p}} \otimes_A \theta_p),\\
\sigma(\theta_{\tilde{s}} \otimes_A \theta_s) &= B_{\tilde{s}}\, (\theta_s \otimes_A \theta_{\tilde{s}} + \theta_{\tilde{s}} \otimes_A \theta_s) - \theta_{\tilde{s}} \otimes_A \theta_s + W_{\tilde{s}}\, (\theta_p \otimes_A \theta_{\tilde{p}} + \theta_{\tilde{p}} \otimes_A \theta_p),
\end{aligned}
$$

$$
\sigma(\theta_a \otimes_A \theta_b) = C_{ab}\, (\theta_a \otimes_A \theta_b + \theta_b \otimes_A \theta_a) - \theta_a \otimes_A \theta_b
$$

SOLUTIONS?

THE UNIQUENESS

Even in the case of a constant metric, such that the lengths of sides are the same in all direction, the solution is not uniquely determined by the requirement of torsion-freeness and metric-compatibility.

In addition to the trivial solution with all W being zero, there are also other possibilities, e.g with $W_h = W_{h^{-1}} = 0$ and $W_g = W_{g^{-1}} = -2$. For N and M with even parities, there are even more sophisticated solutions with alternating functions B.

CONCLUSIONS

WHAT IS THE MESSAGE ?

- Generalization of Riemannian Geometry: possible but need to be adapted to the model ?
- Some fuzzy spaces (circle) are manageable - the curvature is computable, constant curvature solutions can be found !
- The uniqueness of Riemannian geometry is guaranteed (for the circle) under strong assumptions.
- Set of conditions to guarantee the uniqueness is unclear.
- Continuous limit exists but depends on choices of lifting maps.
- The meaning of the continuous limit is unclear.
- The link to the spectral approach is unclear.

A. Bochniak, A. Sitarz, P.Zalecki, Riemannian Geometry of a Discretized Circle and Torus, SIGMA 16 (2020), 143