Local Noncommutativity

Andrzej Sitarz[b,c,2]

[b] *Instytute of Physics, Jagiellonian University, ul. Stanisława Łojasiewicza 11, 30-348 Kraków, Poland.*

[c] *Institute of Mathematics of the Polish Academy of Sciences, Śniadeckich 8, 00-656 Warszawa, Poland.*

E-mail: andrzej.sitarz@uj.edu.pl

ABSTRACT: We propose two simple models of noncommutative space, which are examples of local noncommutativity, that is, a deformed space where the degree of noncommutativity varies. Both models are generalization of spherically symmetric space, with the first that (locally) preserves the q-deformed rotational symmetry, whereas, the second, based on the fuzzy sphere has both local and global rotational invariance.

KEYWORDS: noncommutative geometry, local noncommutativity

Contents

1 Introduction

Within the last decades our understanding of the mathematical principles improved enormously both at the small scale (particle physics) as well as at the big scale (cosmology). However, we are still far from the theory, which could offer a complete, comprehensive view of the entire fundamental physics. The gauge theories, with their roots in the mathematical theory of principal fiber bundles and connections, lead to the formulation of classical and perturbative quantum field theory of interacting fields that describe interactions in the world.

On the other side, a purely geometrical theory of gravity, which is mathematically set in the Riemannian (or pseudo-Riemannian) geometry, lacks a consistent and mathematically sound quantum version, the *evasive* quantum gravity. Although both theories (quantum field theory and classical gravity) are well-tested and define the basics of our current knowledge we still are uncertain whether they are just shadows of a bigger theory. The hypothetical bigger, unique and satisfactory theory would need to combine both gauge theories and gravity into one single mathematical framework.

So far, many the attempts have been directed towards the presentation of General Relativity, so that it is presented like a gauge theory, which could allow quantum formalism following the more or less well understood. An attempt to look in the opposite direction originated some time ago from the works of Alain Connes (see [1] for a review). The basic idea is to look at gauge theories (on the classical level) as modifications of classical gravity. This, however, requires us to enlarge our understanding of geometry and introduce a set of new tools and methods. The answer (at least partial) is proposed in the framework of *Noncommutative Geometry* (see [2] for a comprehensive introduction).

2 Models of noncommutative spaces

There are several examples that are used in the physically motivated models of the noncommutative space-time. We shall present them very briefly and discuss their benefits as well as limitations.

2.1 Global deformations

First or all, we have a family of cocycle deformations both of compact as well as locally compact spaces: mostly so-called theta-deformations and Moyal deformation. The latter, which is probably most studied originates first in the quantum phase-space but has gained more attention through links to the quantization of target space-time in string theory [3].

Another most studied deformation of space-time (and more generally a class of deformation) comes from deformation of classical symmetries, quantum groups. In particular, one of the first result was the classification of possible quantum deformations of the Lorentz (and Poincaré) group [7]. That includes not only q-Minkowski space [] but also most studied κ-Minkowski [6] deformation, much popular due to the natural appearance of a length scale while at the same time keeping the symmetries (though in the deformed form).

Last large family of quantum spaces is based on the nontrivial approximation of smooth manifolds through matrix algebras, known as fuzzy spaces. The idea first discussed and most studied for the fuzzy sphere [10] has an appealing air of simplicity (as matrix algebras are rather easy to manipulate and allow to implement simulations) while at the same time preserving at least some symmetries.

2.2 Dynamical noncommutativity

One of the first ideas of noncommutativity (in the sense of deformations) that is not fixed globally but depends on some parameter (coordinate) of the deformation was proposed by Doplicher, Fredenhagen and Roberts [8] who generalized the usual Moyal-type deformation to the local version:

$$[x^\mu, x^\nu] = \theta^{\mu\nu},$$

where $\theta^{\mu\nu}$ is not treated as a constant but a central element of the algebra.

A string-inspired version of target-space noncommutativity is related to the B-field of string theory and, in the case of a field which is not constant it leads to a version of noncommutative spacetime, however, in the most generic cases, resulting in nonassociative algebras [5]. Only in special cases, where the deformation parameter has values in the center of the algebra one can work with associative star algebras [13]. Such examples are then deeply related to a family of deformations studied by Matsumoto for the three-spheres and can be effectively described as a version of cocycle twisting, where the cocycle is valued in the algebra invariant under the action of the torus [11].

3 Local noncommutativity

We shall introduce here two models, which can be used effectively used in the description of both very short-scale quantum physics and quantum gravity or in the models of the evolution of the universe.

3.1 The q-deformed quantum rotationally symmetric $\mathbb{R}^3$

Let q be a real smooth function defined on a positive part of the real line such that for all r we have $0 \geqslant q(r) \geqslant 1$. We can always assume that for large r the function $q(r) = 1$.

3.1.1 The algebra.

Consider an algebra $\mathcal{A}_{q(r)}$, which is constructed out of smooth functions from $\mathbb{R}_+$ (including 0) into the algebra of standard quantum Podles spheres $S_{q(r)}$. Of, course, q can be 1 over a significant part of the real line thus reducing to the classical case. If $q = 1$ identically then we are fully in the commutative case.

More precisely we should consider a unitization of the algebra of functions that vanish at $r = 0$ as otherwise shall rather be constructing a noncommutative space with a boundary that will be a Podles sphere. We can restrict the algebra further, stating precisely that our algebra shall be generated by elements of the form:

$$f(r)P(A, B),$$

where $P(A, B)$ is a polynomial in the generators of the algebra of the standard quantum Podles sphere A, B:

$$\begin{aligned} AB &= q^2BA, & AB^* &= q^{-2}B^*A, \\ BB^* &= q^{-2}A(1-A), & B^*B &= A(1-q^2A). \end{aligned} \tag{3.1}$$

We define the Hilbert space in a similar manner to be the square summable function from $\mathbb{R}_+$ to the Hilbert space on which $S^2_{q(r)}$ is equivariantly represented, with the restriction that at $r = 0$ they vanish. The scalar product between two spinor fields is taken to be:

$$(\Psi,\Phi) = \int_{r=0}^{\infty} r^2 \left(\Psi(r),\Psi(r)\right) dr,$$

where the scalar product at point r is the usual.

$$\begin{aligned}
\pi_N(B)|l,m\rangle = {}& \quad q^m\sqrt{[l+m+1][l+m+2]}\; r^+(l)\; |l+1,m+1\rangle \\
&+ q^m\sqrt{[l+m+1][l-m]}\; r^0(l)\; |l,m+1\rangle \\
&+ q^m\sqrt{[l-m][l-m-1]}\; r^-(l)\; |l-1,m+1\rangle, \\
\pi_N(B^*)|l,m\rangle = {}& \quad q^{m-1}\sqrt{[l-m+2][l-m+1]}\; r^-(l+1)\; |l+1,m-1\rangle \\
&+ q^{m-1}\sqrt{[l+m][l-m+1]}\; r^0(l)\; |l,m-1\rangle \\
&+ q^{m-1}\sqrt{[l+m][l+m-1]}\; r^+(l-1)\; |l-1,m-1\rangle, \\
\pi_N(A)|l,m\rangle = {}& -q^{m+l+\frac{1}{2}}\sqrt{[l-m+1][l+m+1]}\; r^+(l)\; |l+1,m\rangle \\
+\frac{q^{-\frac{1}{2}}}{1+q^2}&\Big([l-m+1][l+m] + q^2[l-m][l+m+1]\Big)\; r^0(l) + q^{\frac{1}{2}}\,|l,m\rangle \\
&+ q^{m-l-\frac{1}{2}}\sqrt{[l-m][l+m]}\; r^-(l)\; |l-1,m\rangle,
\end{aligned} \tag{3.2}$$

where $[x]$ denotes a q-number

$$[x] = \frac{q^x - q^{-x}}{q - q{-}1},$$

the functions $r^\pm$ and r^0 are:

$$\begin{aligned}
r^0(l) &= q^{-\frac{1}{2}}\frac{(q-\frac{1}{q})[l+|N|+1][l-|N|] \pm q^{\pm 1}[2|N|]}{[2l][2l+2]}, \\
r^+(l) &= \frac{q^{-l-\frac{3}{2}-N}}{[2l+2]}\frac{\sqrt{[l+N+1][l-N+1]}}{\sqrt{[2l+1][2l+3]}}, \\
r^-(l) &= -q^l r^+(l-1).
\end{aligned} \tag{3.3}$$

and, for the standard spinorial representation we take $N = \pm\frac{1}{2}$.

3.1.2 The geometry.

The geometry for this space is introduced through the Dirac operator, built in similar way as the Dirac operator for $\mathbb{R}^3$ is built in the classical case. Assume that we have a Dirac operator D_q over each of the quantum spheres S_q as constructed in [9]. One technical point which differs from the classical case is that in principle we may not fix (yet) the volume of the quantum sphere. For simplicity, however, we assume that the coefficient in from of the spherical Dirac operator is the same as classically,

$$D_3 = i\gamma\left(\partial_r + \frac{1}{r}\right) + \frac{1}{r}D_q. \tag{3.4}$$

$$D_3^2 = -(\partial_r)^2 - \frac{2}{r}\partial_r + \frac{1}{r^2}(D_q)^2 - \frac{i}{r^2}\gamma D_q + \frac{i}{r}\gamma\partial_r(D_q).$$

First of all observe that D_3 has always bounded commutators with all elements of the algebra. Indeed, we know that the generators of the quantum sphere behave in the following way:

$$\begin{aligned}
\tilde{\pi}_N(B)|l,m\rangle_N = &\ \sqrt{1-q^{2(l+m+2)}}\sqrt{1-q^{2(l+m+1)}}\,q^{l-N}\ |l+1,m+1\rangle_N \\
&+ q^{l+m}\sqrt{1-q^{2(l+m+1)}}\ |l,m+1\rangle_N \\
&- q^{2(l+m)}q^{l-N}\ |l-1,m+1\rangle_N + o(q^{2l}), \\
\tilde{\pi}_N(B^*)|l,m\rangle_N = &-q^{2(l+m)+1}q^{l-N}\ |l+1,m-1\rangle_N \\
&+ q^{l+m-1}\sqrt{1-q^{2(l+m)}}\ |l,m-1\rangle_N \\
&+ q^{l-N-1}\sqrt{1-q^{2(l+m)}}\sqrt{1-q^{2(l-m-1)}}\ |l-1,m-1\rangle_N + o(q^{2l}), \\
\tilde{\pi}_N(A)|l,m\rangle_N = &-q^{l+m}q^{l-N+1}\sqrt{1-q^{2(l+m+1)}}\ |l+1,m\rangle_N \\
&+ q^{2(l+m)}\ |l,m\rangle_N \\
&+ q^{l+m}q^{l-N-1}\sqrt{1-q^{2(l+m-1)}}\ |l-1,m\rangle_N + o(q^{2l}),
\end{aligned} \tag{3.5}$$

up to an element of order q^l. Since the parameter q is a function of r we need to make sure that derivation with respect to r shall leave all coefficients bounded.

However, note that for $q(r) < 1$:

$$|\frac{\partial}{\partial_r}q^n(r)| = |n(\partial_r q(r))q^{n-1}| < 1,$$

for sufficiently large n and therefore whenever $q < 1$ then the derivative is also bounded. In the case $q = 1$ the derivative must vanish as we assume that $q(r)$ is smooth and $0 \leqslant q(r) \leqslant 1$.

The model could equally be studied for $N = 0$ (or any integral or half-integral value of N) providing generalization of scalar or higher-spin fields in the model.

3.2 Rotationally symmetric fuzzy $\mathbb{R}^3$

The concept of the fuzzy sphere [10] is an interesting as it provides an approximation of the continuous manifold (a commutative algebra) through a finite-dimensional matrix algebra while at the same time keeping the rotational symmetry intact.

3.3 The algebra

Let us define $M_\infty(\mathbb{C})$ as an algebra of infinite matrices, understood as a a direct limit of matrices of finite rank with the natural embedding of matrices of smaller rank into a bigger rank. By $\rho(m)$ we understand the largest N so that $m \in M_N(\mathbb{C})$ (note that this does not need to coincide with the rank of m).

Consider the following set

$$A = \{f : \mathbb{R}_+ \to M_\infty(\mathbb{C}) \text{ such that } \rho(f(r)) < H(r)\},$$

where $H(r)$ is a an increasing function and, whenever $\rho(f(r)) = \infty$ then the algebra is commutative and for each r is the algebra of smooth functions on a sphere.

As the function ρ is regular under addition and multiplication (that means that $\rho(m_1 m_2) \leqslant \max(\rho(m_1), \rho(m_2))$ and the same applies to the sum). we see that with the standard addition and multiplication the above set becomes an algebra. Yet still we can have a model, which has a strong discontinuity (for example) when passing from the classical to noncommutative regime.

The main idea beyond this model is to have a model with rotational symmetry, simarly as for the usual model of $\mathbb{R}^3$ and, for a large value of the varable r behaving like a classical (commutative) space. We want, however, that it shall demonstrate strong noncommutativity for small r. The interpretation of the parameter r could be as the a radial parameter of the coordinate system that describes a (fuzzy) noncommutative space.

To precisely define the algebra consider the spinorial representation of the sphere algebra. It could be seen as the limit of the q-sphere representation presented above, with the explicit formulas for the generators of spherical harmonics given by (here again $N = \pm\frac{1}{2}$),

$$\begin{aligned}
\pi_N(b)|l,m\rangle =& \sqrt{(l+m+1)(l+m+2)}\,\alpha_l^+\,|l+1,m+1\rangle \\
&+ \sqrt{(l+m+1)(l-m)}\,\alpha_l^0\,|l,m+1\rangle \\
&- \sqrt{(l-m)(l-m-1)}\,\alpha_{l-1}^+\,|l-1,m+1\rangle, \\
\pi_N(b^*)|l,m\rangle =& -\sqrt{(l-m+2)(l-m+1)}\,\alpha_l^+\,|l+1,m-1\rangle \\
&+ \sqrt{(l+m)(l-m+1)}\,\alpha_l^0\,|l,m-1\rangle \\
&+ \sqrt{(l+m)(l+m-1)}\,\alpha_{l-1}^+\,|l-1,m-1\rangle, \\
\pi_N(a)|l,m\rangle =& -\sqrt{(l-m+1)(l+m+1)}\,\alpha_l^+\,|l+1,m\rangle \\
&+ m\,\alpha_l^0\,|l,m\rangle \\
&- \sqrt{(l-m)(l+m)}\,\alpha_{l-1}^+\,|l-1,m\rangle.
\end{aligned} \tag{3.6}$$

where α_l^+, α_l^0 are

$$\begin{aligned}
\alpha_l^0 &= \frac{N}{l(l+1)}, \\
\alpha_l^+ &= \sqrt{1 - \frac{N^2}{(l+1)^2}}\,\frac{1}{\sqrt{(2l+1)(2l+3)}}
\end{aligned} \tag{3.7}$$

Consider now a family of diagonal operators on the Hilbert space, which are defined as follows:

$$P(r)|l,m\rangle = \eta\left(\frac{l}{H(r)}\right)|l,m\rangle$$

where η is smooth version of the step function that is

$$\eta(x) = \begin{cases} 1 & x < 1-\epsilon, \\ 0 & x > 1+\epsilon \end{cases},$$

for a small $\epsilon > 0$.

Though $P(r)$ is not a projection for an arbitrary r it always is a finite rank operator whenever $H(r) < \infty$. Therefore the algebra generated by:

$$f(r)P(r)aP(r),\ g(r)P(r)bP(r),\ h(r)P(r),$$

for any functions f, g, h such that f, g vanish at $r = 0$, will be a finite-dimensional algebra for any such r. It is easy to see that in fact it will be a simple algebra, that is, isomorphic to a full matrix algebra.

3.3.1 The geometry

Consider now a family of operators, which are the standard Dirac operator on the sphere, truncated through the projection $P(r)$:

$$D_2(r)|l, m\rangle = (l+N)P(r)|l, m\rangle, \tag{3.8}$$

Certainly, if $P(r)$ is finite rank then $D_2(r)$ is again a finite rank operator. We propose that the geometry of the consructed model of a fuzzy three-space is given by:

$$D_3 = \gamma\left(\partial_r + \frac{1}{r}\right) + \frac{1}{r}D_2(r). \tag{3.9}$$

Next we shall prove that it is a good candidate for the Dirac operator, that is it has bounded commutators with the elements of the algebra. Indeed, for functions $f(r), g(r), h(r)$ that are smooth, their commutator with ∂_r gives again a bounded function. Similarly, by definition $P(r)$, which for any r is finite rank has a bounded derivative.

The square of the Dirac operator becomes

$$D_3^2 = -(\partial_r)^2 - \frac{2}{r}\partial_r + \frac{1}{r^2}(D_2(r))^2 - \frac{i}{r^2}\gamma D_2(r) + \frac{i}{r}\gamma\partial_r(D_2(r)).$$

4 The application of local noncommutativity models

In what follows we give a brief sketch of possible applications of the above geometrical models to typical examples of physical models. Being aware of all limitations we do not treat the models as definitive but rather as an effective way to approach possible effects of noncommutativity on physics. Note that unlike in the "global" deformation models here we work with parameters that vary in space and therefore we can link the evidence of commutative space-time (see for example [14]) with the concept of quantum-space time at Planck scale.

4.1 The wave function on local q-space

Let us consider a wave equation on the sections of the spinors (though the scalar version is also possible) split into the spherical eigenfunctions of the Dirac operator. Assuming that we take the exact form of the Dirac operator,

$$(D_q)^2|l,m\rangle = [l+N]_{q(r)}|l,m\rangle, \tag{4.1}$$

and postulate the solution of the wave equation in the form

$$\Psi_{l,m}(r,t) = F(r,t)|l,m\rangle, \tag{4.2}$$

we obtain using (3.9) and the square of the Dirac the following equation for F in the scalar case (by analogy with the classical case we interpret the scalar version of the Laplace operator as $N=0$ case with the grading γ in D_l^2 set to i).

$$\begin{aligned}(\partial_t)^2F(r,t) =& -(\partial_r)^2F(r,t) - \frac{2}{r}\partial_r F(r,t) \\ &+ \left(\frac{[l]^2_{q(r)}}{r^2} + \frac{[l]_{q(r)}}{r^2} - \partial_r[l]_{q(r)}\right)F(r,t).\end{aligned} \tag{4.3}$$

Of course, for the limiting case of constant $q(r)=1$ we recover the usual equation for the wave function in spherical coordinates (as the last term vanishes since q is constant and $[n]_q = n$ at $q=1$).

There are, however, two interesting cases of possible effects that could depend on q. First of all, if $q<1$ but constant (locally) in r, we obtain an equation of motion, which is similar to the classical one, however, with the q-dependent parameters. In particular, the solutions with a fixed frequency will be of the type

$$F(r,t) = e^{i\omega t}\sqrt{\frac{\pi}{2\omega}}B_\alpha(\omega x),$$

where $B_\alpha(x)$ is a Bessel function of order

$$\alpha = [l]_q + \frac{1}{2}.$$

We see that the solution is exactly the same as in the classical case, with the only difference in the order of the Bessel function. Since the order enters the asymptotics of the solution (Bessel function) only as the phase factor, we see that the presence of the noncommutative region can cause a phase shift in the wave passing through this region, which can be potentially observable.

Another interesting situation is $q(r)$, which is very slowly varying with r, as the which triggers the term in the equation that depends on the derivative of the eigenvalues with respect to the radial coordinate.

Assuming a very small ϵ correction to $\log q(r)$ which is linear in r we obtains that the correction to the wave equation in of order ϵ^2:

$$\epsilon^2 \left(\frac{1}{6} l(l^2 - 1)(2l - 1) F(r, t) \right),$$

which results in the respective change in the frequency of the wave of the same order:

$$\omega' \sim \omega \left(1 - \epsilon^2 \frac{1}{6} l(l^2 - 1)(2l - 1) \right).$$

Thus, a slowly varying noncommutative region can possibly cause a slight frequency shift of the traveling wave. As the shift depends on the l-parameter, of the multipole expansion of the wave then we have a very specific pattern that can in principle be verified in observations.

4.2 The wave function on fuzzy three-space

The wave equation for the fuzzy version of local noncommutativity requires special attention as the Hilbert space, in which we seek the solutions is in fact very deeply depending on the actual size of the algebra at this point.

Therefore, for r such that $P(r)$ is finite rank we will have only a finite number of multipole solutions. Let us fix l and consider a solution of the wave equation for the fuzzy $\mathbb{R}^3$ in the region that $|l, m\rangle$ is in the range of $P(r)$. Then the solution is not different form the classical one, that is a combination of fundamental solutions of Bessel functions. However, if $|l, m\rangle$ is not in the range of $P(r)$ the solution must vanish identically. Therefore, we infer that strong noncommutativity region will enforce reflection of all waves that have high l in the multipole expansion.

On the other hand the emission and propagation of waves from the highly noncommutative region will be naturally limited to low l (in the multipole expansions). Such effects, though, discussed here only in a qualitative way are, in principle, detectable. For example, the microwave background radiation can provide significant indications whether high multipole components of the radiation are present and therefore put an experimental limit on the possible noncommutativity of the very early universe.

5 Conclusions

The presented two models are nothing else but examples, or toy models, of the way a local noncommutativity can be introduced and explored. We have chosen two possible way to implement locality of the noncommutative space, while preserving either the spherical symmetry or the deformed version of spherical symmetry. In either case the noncommutativity can occur at the scale that is beyond reach of current experiments (even by a large factor exceeding the Planck scale) yet the model can have the observational consequences for the long-distance effects and observations far from the noncommutative regions.

The models are based on the two well-studied and mathematically well-understood models of a noncommutative spheres, being realized in a three-dimensional space that, for large r is the observed local $\mathbb{R}^3$. Our intention was to present a model, which is mathematically consistent and provides a good example of a spectral triple. We have shown that the proposed algebras and the Dirac operators satisfy the basic conditions required (bounded commutators) though it is easy to see that further conditions (like the first order condition) will be satisfied as well provided that appropriate spectral triples are built for the spheres.

Of course, a lot depends on the assumed model of noncommutativity, that is on the behavious of the deformation parameter $q(r)$ or the rank of the fuzzy sphere $P(r)$. In principle that can be treated either as a free parameter for the equations of motions determining the metric (which relates to the algebra and its structure) or as a given parameter for the model.

We postpone the analysis of the possible gravity action that involves both the Dirac operator and the noncommutativity function to our further study. It is desirable to see whether a noncommutative models could effectively replace solutions in General Relativity (like the Schwarzschild solution) and cosmology (Robertson-Walker) where the singularity disappears and is replaced by noncommutativity while approaching $r = 0$.

Our intention was to show that even in such simplified models the noncommutativity, which is local, and therefore very well compatible with the observations setting limits on any potential global noncommutative parameters, can have some easy observational consequences. In the first model that can be visible though the phase shift of waves with different spherical multipole moments as well as possible frequency shifts. Second model suggests the cut-off for higher multipole number waves and both can be investigated for early universe radio sources as well as microwave bagckround radiation.

Acknowledgements

Publication supported by the John Templeton Foundation Grant âĂđConceptual Problems in Unification TheoriesâĂİ (No. 60671)

References

[1] A.Connes *A short survey of noncommutative geometry.* J. Math. Phys. 41, no. 6, 3832–3866, (2000)

[2] A. Connes, M. Marcolli: *Noncommutative Geometry, Quantum Fields and Motives*, Colloquium Publications, Vol. 55, Providence. RI: Amer. Math. Soc., 2008.

[3] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, JHEP 9909, 032 (1999)

[4] E. Witten, *Noncommutative Geometry and String Field Theory*, Nucl. Phys. B 268, 253 (1986)

[5] R.Szabo, *Quantum Field Theory on Noncommutative Spaces*,Physics Reports 378: (2003), 207-99

[6] J.Lukierski, A.Nowicki, H.Ruegg, *New quantum Poincaré algebra and κ-deformed field theory*, Phys. Lett. B293 (1992), 344-352

[7] S.L.Woronowicz, S.Zakrzewski, *Quantum deformations of the Lorentz group. The Hopf ∗-algebra level*, Comp. Math. 90 (1994), 211-243.

[8] S.Doplicher, K.Fredenhagen, J.Roberts *Space-time quantization induced by classical gravity*, Phys. Lett. B, 1994, V.331, 39-44.
S.Doplicher, K.Fredenhagen, J.Roberts, *The quantum structure of space-time at the Planck scale and quantum fields*, Comm. Math. Phys., 1995, V.172, 187-220

[9] L. Dabrowski, A.Sitarz *Dirac operator on the standard Podles quantum sphere*, Banach Center Publications 61 (2003), 49-58

[10] J.Madore, *The fuzzy sphere*, Classical Quantum Gravity, 1992, V.9, 69-88

[11] K.Matsumoto, *Noncommutative three-dimensional spheres.* Japan.J.Math. (N.S.) 17, no.2, 333–356, (1991)
K.Matsumoto, *Noncommutative three-dimensional spheres. II. Noncommutative Hopf fibering.* Yokohama Math.J. 38 , no.2, 103–111 (1991)
K.Matsumoto, *Noncommutative* 3*-spheres.* Current topics in operator algebras (Nara, 1990), 234–245, World Sci. Publishing, River Edge, NJ, 1991
K.Matsumoto, J.Tomiyama, *Noncommutative lens spaces.* J.Math.Soc.Japan 44, no.1, 13–41, (1992)

[12] A.Sitarz, *More Noncommutative 4-Spheres*, Lett.Math.Phys. 55, 127–131, (2001)

[13] A.Sitarz *Twists and spectral triples for isospectral deformations.* Lett.Math.Phys. , 58, 69–79, (2001)

[14] G.Amelino-Camelia et al, *On the IR/UV mixing and experimental limits on the parameters of canonical noncommutative spacetimes* JHEP01 (2004) 037