ELEMENTS OF MODERN MATHEMATICS

LIST OF TOPICS:

Topological manifolds

Basic notions. Homotopy groups. Chain and cochain complexes. Basic definitions for homology and cohomology. Čech cohomology. Basic computations.

Differential manifolds

Manifold: definition. Maps, atlases. Examples of manifolds. Differential structures. Diffeomorphisms. Smooth maps between manifolds.
Tangent space. Vector fields: definition through tangent space (sections of the tangent space) and operational (derivations on smooth functions). Cotangent bundle.
Pushforward and operations on vector fields.
Differential forms. External derivative. De Rham cohomology complex. Pullback of differential forms.

Fibre budles

Basic definition of a fibre bundle. Vector budle. Basic examples (Möbius band). Triviality of vector bundles.

Lie groups

Basic definition of a Lie group. Vector fields over Lie groups: left invariant. Operations on Lie groups. Adjoint action.

Principal fibre bundles

Definition. Triviality of principal fibre bundles. Right action of the Lie group. Exmples: the Hopf fibration $SU(2) \to S^2$ with local trivialisations. Vertical vectors and fundamental vertical vector fields. Definition of the connection as a bundle of horizontal vectors.

Different Views on Connections

Connection form, pullback of the connection form and its behavior with respect to local trivialisations. Curvature form and its pullback.
Associated bundles. Connection on associated bundles. General notion of a connection on sections of a vector bundle. Curvature of the connection.
Linear connections, metric compatibility of connections. Torsion of the metric connection.

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Problems set 4 for 22.11.16

Problem 1: homotopy groups

1. Let γ, ρ be continuous loops: $[0,1] \to X$, such that both start and end at x_0. Show that the homotopy classes of loops are a group (show that the product $[\gamma] * [\rho]$ does not depend on the representant of each class, show the existence of an inverse and a neutral element).
2. Let x_0, x_1 be two points in X that could be joined by a continuous path. Construct an isomorphism between $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$.
3. Show that the higher homotopy groups are abelian.
4. Show that $\pi_0(X)$ is the set of connected components of X. Is there a group structure ?
5. Compute $\pi_1(\mathbb{R}^n)$, $\pi_1(S^1)$, $\pi_1(K^2)$ where K^2 is a Klein bottle.
6. Construct explicit examples of maps $f : S^2 \to S^2$ (or $g : S^3 \to S^3$), which are in a nontrivial homotopy class. If the respective homotopy groups are $\mathbb{Z}$ can it be constructed for any $k \in \mathbb{Z}$?

Problem 2: chain and cochain complexes.

1. Please show that the cochain complex:

$$C^n(X) = C(X^{n+1}, \mathbb{C}), \qquad (\delta_n f)(x_1, \ldots, x_{n+2}) = \sum_{k=1}^{n+2} (-1)^k f(x_1, \ldots, \hat{x}_k, \ldots, x_{n+2}),$$

   is acyclic, that is, its homology vanishes.

Problem 3: Čech cohomology

1. Using two covers of a circle by two sets:

$$(-\frac{5}{4}\pi, \frac{5}{4}\pi), (\frac{1}{4}\pi, \frac{7}{4}\pi),$$

   and by three sets:

$$(-\frac{2}{3}\pi, \frac{1}{3}\pi), (0, \pi), (\frac{2}{3}\pi, \frac{5}{3}\pi),$$

   please compute in each case explicitly the Čech cochain complex (taking functions valued in $\mathbb{Z}$ or $\mathbb{R}$) the coboundary and the respective cohomologies.

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Problems set 5 for 29.11.16

Problem 1: Differential structures

1. Consider $\mathbb{R}$ with two atlases: first, the usual one $\mathbb{R}, \text{id}$, and a second one with $(\mathbb{R}, \Psi_3 : x \to x^3)$. Please show that these atlases are not compatible with each other as $\text{id} \circ \Psi_3^{-1}$ is not differentiable.
2. Nonetheless show that there exists a *diffeomorphism* between $\mathbb{R}$ with one differential structure and $\mathbb{R}$ with the other one.

Problem 2: Tangent space and vector fields

1. Please show that the equivalence classes of paths (as defined during the lecture) form a vector space. $\textbf{Hint:}$ use the structure of the vector space in $\mathbb{R}^n$.
2. Let f be a smooth function. If χ is a smooth vector field (in the sense $\chi : M \to TM$) what is the argument to show that $\chi(f)$ is smooth ?
3. Show that χ acts locally: that is, if f vanishes in some open set U then $\chi(f)$ also is identically zero on U.

Problem 3: Manifolds and fibre bundles

1. Show that the Moebius band is a vector bundle.
2. Show that there are no nontrivial complex line bundles over S^1.

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Problems set 6 for 6.12.16

Problem 1: f_*

1. Provide an example where a map between manifolds $f : M \to N$ is a bijection but f_* is not a well defined smooth map between vector fields. Where is the problem ?
2. Assuming that f is a diffeomorphism is it true that $(f_*)^{-1} = (f_*^{-1})$?

Problem 2: Transformations

1. If ϕ_X is a (local) one-parameter group of tranformations for the vector field X, similarly ϕ_Y, for Y, please show that
$$\phi_X(t) \circ \phi_Y(s) = \phi_Y(s) \circ \phi_X(t),$$
is equivalent to $[X, Y] = 0$.
2. Find a vector field associated to translations in $\mathbb{R}^n$ and rotation in $\mathbb{R}^2$.

Problem 3: Direct sums

Show that the if W is the Moebius band vector bundle over S^1 then $W \oplus W$ is isomorphic to the trivial bundle.

Problem 4: Lie derivative

1. If $\mathcal{L}_X$ is a Lie derivatirve with respect to the vector field X, please check how $\mathcal{L}_{fX}$ depends on f and $\mathcal{L}_X$.
2. Using the definition of $\mathcal{L}_X$ please find $\mathcal{L}_X(\omega)$, where ω is a one-form.

Problem 5: Differential forms

1. Please compute the de Rham complex and cohomology for S^1 and S^2.
2. For a diffeomorphism $f : M \to N$ we can define both f_* and f^* of a one-form. Please compute $f_*(d\mu)$ and $f^*(d\nu)$, where $\mu \in C^\infty(M), \nu \in \mathbb{C}^\infty(N)$.
3. Define left-invariant forms (of an arbitrary order) on a Lie group and show that the de Rham complex restricts to left-invariant forms.

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Problems set 7 for 13.12.16

Problem 1: PFB

1. Is the Möbius band a principal fibre bundle (PFB) with $\mathbb{R}$ as the group ?
2. When is a vector bundle a principal fibre bundle ?
3. Using the identification $S^3 = SU(2)$ please find two local sections $S^2 \to S^3$ that give $S^3 \to S^2 = SU(2)/U(1)$ structure of a PFB.
4. Let U_a be an open cover of M, σ_a be the local sections and $h_{ab} : U_a \cap U_b \to G$ transition functions. Please find all conditions that the collection of transition functions satisfy.

Problem 2:

Let R_g, L_g and ad_g be, respectively, left, right and adjoint translations on a Lie group G. Please find a basis of right, left and ad-invariant one forms o $S^1, \mathbb{R}^n, SU(2)$.

Problem 3: Lie algebra

Let G be a Lie group and $\mathcal{L}$ its Lie algebra of left-invariant vector fields.

1. Is the algebra isomorphic to the algebra of right-invariant fields ?
2. If ad_g is the adjoint map - how does the $(ad_g)_*$ act ?

Problem 4: Cartan form

Let f, h be maps $M \to G$. Consider the pull-back form:

$$(f \cdot h)^* \Theta$$

and express it through $f^*\Theta$ and $h^*\Theta$.
Reminder: $\Theta(X) = X$ whenever X is a left-invariant field.

Problem 5: Connection

For a vector field X on M, denote by X_h its lift to horizontal vector field on $P(M, G)$. Taking and equivariant map $\Phi : P \to V$ define:

$$\nabla_X(\Phi) = X_h(\Phi).$$

1. Please show that this definition is consistent and $\nabla_X(\Phi)$ is an equivariant map.
2. Show that ∇_X is a covariant derivative (obeys Leibniz rule).

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Problems set 8 for 20.12.16

Problem 1: Lie algebra

Let G be a Lie group and $\mathcal{L}(G)$ the Lie algebra of left-invariant vector fields. Please,

1. check whether $(R_g)_*(X) \in \mathcal{L}(G)$ for $X \in \mathcal{L}(G)$?
2. check whether $(ad_g)_*(X) \in \mathcal{L}(G)$ for $X \in \mathcal{L}(G)$?
3. check whether $(ad_g)_*(X)$ relates to $\mathcal{L}(ad_g)$

Problem 2: Magnetic monopole connection

Find an example of a connection for the principal fibre $S^3 \to S^2$.

Problem 3: Covariant derivative

Let ω be a connection one-form on a principal fibre bundle and ∇ be covariant derivative on an associated vector bundle. Could one express ∇ using ω ?

Problem 4: Trivial bundles

Let $P(M,G)$ be a trivial bundle. Please find explicit formulae for the connection, curvarture, their pull-backs, covariant derivative and its curvature.