

TWISTING REALITY

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PLAN

DIFFERENTIAL GEOMETRY

Classical differential geometry:

- an orientable manifold M , smooth functions, $C^\infty(M)$,
- differential algebra $\Omega(M)$, metric $g^{\mu\nu}$, Laplace operator Δ ,
- spin^c structure(s), real spin structure, Dirac operator

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- properties of the Dirac operator (ellipticity...)

Problems are to calculate:

- the eigenvalues of the Dirac operator
- the invariants of the manifolds/structures

THE GEOMETRY ACCORDING TO CONNES

Spectral geometry:

- an algebra \mathcal{A} , its representation π on Hilbert space \mathcal{H} ,
- an unbounded operator D such that $[D, \pi(a)]$ is bounded for any $a \in \mathcal{A}$,
- grading γ , real structure J
- (anti)commutation relations between D, γ, J, π
- 0-order and order-1 conditions:

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- **regularity** axioms
- **finiteness** axioms
- **dimension** axioms

COMMUTATIVE AND NONCOMMUTATIVE

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A. Connes, *Noncommutative geometry and reality*, J. Math. Phys. 36, 6194, (1995)

THEOREM [CONNES]

If $\mathcal{A} = C^\infty(M)$, M a spin Riemannian compact manifold, $\mathcal{H} = L^2(S)$ (sections of spinor bundle) and D the Dirac operator on M then to $(\mathcal{A}, \mathcal{H}, D)$ is a spectral triple (with a real structure).

Commutative geometries, which satisfy Connes' axioms are in 1:1 correspondence with Riemannian spin manifolds with a given spin structure and metric.

A. Connes, *On the spectral characterization of manifolds*, J. Noncom. Geom. 7, 1–82 (2013)

GENUINE NONCOMMUTATIVE REAL SPECTRAL TRIPLES

EXAMPLES OF REAL SPECTRAL GEOMETRIES

- The Noncommutative Torus: $UV = e^{2\pi i\theta} VU$
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HOW TO CONSTRUCT THEM?

There is **so far** no general method. Only examples.

REDISCOVERING THE SPIN STRUCTURE.

The question: are there spin structures in NCG ?

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THEOREM (PASCHKE & SITARZ, LMP, 77, 3,(2006))

There are four inequivalent equivariant spin structures on the 2-dimensional noncommutative torus, with a unique choice of equivariant Dirac operator for each spin structure:

$$d_{\mu,\nu}^+ = \tau_\mu \mu + \tau_\nu \nu,$$

which satisfies the Hochschild cycle condition, provided that $\tau_\mu \tau_\nu^ \neq \tau_\mu^* \tau_\nu$. The spectrum of the equivariant Dirac operator depends on the spin structure.*

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More results followed.

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Problem: Why only flat or round geometries ?

REAL STRUCTURE AND POISSON GEOMETRY

E. Hawkins, *Noncommutative rigidity*, Commun. Math. Phys. 246, 211–235 (2004)

In fact, with the help of Poisson geometry, it was showed by Eli Hawkins that if a noncommutative **real** spectral triple is a deformation of the real spectral triple of functions on a 2-dimensional smooth manifold, then the underlying Riemannian manifold can only be either the flat torus or the round sphere.

Commun. Math. Phys. 246, 211–235 (2004)
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Communications in
Mathematical
Physics

Noncommutative Rigidity

Eli Hawkins

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Abstract: Using very weak criteria for what may constitute a noncommutative geometry, I show that a pseudo-Riemannian manifold can only be smoothly deformed into noncommutative geometries if certain geometric obstructions vanish. These obstructions can be expressed as a system of partial differential equations relating the metric and the Poisson structure that describes the noncommutativity. I illustrate this by computing the obstructions for well known examples of noncommutative geometries and quantum groups. These rigid conditions may cast doubt on the idea of noncommutatively deformed space-time.

1. Introduction

One plausible way to try and construct examples of noncommutative geometry is to start with an ordinary, commutative manifold and deform it. One can try to construct noncommutative algebras that in some sense approximate the algebra of smooth functions on the manifold, and then to construct noncommutative geometries which approximate the geometry of the original manifold. There has been considerable success with the first step. Techniques of geometric quantization can be applied in many cases to construct a sequence of algebras which approximate the algebra of functions in a very strong sense. In a much weaker sense, the formal deformation quantization constructions of Fedosov [16] and Kontsevich [21] give noncommutative approximations to any manifold.

Another motive for considering deformations is physical. There are many reasons to suspect that pseudo-Riemannian geometry might not accurately describe the small scale structure of space-time. Noncommutative geometry is a plausible route toward a better description. However, the fact that pseudo-Riemannian geometry is a sufficient description of space-time for most purposes, suggests that noncommutativity might be treated as a perturbation.

If so, then this noncommutativity would be described in the leading order by a Poisson structure. Much optimism about this direction was generated by Kontsevich's remarkable

ARE THERE ANY INTERESTING NC GEOMETRIES ?

A SOFTER VERSION OF *geometry*?

The facts:

- ① for the examples of q -deformed algebras (Podleś spheres, $SU_q(2)$) - there are no spectral geometries **in the exact sense** – but – there are geometries in which some of the commutation relations are **satisfied up to compact operators**:

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$$\left[J\pi(a)J^{-1}, \pi(b) \right] \in \mathcal{K}_q,$$

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Remark: Leads to nontrivial classical "triples".

REALITY TWISTED BY A LINEAR AUTOMORPHISM

Let A be a complex $*$ -algebra and let (H, π) be a (left) representation of A on a complex vector space H . A linear automorphism ν of H defines an algebra automorphism

$$\bar{\nu} : \text{End}(H) \rightarrow \text{End}(H), \quad \phi \mapsto \nu \circ \phi \circ \nu^{-1}.$$

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The inverse of $\bar{\nu}$ is $\phi \mapsto \nu^{-1} \circ \phi \circ \nu$. Since $\bar{\nu}$ is an algebra map, the composite

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is an algebra map too, and hence it defines a new representation (H, π^ν) of A . The map ν is an isomorphism that intertwines (H, π) with (H, π^ν) .

We could also require that $\pi^\nu(a) \in \pi(A)$ so for faithful π the map $\bar{\nu}$ defines an (algebra) automorphism of A

TWISTED REALITY

DEFINITION (TWISTED REAL SPECTRAL TRIPLE)

Let A be a $*$ -algebra, (H, π) a representation of A , D a linear operator on H , and let ν be a linear automorphism of H . We say that the triple (A, H, D) admits a ν -*twisted real structure* if there exists an anti-linear map $J : H \rightarrow H$ such that $J^2 = \epsilon \text{id}$, and, for all $a, b \in A$,

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$$\nu J\nu = J,$$

TWISTED REALITY

If (A, H, D) admits a grading operator $\gamma : H \rightarrow H$:

$$\gamma^2 = \text{id}, \quad [\gamma, \pi(\mathbf{a})] = 0, \quad \gamma D = -D\gamma, \quad \nu^2 \gamma = \gamma \nu^2,$$

then the twisted real structure J is also required to satisfy

$$\gamma J = \epsilon'' J \gamma,$$

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In case of H being a Hilbert space the automorphism ν is also assumed to be densely defined and selfadjoint, with the requirement that $\bar{\nu}$ maps $\pi(A)$ into bounded operators.

The signs $\epsilon, \epsilon', \epsilon''$ determine the KO -dimension modulo 8 in the usual way and the operator J is antiunitary.

TWISTED REAL SPECTRAL TRIPLES

We shall say that a spectral triple admits a ν -twisted real structure, or simply that is a ν -twisted real spectral triple.

The commutant condition is called the *order-zero condition* and the one with the Dirac operator is called the *twisted order-one condition*. We shall call the modified condition the *the twisted ϵ' -condition*.

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REMARK

This is an *extension* not a *replacement*. In the case of $\nu = \text{id}$ we get the usual, well known, spectral triples.

ON TWISTING REAL SPECTRAL TRIPLES BY ALGEBRA AUTOMORPHISMS

by Giovanni Landi, Pierre Martinetti

On twisting real spectral triples by algebra automorphisms

Giovanni Landi, Pierre Martinetti

Abstract

We systematically investigate ways to twist a real spectral triple via an algebra automorphism and in particular, we naturally define a twisted partner for any real graded spectral triple. Among other things we investigate consequences of the twisting on the fluctuations of the metric and possible applications to the spectral approach to the standard model of particle physics.

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1st January 2016

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Twisted and graded twisted spectral triples were defined in [7] by replacing the boundedness of the commutator $[D, a]$ with the requirement that the twisted commutator

$$[D, a]_\rho := Da - \rho(a)D, \quad (2.6)$$

for an automorphism $\rho \in \text{Aut}(\mathcal{A})$, be bounded for any $a \in \mathcal{A}$. Furthermore, the automorphism ρ is not taken to be a $*$ -automorphism, but rather to satisfy

$$\rho(a^*) = (\rho^{-1}(a))^*. \quad (2.7)$$

Such an automorphism was named *regular* in [13]. The requirement (2.7) has origin in the additional assumption (coming from considerations in index theory in [1]) that the algebra \mathcal{A} has a 1-parameter group of automorphisms $\{\rho_t\}_{t \in \mathbb{R}}$ and that ρ coincides with the value at $t = i$ of the analytic extension of $\{\rho_t\}_{t \in \mathbb{R}}$. In typical examples (for instance the spectral triples associated to codimension 1 foliations) the 1-parameter group of automorphisms is the modular automorphism group of a twisted trace. Such twisted traces appear naturally with twisted spectral triples. Indeed, if $(\mathcal{A}, \mathcal{H}, D)$ is a ρ -twisted spectral triple with $D^{-1} \in \mathcal{L}^{\infty, \infty}$, the Dixmier ideal, from [7, Prop. 3.3] the functional

$$\mathcal{A} \ni a \mapsto \varphi(a) = \int a D^{-n} := \text{Tr}_\omega(a D^{-n}), \quad (2.8)$$

¹When possible we omit the representation symbol and identify $a \in \mathcal{A}$ with its representation $\pi(a) \in \mathcal{L}(\mathcal{H})$.

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with Tr_ω the Dixmier trace, is a ρ^{-n} -trace, that is $\varphi(ab) = \varphi(b\rho^{-n}(a))$ for all $a, b \in \mathcal{A}$.

Now, the algebras \mathcal{A} and \mathcal{A}^ρ have isomorphic automorphism groups. An isomorphism is:

$$\text{Aut}(\mathcal{A}) \ni \rho \mapsto \rho^\circ \in \text{Aut}(\mathcal{A}^\rho), \quad \rho^\circ(b^*) := (\rho^{-1}(b))^*, \quad \forall b^* \in \mathcal{A}^*. \quad (2.9)$$

The use of ρ^{-1} instead of ρ is to parallel condition (2.7). In a sense, the above means

$$\rho^\circ(Jb^*J^{-1}) = J(\rho^{-1}(b))^*J^{-1} = J\rho(b^*)J^{-1}, \quad (2.10)$$

and the second equality is due to condition (2.7). We are then led to the following.

Definition 2.1. A real twisted spectral triple of KO-dimension k is the datum of a twisted spectral triple $(\mathcal{A}, \mathcal{H}, D; \rho)$ together with an antilinear isometry operator J satisfying the rule of signs (2.1), the zero-order condition (2.2), and the twisted first-order condition

$$[[D, a]_\rho, Jb^*J^{-1}]_{\rho^\circ} = 0, \quad \forall a, b \in \mathcal{A}. \quad (2.11)$$

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Definition 2.1. A real twisted spectral triple of KO-dimension k is the datum of a twisted spectral triple $(\mathcal{A}, \mathcal{H}, D; \rho)$ together with an antilinear isometry operator J satisfying the rule of signs (2.1), the zero-order condition (2.2), and the twisted first-order condition

THE FLUCTUATIONS OF THE DIRAC

Let Ω_D^1 be a bimodule of one forms:

$$\Omega_D^1 := \left\{ \sum_i \pi(a_i)[D, \pi(b_i)] \mid a_i, b_i \in A \right\}.$$

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For our case of ν -twisted real spectral triple we set the fluctuated Dirac operator D_α to be:

$$D_\alpha := D + \alpha + \epsilon' \nu J \alpha J^{-1} \nu,$$

with the requirement that $\alpha + \epsilon' \nu J \alpha J^{-1} \nu$ is selfadjoint.

FLUCTUATIONS

PROPOSITION

If (A, H, D) with $J \in \text{End}(H)$ is a ν -twisted real spectral triple, then (A, H, D_α) with (the same) J is also a ν -twisted real spectral triple. If (A, H, D) is even with grading γ , then (A, H, D_α) is even with (the same) grading γ . The composition of twisted fluctuations is a twisted fluctuation.

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PROOF

As a perturbation of D by a bounded selfadjoint operator, the fluctuated Dirac operator D_α is selfadjoint, has bounded commutators with $\pi(a) \in A$ and has compact resolvent. First, we shall demonstrate that a fluctuation of the fluctuated Dirac operator is also a fluctuation. In other words the bimodule of one forms is independent of the choice of α . Let $a \in A$ and $\alpha \in \Omega_D^1$.

PROOF (CONTINUED)

We compute:

$$\begin{aligned}[\alpha', \pi(\mathbf{a})] &= \nu \mathbf{J} \alpha \mathbf{J}^{-1} \nu \pi(\mathbf{a}) - \pi(\mathbf{a}) \nu \mathbf{J} \alpha \mathbf{J}^{-1} \nu \\ &= \nu \mathbf{J} \alpha \mathbf{J}^{-1} \nu \pi(\mathbf{a}) - \nu \pi(\bar{\nu}^{-1}(\mathbf{a})) \mathbf{J} \alpha \mathbf{J}^{-1} \nu \\ &= \nu \mathbf{J} \alpha \mathbf{J}^{-1} \nu \pi(\mathbf{a}) - \nu \mathbf{J} \alpha \mathbf{J}^{-1} \pi(\bar{\nu}(\mathbf{a})) \nu \\ &= \nu \mathbf{J} \alpha \mathbf{J}^{-1} \nu \pi(\mathbf{a}) - \nu \mathbf{J} \alpha \mathbf{J}^{-1} \nu \left(\nu^{-1} \pi(\bar{\nu}(\mathbf{a})) \nu \right) = 0.\end{aligned}$$

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Therefore for any $\alpha \in \Omega_D^1$ and $\mathbf{a} \in A$ we have:

$$[D_\alpha, \pi(\mathbf{a})] = [D, \pi(\mathbf{a})] + [\alpha, \pi(\mathbf{a})],$$

PROOF (CONTINUED)

To finish the proof it remains only to check that D_α satisfies the compatibility relation with J , that is:

$$D_\alpha J\nu = \epsilon'\nu J D_\alpha.$$

Since D itself satisfies it, we compute it for $\alpha + \epsilon'\alpha'$:

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$$\begin{aligned}(\alpha + \epsilon'\alpha')J\nu &= (\alpha J\nu + \epsilon'\nu J\alpha J^{-1}\nu J\nu) \\ &= \alpha J\nu + \epsilon'\nu J\alpha \\ &= \epsilon'\nu J(\alpha + \epsilon'J^{-1}\nu^{-1}\alpha J\nu) \\ &= \epsilon'\nu J(\alpha + \epsilon'\alpha').\end{aligned}$$

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Like in the case of the real spectral triples the twisted one admit a family of fluctuated Dirac operators.

EXAMPLE 1: CONFORMAL PERTURBATIONS

Let us assume that we have a real spectral triple (A, H, D, J) with reality operator J and fixed signs ϵ, ϵ' . Let $k \in \pi(A)$ be a positive and invertible bounded operator such that k^{-1} is also bounded, and let us denote by $k' = JkJ^{-1}$.

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PROPOSITION

If (A, H, D) with J is a real spectral triple, which satisfies order one condition, then for:

$$D_k = k' D k', \quad \nu(h) = k^{-1} k' h,$$

the triple (A, H, D_k) with J is a ν -twisted real spectral triple. If furthermore (A, H, D) is even with grading γ , then (A, H, D_k) is even with (the same) grading γ .

EXAMPLE 1: CONFORMAL PERTURBATIONS

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Since k and k' are bounded operators it is clear that $\bar{\nu}$ maps bounded operators to bounded operators, and we have for all $a \in A$:

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We show now that D_k satisfies the twisted order-one condition :

$$\begin{aligned} J\pi(b)J^{-1}[D_k, \pi(a)] &= J\pi(b)J^{-1}JkJ^{-1}[D, \pi(a)]JkJ^{-1} \\ &= k'[D, \pi(a)]k'J(k^{-2}\pi(b)k^2)J^{-1} = [D_k, \pi(a)]J\pi(\bar{\nu}^2(b))J^{-1}. \end{aligned}$$

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Next we check compatibility between J and ν :

$$\nu J\nu = k^{-1}JkJ^{-1}Jk^{-1}JkJ^{-1} = J.$$

EXAMPLE 1: CONFORMAL PERTURBATIONS

PROOF (CTD)

Finally, if $JD = \epsilon' DJ$ then for D_k we have:

$$JD_k = JK'J^{-1}JDk' = \epsilon'kDJk' = \epsilon'k(k')^{-1}D_k(k')^{-1}kJ,$$

so that the following is satisfied

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REMARK

In the 'classical' case of a manifold M and (commutative) $A = C^\infty(M)$ with Ad_J being the complex conjugation, the conformal twists are always trivial as $JkJ^{-1} = k$ for a positive k and hence $\nu = \text{id}$.

TWISTED REALITY ON QUANTUM CONES

The coordinate algebra of the *quantum disc* $\mathcal{O}(D_q)$ is a complex $*$ -algebra generated by z , subject to the relation

$$z^*z - q^2zz^* = 1 - q^2,$$

where $q \in (0, 1)$. $\mathcal{O}(D_q)$ can be understood as a \mathbb{Z} -graded algebra

$$\mathcal{O}(D_q) = \bigoplus_{n \in \mathbb{Z}} \mathcal{O}(D_q)_n,$$

with the degrees given on the generators by $|z| = -|z^*| = 1$. For any $N > 1$, the *quantum cone* $\mathcal{O}(C_q^N)$ is defined as the $*$ -algebra generated by y and self-adjoint x subject to relations:

$$xy = q^{2N}yx, \quad yy^* = \prod_{l=0}^{N-1} (1 - q^{-2l}x), \quad y^*y = \prod_{l=1}^N (1 - q^{2l}x).$$

TWISTED REALITY ON QUANTUM CONES

We will view $\mathcal{O}(C_q^N)$ as a subalgebra of $\mathcal{O}(D_q)$ in this way. Using the \mathbb{Z} -grading of $\mathcal{O}(D_q)$ we define a degree counting algebra automorphism,

$$\nu : \mathcal{O}(D_q) \rightarrow \mathcal{O}(D_q), \quad a \mapsto q^{|a|} a,$$

for all homogeneous elements of $\mathcal{O}(D_q)$. Obviously, ν preserves the degrees of homogeneous elements, and hence restricts to an automorphism of $\mathcal{O}(C_q^N)$. It is also compatible with the $*$ -structure in the sense that,

$$\nu \circ * \circ \nu = *.$$

The maps $\partial_-, \partial_+ : \mathcal{O}(D_q) \rightarrow \mathcal{O}(D_q)$, defined on generators of the disc algebra by

$$\partial_-(z) = z^*, \quad \partial_-(z^*) = 0, \quad \partial_+(z) = 0, \quad \partial_+(z^*) = q^2 z,$$

extend to the whole of $\mathcal{O}(C_q^N)$ as ν^2 -skew derivations, i.e. by the twisted Leibniz rule,

$$\partial_{\pm}(ab) = \partial_{\pm}(a)\nu^2(b) + a\partial_{\pm}(b),$$

TWISTED REALITY ON QUANTUM CONES

Set

$$H_+ = \bigoplus_{n \in \mathbb{Z}} \mathcal{O}(D_q)_{nN+1}, \quad H_- = \bigoplus_{n \in \mathbb{Z}} \mathcal{O}(D_q)_{nN-1}, \quad H = H_+ \oplus H_-.$$

Since $\mathcal{O}(C_q^N)$ is spanned by homogeneous elements of degrees that are multiples of N , and ν preserves the degrees, both H_{\pm} and hence also H are left $\mathcal{O}(C_q^N)$ -modules (representations) by the action

$$\pi(a)(h_{\pm}) = \nu^2(a)h_{\pm}, \quad \text{for all } a \in \mathcal{O}(C_q^N), h_{\pm} \in H_{\pm}.$$

Furthermore, ν restricts to an automorphism of the vector space H . Note, however, that since ν is an automorphism of the graded algebra $\mathcal{O}(D_q)$ and the action of $\mathcal{O}(C_q^N)$ is defined by the multiplication in $\mathcal{O}(D_q)$, the induced automorphism $\bar{\nu}$ of the (linear) endomorphism ring takes a particularly simple form on $\pi(a)$,

$$\bar{\nu}(\pi(a)) = \pi(\nu(a)), \quad \text{for all } a \in \mathcal{O}(C_q^N).$$

TWISTED REALITY ON QUANTUM CONES

Since ∂_{\pm} have degrees ± 2 ,

$$\partial_{\pm}(H_{\mp}) \subseteq H_{\pm},$$

and the $*$ -operation reverses degrees we can define operators

$$D : H \rightarrow H, \quad (h_+, h_-) \mapsto \left(-q^{-1} \partial_+(h_-), q \partial_-(h_+) \right),$$

$$J : H \rightarrow H, \quad (h_+, h_-) \mapsto (-h_-^*, h_+^*),$$

and also the grading operator,

$$\gamma : H \rightarrow H, \quad (h_+, h_-) \mapsto (h_+, -h_-).$$

TWISTED REALITY ON QUANTUM CONES

PROPOSITION

With these definitions, $(\mathcal{O}(C_q^N), H, D)$ is an (algebraic) even spectral triple of KO-dimension two with ν -twisted real structure J and grading γ .

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PROPOSITION

With these definitions, $(\mathcal{O}(C_q^N), H, D)$ is an (algebraic) even spectral triple of KO-dimension two with ν -twisted real structure J and grading γ .

PROOF

The order zero condition follows immediately by the definition of J and the fact that $*$ is an anti-algebra involution, the compatibility between J and ν is straightforward. Checking of the twisted order one condition and twisted commutativity is a bit more involved. Specifically, for all $h_{\pm} \in H_{\pm}$,

$$\nu J D(h_{\pm}) = q^{\pm 1} \nu (\partial_{\mp}(h_{\pm})^*) = q^{\pm 1} \nu^{-1} (\partial_{\pm}(h_{\pm}^*)),$$

on the other hand,

$$D J \nu(h_{\pm}) = \pm D(\nu^{-1}(h_{\pm}^*)) = q^{\mp 1} \partial_{\pm}(\nu^{-1}(h_{\pm}^*)) = q^{\pm 1} \nu^{-1}(\partial_{\pm}(h_{\pm}^*)),$$

TWISTED REALITY ON QUANTUM CONES

PROOF (CTD)

To prove the twisted first order condition observe that, for all $a, b \in \mathcal{O}(C_q^N)$ and $h_{\pm} \in H_{\pm}$,

$$\begin{aligned} [D, \pi(a)](h_{\pm}) &= \pm q^{\pm 1} \left(\partial_{\mp} \left(\nu^2(a) h_{\pm} \right) - \nu^2(a) \partial_{\mp} (h_{\pm}) \right) \\ &= \pm q^{\pm 1} \partial_{\mp} \left(\nu^2(a) \right) \nu^2(h_{\pm}) = \pm q^{\pm 5} \nu^2(\partial_{\mp}(a) h_{\pm}), \end{aligned}$$

by the q -skew-derivation properties, and

$$J\pi(b)J^{-1}(h_{\pm}) = h_{\pm} \nu^2(b)^* = h_{\pm} \nu^{-2}(b^*),$$

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then

$$J\bar{\nu}^2(\pi(b))J^{-1}(h_{\pm}) = h_{\pm}\nu^{-4}(b^*).$$

Putting all this together one easily checks that J and D satisfy the twisted order-one condition.

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THANK YOU