Spectral Action and Noncommutative Geometry

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11.09.2013, Kraków

1. THE ACTION.
   - Why ?
2. NONCOMMUTATIVE GEOMETRY: THE SETUP
   - Geometry
   - Spectral triples
   - The action: going spectral
   - Calculating the spectral action
   - The spectral action.
3. SPECTRAL ACTION ON NONCOMMUTATIVE SPACES
   - Noncommutative Tori
   - The quantum group $SU_q(2)$
   - Any use in the real world ?
4. CONCLUSIONS

WHAT IS THE ACTION?

L'Action est proportionnelle au produit de la masse par la vitesse et par l'espace. Maintenant, voici ce principe, si sage, si digne de l'Etre supreme : lorsqu'il arrive quelque changement dans la Nature, la quantité d'Action employée pour ce changement est toujours la plus petite qu'il soit possible.

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- *The action is a function (a functional) on the space of physical parameters*
- *The extremum of the action principle allows to derive the equations of motion.*
- *It helps to describe most of physical processes - in particular (we believe) it describes all fundamental physics.*

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- Can we calculate the spectral action ?
- What can we say about the result ?
- Does the result make sense for physics ?

Geometry *a la Connes*...

Define what is the geometry

The main ingredients are an algebra $\mathcal{A}$, its faithful representation π on a Hilbert space $\mathcal{H}$, a selfadjoint operator $\mathcal{D}$, which satisfies several conditions.

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HOW DOES IT RELATE TO THE CLASSICAL ONE?

If $\mathcal{A} = C^\infty(M)$, M a spin Riemannian compact manifold, $\mathcal{H} = L^2(S)$ (sections of spinor bundle) and $\mathcal{D}$ the Dirac operator on M then to $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a model for such geometry.

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WHAT DO WE GAIN ?

The notion of geometry is *encoded* in the Dirac operator, so instead of talking about different fields we talk about one (geometric object), and we can extend the notion of geometry to some modifications of space-time.

DEFINE: A SPECTRAL TRIPLE

Algebra $\mathcal{A}$, its faithful representation π on a Hilbert space $\mathcal{H}$, a selfadjoint operator D, satisfying several conditions:

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5. ...+ ,, regularity" type conditions

THEOREM

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RECONSTRUCTION THEOREM (CONNES, GRACIA-BONDIA, RENNIE, VARILLY)

If $\mathcal{A}$, (commutative algebra), $(\mathcal{A}, \mathcal{H}, D)$ is a spectral triple satisfying all conditions then M is a spin manifold and $\mathcal{A} = C^\infty(M)$, $\mathcal{H} = L^2(S)$, and D is the Dirac operator on M.

EXAMPLES OF GENUINE NONCOMMUTATIVE MANIFOLDS

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- Quantum spheres.
  Iochum-Levy-A.S. *Spectral action on SUq(2), Commun.Math.Phys. 289,107 –155 (2009)*

GOING SPECTRAL

THE SPECTRAL PROPERTIES OF THE DIRAC OPERATOR

It is known that the spectrum of the Dirac is discrete (separate eigenvalues), with finite multiplicities and no point of convergence apart form ∞.
Roughly speaking - spectrum of Dirac squared is like the spectrum of Laplace operator. The properties do not change if we modify the operator (change the metric, add connections, torsion etc etc.)

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THE ACTION FUNCTIONAL (1)

Let us suppose that the action is a functional of the Dirac and counts just the number of possible physical states: it counts the number of eigenvalues of $\mathcal{D}$:

$$S(\mathcal{D}, \Lambda) = \#\{|\lambda| < \Lambda, Dv = \lambda v\}$$

Going spectral (2)

The exotic traces

Although the usual trace does not extend to the operators like Dirac (or its powers) there are some *exotic traces* which might be interpreted as regularized traces (something of the form of ζ-function regularization.

GOING SPECTRAL (2)

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THE ACTION FUNCTIONAL (2)

Using this exotic traces we can postulate the action to be (for example)

$$S(\mathcal{D}, \Lambda) = \sum_n \Lambda^n \operatorname{Tr} \mathcal{D}^{-n},$$

where Tr is that exotic trace. For most n that would be 0 but some terms would be nonzero.

THE IDEA AND THE REALITY

THE IDEA IS SOMEHOW OLD...

The idea goes back even to A. D. Sakharov (Spectral density of eigenvalues of the wave equation and vacuum polarization, Teor. Mat. Fiz. 23 (1975) 178)
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HOW IS IT?

The calculation is tedious but straightforward.

THE TOOLS

THE HEAT KERNEL

Instead of looking at $f(\mathcal{D})$ (where f might be a cut-off function) let us look at:

$$S_{HK}(\mathcal{D}, \Lambda) = \operatorname{tr} e^{-\frac{\mathcal{D}^2}{\Lambda^2}}.$$

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WHY IS IT HEAT OPERATOR?

$$\Psi(t) = e^{-t\mathcal{D}^2}\Psi_0 \quad \Longrightarrow \quad (\partial_t + \mathcal{D}^2)\Psi(t) = 0.$$

So we have the operator solving the heat equation. The question is, how does its trace vary with $t = \frac{1}{\Lambda^2}$? especially at $t \to 0$ ($\Lambda \to \infty$).

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THE RESULT:

Known in mathematics - *heat kernel asymptotics* for operators of Laplace type (so the square of the Dirac is the one included).

HEAT-KERNEL ASYMPTOTICS

THEOREM (GILKEY)

There exists a unique connection on E, ∇, and a unique endomorphism S, such that:

$$L = -g^{ab}\nabla_a\nabla_b - Z,$$

Heat-kernel asymptotics

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Theorem (Gilkey)

$$\mathrm{Tr}\, e^{-tL} = \sum_{k=0}^{n} \left(\frac{1}{4\pi t}\right)^{\frac{k-n}{2}} \int_M a_{[k]}(x) + o(t).$$

where $a_{[k]}(x)$ are functions on M (De Witt–Seeley–Gilkey).

HEAT-KERNEL ASYMPTOTICS

$$a_{[0]} = \mathrm{rank}(E), \quad a_{[1]} = 0, \quad a_{[2]} = \mathrm{tr}(-\frac{1}{6}R + Z).$$

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REMARK

First term gives nothing else but the so-called Weyl's theorem about the growth of eigenvalues of the Laplace operator.

ACTION FUNCTIONAL

Take the space of all *inner fluctuations* of Dirac operators:

$$\mathrm{D}_{\mathcal{A}} = \{D + A\},$$

where A is a self-adjoint one-form $A = \sum_i a_i [D, b_i]$.
For a fixed function f (cutoff) consider the following functional on the space of $\mathrm{D}_{\mathcal{A}}$:

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INNER FLUCTUATIONS...

Inner fluctuation of inner fluctuation are inner fluctuations!

RIGHT TOOL: ASYMPTOTIC EXPANSION!

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where Sd^+ is the positive part of the dimension spectrum of $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ and $f_k = \frac{1}{2} \int_0^\infty f(t)\, t^{k/2-1}\, dt$.

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THE DIMENSION SPECTRUM

$Sd \subset \mathbb{C}$ is the set of all poles of the function: $s \mapsto \mathrm{Tr}(P|D|^{-s})$ where P is any operator in the algebra generated by A,$[D, A]$,$[|D|, A]$.

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ZETA FUNCTION

$\zeta'_X(s) = \mathrm{Tr}(|X|^{-s}) = \zeta_X(s) - \mathrm{Tr}(p_X)$ where p_X is the projection on the kernel of X.

The noncommutative integral

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- There are no fluctuactions for real spectral triples.
- Spectral action gives pure gravity:

$$S(D_g) = \frac{f_0}{4\pi^2}\Lambda^4 \int \sqrt{g}\, d^4x + \frac{f_2}{48\pi^2}\Lambda^2 \int R\sqrt{g}\, d^4x$$
$$+ \frac{f_4}{16\,\pi^2}\frac{1}{360}\int (5R^2 - 8R_{\mu\nu}R^{\mu\nu} - 7R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma})\sqrt{g}\, d^4x + \cdots$$

THEOREM (CHAMSEDDINE-CONNES)

If the spectral triple is regular with simple dimension spectrum and $\mathrm{Tr}\,|D|^{-s}$ *is regular at* $s = 0$ *then the variation of the scale invariant part of the spectral action is:*

$$\zeta'_{\mathcal{D}+A}(0) - \zeta'_{\mathcal{D}}(0) = \sum_{q=1}^{n} \frac{(-1)^q}{q} \fint (A\mathcal{D}^{-1})^q,$$

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IN 4 DIMENSIONS

If there is no tadpole (that is $⨍ AD^{-1} = 0$ for any A) then the scale invariant part is:

$$\zeta'_{\mathcal{D}+A}(0) - \zeta'_{\mathcal{D}}(0) = \frac{1}{4}\int_{\tau_0} (dA + A^2)^2 - \frac{1}{2}\int_{\psi} (AdA + \frac{2}{3}A^3).$$

where τ_0 is a Hochschild cocycle and ψ is cyclic cocycle.

SPECTRAL TRIPLE ON NC TORI (N-DIMENSIONAL)

- Noncommutative Torus $C^\infty(\mathbb{T}^n_\Theta)$ is an algebra, which is generated by unitary operators, which commute up to a scalar phase:

$$U_k \, U_l = e^{2\pi i \theta_{kl}} \, U_l \, U_k,$$

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- Let τ be the trace on $C^\infty(\mathbb{T}^n_\Theta)$ defined by $\tau\big(\sum_{k\in\mathbb{Z}^n} a_k\, U_k\big) := a_0$ and $\mathcal{H}_\tau$ be the GNS Hilbert space obtained by completion of $C^\infty(\mathbb{T}^n_\Theta)$ with respect of the norm induced by the scalar product $\langle a, b\rangle := \tau(a^* b)$. On $\mathcal{H}_\tau = \{\sum_{k\in\mathbb{Z}^n} a_k\, U_k \;:\; \{a_k\}_k \in l^2(\mathbb{Z}^n)\}$, we consider the left and right regular representations of $C^\infty(\mathbb{T}^n_\Theta)$ by bounded operators, that we denote respectively by $L(.)$ and $R(.)$.

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- Let δ_μ, $\mu \in \{1, \ldots, n\}$, be the n (pairwise commuting) canonical derivations, defined by

$$\delta_\mu(U_k) := i k_\mu U_k.$$

Spectral triple on NC Tori (n-dimensional)

- Let $\mathcal{A}_\Theta := C^\infty(\mathbb{T}^n_\Theta)$ acting on $\mathcal{H} := \mathcal{H}_\tau \otimes \mathbb{C}^{2^m}$ with $n = 2m$ or $n = 2m+1$,
  Each element of $\mathcal{A}_\Theta$ is represented on $\mathcal{H}$ as $L(a) \otimes 1_{2^m}$ where L (resp. R) is the left (resp. right) multiplication.

SPECTRAL TRIPLE ON NC TORI (N-DIMENSIONAL)

- Let $\mathcal{A}_\Theta := C^\infty(\mathbb{T}^n_\Theta)$ acting on $\mathcal{H} := \mathcal{H}_\tau \otimes \mathbb{C}^{2^m}$ with $n = 2m$ or $n = 2m + 1$,
  Each element of $\mathcal{A}_\Theta$ is represented on $\mathcal{H}$ as $L(a) \otimes 1_{2^m}$ where L (resp. R) is the left (resp. right) multiplication.
- The Tomita conjugation $J_0(a) := a^*$ satisfies $[J_0, \delta_\mu] = 0$ and we define $J := J_0 \otimes C_0$ where C_0 is an operator on $\mathbb{C}^{2^m}$.

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- The Tomita conjugation $J_0(a) := a^*$ satisfies $[J_0, \delta_\mu] = 0$ and we define $J := J_0 \otimes C_0$ where C_0 is an operator on $\mathbb{C}^{2^m}$.
- The Dirac operator is given by

$$\mathcal{D} := -i\,\delta_\mu \otimes \gamma^\mu,$$

Lemma

LEMMA

- *if* $\frac{1}{2\pi}\Theta$ *is diophantine then the dimension of the NC-Torus is dimension* $\{n - \mathbb{N}\}$ *and all poles are simple*

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THEOREM

Consider the n-NC-torus $\big(C^\infty(\mathbb{T}^n_\Theta), \mathcal{H}, \mathcal{D}\big)$ *where* $n \in \mathbb{N}$ *and* $\frac{1}{2\pi}\Theta$ *is a real skew-symmetric real diphantine matrix. For any selfadjoint one-form* $A = L(-iA_\alpha) \otimes \gamma^\alpha$*, the full spectral action of* $\mathcal{D}_A = \mathcal{D} + A + \epsilon JAJ^{-1}$ *is*
(i) for $n = 2$*,*

$$\mathcal{S}(\mathcal{D}_A, f, \Lambda) \; = 4\pi \, f_2 \, \Lambda^2 + \mathcal{O}(\Lambda^{-2}),$$

(II) FOR $n = 4$

$$\mathcal{S}(\mathcal{D}_A, f, \Lambda) = 8\pi^2 f_4 \Lambda^4 - \frac{4\pi^2}{3} f(0)\, \tau(F_{\mu\nu} F^{\mu\nu}) + \mathcal{O}(\Lambda^{-2}).$$

where $F_{\mu\nu} = \delta_\mu(A\nu) - \delta_\nu(A\mu)$.

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- V. Gayral, B. Iochum and D. Vassilevich, “Heat kernel and number theory on NC-torus”, CMP., Commun.Math.Phys. **273** (2007), 415–443.
- D. Essouabri, B. Iochum, C. Levy, A.S. “Spectral action on noncommutative torus” J.Noncommut.Geom. **2** (2008), 53–123.

Idea of the proof...

Examples (4d)

$$\int\!\!\!\!\!\!- \, A^q \mathcal{D}^{-q} = \delta_{q,4}\, \frac{\pi^2}{12} \tau\left(A_{\alpha_1} \cdots A_{\alpha_4}\right) \mathrm{Tr}(\gamma^{\alpha_1} \cdots \gamma^{\alpha_4} \gamma^{\mu_1} \cdots \gamma^{\mu_4}) \delta_{\mu_1,\ldots,\mu_4} \,.$$

where

$$\delta_{\mu_1,\ldots,\mu_4} := \delta_{\mu_1\mu_2}\delta_{\mu_3\mu_4} + \delta_{\mu_1\mu_3}\delta_{\mu_2\mu_4} + \delta_{\mu_1\mu_4}\delta_{\mu_2\mu_3}$$

We calculate for

$$A = -iA_\alpha \otimes \gamma^\alpha = -i \sum_{l \in \mathbb{Z}^n} a_{\alpha,l}\, U_l \otimes \gamma^\alpha$$

Idea of the proof...

Example (4D)

$$\int\!\!\!\!\!\!- A^n \mathcal{D}^{-n} = \operatorname*{Res}_{s=0} (-i)^n \Big({\sum_{k \in \mathbb{Z}^n}}' \frac{k_{\mu_1} \ldots k_{\mu_n}}{|k|^{s+2n}} \Big)$$

$$\tau\Big(\sum_{l \in (\mathbb{Z}^n)^n} \widetilde{a}_{\alpha,l} U_{l_1} \cdots U_{l_n} \Big) \; \mathrm{Tr}(\gamma^{\alpha_1} \ldots \gamma^{\alpha_n} \gamma^{\mu_1} \ldots \gamma^{\mu_n})$$

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Lemma

If $\frac{1}{2\pi}\Theta$ is diophantine, the function $s \mapsto \operatorname{Tr}\left(\varepsilon J A J^{-1} A |D|^{-s}\right)$ extends meromorphically on the whole plane with only one possible pole at $s = n$. Moreover, this pole is simple and

$$\operatorname*{Res}_{s=n} \operatorname{Tr}\left(\varepsilon J A J^{-1} A |D|^{-s}\right) = a_{\alpha,0}\, a_0^{\alpha}\, 2^{m+1} \pi^{n/2}\, \Gamma(n/2)^{-1}.$$

PROOF.

With $A = L(-iA_\alpha) \otimes \gamma^\alpha$, we get $\epsilon JAJ^{-1} = R(iA_\alpha) \otimes \gamma^\alpha$, and by multiplication $\varepsilon JAJ^{-1}A = R(A_\beta)L(A_\alpha) \otimes \gamma^\beta\gamma^\alpha$. Thus,

$$\begin{aligned}\mathrm{Tr}\left(\varepsilon JAJ^{-1}A|D|^{-s}\right) &\sim_c \sum_{k\in\mathbb{Z}^n}{}' \langle U_k, A_\alpha U_k A_\beta\rangle\, |k|^{-s}\, \mathrm{Tr}(\gamma^\beta\gamma^\alpha) \\ &\sim_c \sum_{k\in\mathbb{Z}^n}{}' \sum_l a_{\alpha,l}\, a_{\beta,-l}\, e^{ik.\Theta l}\, |k|^{-s}\, \mathrm{Tr}(\gamma^\beta\gamma^\alpha) \\ &\sim_c 2^m \sum_{k\in\mathbb{Z}^n}{}' \sum_l a_{\alpha,l}\, a^\alpha_{-l}\, e^{ik.\Theta l}\, |k|^{-s}.\end{aligned}$$

Theorem (2.5 [EILS]) (*ii*) entails that $\sum'_{k\in\mathbb{Z}^n} \sum_l a_{\alpha,l}\, a^\alpha_{-l}\, e^{ik.\Theta l}\, |k|^{-s}$ extends meromorphically on the whole plane $\mathbb{C}$ with only one possible pole at $s = n$:

$$\operatorname*{Res}_{s=n} \sum_{k\in\mathbb{Z}^n}{}' \sum_l a_{\alpha,l}\, a^\alpha_{-l}\, e^{ik.\Theta l}\, |k|^{-s} = a_{\alpha,0}\, a^\alpha_0 \operatorname*{Res}_{s=n} Z_n(s).$$

THE QUANTUM 3D SPHERE.

THE ALGEBRA

$\mathcal{A}(SU_q(2))$ is the *-algebra generated by a and b, subject to relations $(0 < q < 1)$:

$$ba = qab, \qquad b^* a = q\, ab^*, \qquad bb^* = b^* b,$$
$$a^* a + q^2\, b^* b = 1, \qquad aa^* + bb^* = 1,$$

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REMARK

The algebra is one of the most studied: representations, K-theory, cyclic cohomology, twisted cyclic cohomology, covariant differential calculi, bicovariant differential calculi, quantum Hopf fibration...

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REFERENCE:

B.Iochum, C.Levy, A.S. *Spectral action on $SU_q(2)$*

THE $SU_q(2)$ SPECTRAL TRIPLE...

HILBERT SPACE...

We take the Hilbert space with orthonormal basis

$$|j\mu n\rangle\!\rangle := \begin{pmatrix} |j\mu n\!\uparrow\rangle \\ |j\mu n\!\downarrow\rangle \end{pmatrix}$$

with the convention that the lower component is zero when $n = \pm(j + \frac{1}{2})$ or $j = 0$.

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...AND REPRESENTATION

We set the approximate representation, that is:

$$\underline{\pi}(x) - \pi(x) \in \mathcal{K}, \quad x \in \mathcal{A}(SU_q(2)),$$
$$\underline{\pi}(a) := a_+ + a_-, \quad \underline{\pi}(b) := b_+ + b_-$$

...WITH THE FOLLOWING DEFINITIONS:

$$a_+ \,|j\mu n\rangle\rangle := q_{j^+ + \mu^+} \begin{pmatrix} q_{j^+ + n^+ + 1} & 0 \\ 0 & q_{j^+ + n} \end{pmatrix} |j^+ \mu^+ n^+\rangle\rangle,$$

$$a_- \,|j\mu n\rangle\rangle := q^{2j+\mu+n+\frac{1}{2}} \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix} |j^- \mu^+ n^+\rangle\rangle,$$

$$b_+ \,|j\mu n\rangle\rangle := q^{j+n-\frac{1}{2}} q_{j^+ + \mu^+} \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix} |j^+ \mu^+ n^-\rangle\rangle,$$

$$b_- \,|j\mu n\rangle\rangle := -q^{j+\mu} \begin{pmatrix} q_{j^+ + n} & 0 \\ 0 & q_{j^- + n} \end{pmatrix} |j^- \mu^+ n^-\rangle\rangle.$$

where $x^\pm = x \pm \frac{1}{2}$ and $q_n = \sqrt{1 - q^{2n}}$.

THE DIRAC OPERATOR AND REALITY

Is the same as in the classical case of a 3-sphere with the round metric, so with the same spectrum and multiplicities:

$$\mathcal{D}\,|j\mu n\rangle\!\rangle := \begin{pmatrix} 2j+\frac{3}{2} & 0 \\ 0 & -2j-\frac{1}{2} \end{pmatrix} |j\mu n\rangle\!\rangle,$$

$$J|j\mu n\rangle\!\rangle := \begin{pmatrix} i^{2(2j+\mu+n)} & 0 \\ 0 & i^{2(2j-\mu-n)} \end{pmatrix} |j(-\mu)(-n)\rangle\!\rangle$$

REMARK

This spectral triple is a unique spectral geometry equivariant under left and right actions of $\mathcal{U}_q(su(2))$ and satisfying most of the conditions of NCG.

THE SPECTRUM AND NC INTEGRAL...

LEMMA

The dimension spectrum of $\mathcal{A}(SU_q(2)), \mathcal{H}, D$ is $\{1,2,3\}$ and for any operator T from the algebra generated by $\mathcal{A}, \delta^n(\mathcal{A})$ we have:

$$\int\!\!\!\!\!- T|\mathcal{D}|^{-3} = 2(\tau_1 \otimes \tau_1)\big(r(T)^\circ\big),$$

$$\int\!\!\!\!\!- T|\mathcal{D}|^{-2} = 2\big(\tau_1 \otimes \tau_0 + \tau_0 \otimes \tau_1\big)\big(r(T)^\circ\big),$$

$$\int\!\!\!\!\!- T|\mathcal{D}|^{-1} = \big(2\,\tau_0 \otimes \tau_0 - \tfrac{1}{2}\,\tau_1 \otimes \tau_1\big)\big(r(T)^\circ\big).$$

$$\int\!\!\!\!\!- FT|\mathcal{D}|^{-3} = F \int\!\!\!\!\!- T|\mathcal{D}|^{-2} \qquad = 0$$

$$\int\!\!\!\!\!- FT|\mathcal{D}|^{-1} = \big(\tau_0 \otimes \tau_1 - \tfrac{1}{2}\,\tau_1 \otimes \tau_0\big)\big(r(T)^\circ\big).$$

$$r(a_+) := \pi_+(a) \otimes \pi_-(a), \qquad r(a_-) := -q\, \pi_+(b) \otimes \pi_-(b^*),$$
$$r(b_+) := -\pi_+(a) \otimes \pi_-(b), \qquad r(b_-) := -\pi_+(b) \otimes \pi_-(a^*).$$

$$\pi_\pm(a)\, \varepsilon_n := q_{n+1}\, \varepsilon_{n+1},$$
$$\pi_\pm(b)\, \varepsilon_n := \pm q^n\, \varepsilon_n,$$

$$q_n := \sqrt{1 - q^{2n}}$$

$$\sigma\big(\pi_\pm(a)\big)(z) := z,$$
$$\sigma\big(\pi_\pm(a^*)\big)(z) := \bar{z},$$
$$\sigma\big(\pi_\pm(b)\big)(z) = \sigma\big(\pi_\pm(b^*)\big)(z) := 0.$$

$$\tau_0(x) := \lim_{N\to\infty} \big(\operatorname{Tr}_N x - (N+1)\tau_1(x)\big),$$
$$\tau_1(x) := \tfrac{1}{2\pi} \int_0^{2\pi} \sigma(x)(e^{i\theta})\, d\theta,$$

SCALE-INVIARIANT TERMS

$$\phi_n(a_0, \cdots, a_n) := \fint a_0[\mathcal{D}, a_1]\mathcal{D}^{-1} \cdots [\mathcal{D}, a_n]\mathcal{D}^{-1}.$$

WE HAVE:

- $b\phi_1 = -\phi_2, \quad b\phi_2 = 0, \quad b\phi_3 = 0$
- $B\phi_1 = 0, \quad B_0\phi_2 = -(1-\lambda)\phi_1.$
- $bB_0\phi_2 = 2\phi_2 + B_0\phi_3$
- $B\phi_2 = 0.$
- $B_0\phi_3 = Nb'\phi_1, \quad B\phi_3 = 3B_0\phi_3.$

SCALE-INVIARIANT TERMS

LEMMA

$$S_{inv}(A) = -⨍ AD^{-1} + \frac{1}{2} ⨍ AD^{-1}AD^{-1} - \frac{1}{3} ⨍ AD^{-1}AD^{-1}AD^{-1}$$
$$= -\frac{1}{2}\int_{N\phi_1} A + \frac{1}{2}\int_{\phi_2}(dA + A^2) - \frac{1}{2}\int_{\phi_3}(AdA + \frac{2}{3}A^3).$$

REMARK

Then in the case of vanishing tadpole $\phi_1 \equiv 0$ only the Chern-Simons term remains!

TADPOLE AND OTHER BEASTS...

Q & A:

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Does the tadpole vanish?

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TADPOLE AND OTHER BEASTS...

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Can we still calculate something? Yes!

WHAT IS THE SOLUTION?

We need to look at the differential forms at each dimension separately! That is, we look at one forms that contribute to the tadpole, that contribute to the term in dimension two and that contribute to the Chern-Simons term!

Differential calculus...

Consider all operators for which the following condition holds:

The ideal

$$\mathcal{R} := \left\{ T \in \Psi_0(\mathcal{A}) : ⨍ tT|D|^{-3} = ⨍ tT|D|^{-2} = 0, \ \forall t \in \Psi_0(\mathcal{A}) \right\}$$

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Lemma

$\mathcal{R}$ is an ideal within $\Psi_0(\mathcal{A})$.

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Lemma

For any $t \in \Psi_0(\mathcal{A})$ and $T \in \mathcal{R}$, we have

$$\fint t\,T\,|D|^{-1} = \fint T\,t\,|D|^{-1}.$$

DIFFERENTIAL CALCULUS...

The following rules of the first-order differential calculus hold, up to the ideal $\mathcal{R}$ and forgetting $\underline{\pi}$

$$
\begin{array}{lll}
a\,da \simeq da\,a, & a^*\,da \simeq -da^*\,a, & b\,da \simeq q\,da\,b, \\
a\,da^* \simeq -da\,a^*, & a^*\,da^* \simeq da^*\,a^*, & b\,da^* \simeq q^{-1}\,da^*\,b, \\
a\,db \simeq q^{-1}\,db\,a, & a^*\,db \simeq q\,db\,a^*, & b\,db \simeq db\,b, \\
a\,db^* \simeq q^{-1}\,db^*\,a, & a^*\,db^* \simeq q\,a^*\,db^*, & b\,db^* \simeq db^*\,b,
\end{array}
$$

and

$$
a^*\,da - q^2 da\,a^* \simeq (1-q^2)\,F, \qquad q^2\,a\,da^* - da^*\,a \simeq (1-q^2)\,F.
$$

Moreover, the following identity holds, up to the ideal of trace-class operators

$$
a^*\,da + q^2\,b\,db^* + q^2\,a\,da^* + q^2\,b^*\,db = (1-q^2)\,F.
$$

HOW DOES IT WORK?

THE CHERN-SIMONS TERM

Take the one form:

$$A = x_0 F + x_a da + da^* x_{a^*} + x_b db + db^* x_{b^*},$$

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LEMMA

$$\fint A dA\, D^{-3} = \int \sigma(x + x_a a + a^*(x_a)^*)^2,$$

$$\fint A^3\, D^{-3} = \int \sigma(x + x_a a + a^*(x_a)^*)^3.$$

OTHER TERMS OF THE SPECTRAL ACTION

TERMS FROM Λ-EXPANSION

In the full spectral action with the reality operator, of $SU_q(2)$ for a one-form $\mathbb{A}$ and A its associated δ-one-form, the coefficients are:

$$\fint |D_{\mathbb{A}}|^{-3} = 2\,,$$

$$\fint |D_{\mathbb{A}}|^{-2} = -4 \fint A|\mathcal{D}|^{-3},$$

$$\fint |D_{\mathbb{A}}|^{-1} = -\tfrac{1}{2} + 2\Big(\fint A^2|\mathcal{D}|^{-3} - \fint A|\mathcal{D}|^{-2}\Big) + \Big|\fint A|\mathcal{D}|^{-3}\Big|^2,$$

$$\zeta_{D_{\mathbb{A}}}(0) = -2\fint A|\mathcal{D}|^{-1} + \fint A^2|\mathcal{D}|^{-2} - \tfrac{2}{3}\fint A^3|\mathcal{D}|^{-3}$$
$$+ \overline{\fint A|\mathcal{D}|^{-3}}\Big(\tfrac{1}{2}\fint A|\mathcal{D}|^{-2} - \fint A^2|\mathcal{D}|^{-3}\Big) + \tfrac{1}{2}\fint A|\mathcal{D}|^{-3}\,\overline{\fint A|\mathcal{D}|^{-2}}\,.$$

OTHER TERMS OF THE SPECTRAL ACTION

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LEMMA

$$\fint |D_A|^{-2} = \fint |\mathcal{D}|^{-2} = 0,$$

unless $A = \alpha(a\delta(a^*) - \delta(a^*)a)$ *when*

$$\fint |D_A|^{-2} = 8\alpha.$$

$\mathbb{A}$	$\fint A\,\lvert\mathcal{D}\rvert^{-3}$	$\fint A^2\,\lvert\mathcal{D}\rvert^{-3}$	$\fint A^3\,\lvert\mathcal{D}\rvert^{-3}$	$\fint A\,\lvert\mathcal{D}\rvert^{-2}$	$\fint A^2\,\lvert\mathcal{D}\rvert^{-2}$	$\fint A\,\lvert\mathcal{D}\rvert^{-1}$
$a^* da$	2	2	2	$\frac{4q^2}{q^2-1}$	$\frac{4q^2(q^2+2)}{q^4-1}$	$\frac{3q^2+1}{2(q^2-1)}$
$b^* db$	0	0	0	0	$\frac{-4}{q^4-1}$	$\frac{-2}{q^2-1}$
ada^*	-2	2	-2	$\frac{-4}{q^2-1}$	$\frac{4(2q^2+1)}{q^4-1}$	$\frac{q^2+3}{2(q^2-1)}$
bdb^*	0	0	0	0	$\frac{-4}{q^4-1}$	$\frac{-2}{q^2-1}$

THE SPECTRAL ACTION

ANY USE FOR PHYSICS?

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$$S(D) = \mathrm{Tr}\, f(D^2),$$

where f is a a cut-off function. For the geometries of the type $M \times F$, where M is a Riemannian manifold and F is a discrete geometry we obtain, for the bosonic part:

$$\begin{aligned} S &= \frac{1}{\pi^2}(48\, f_4\, \Lambda^4 - f_2\, \Lambda^2\, c + \frac{f_0}{4}\, d) \int \sqrt{g}\, d^4 x \\ &+ \frac{96\, f_2\, \Lambda^2 - f_0\, c}{24\pi^2} \int R \sqrt{g}\, d^4 x \\ &+ \frac{f_0}{10\, \pi^2} \int (\frac{11}{6}\, R^* R^* - 3\, C_{\mu\nu\rho\sigma}\, C^{\mu\nu\rho\sigma}) \sqrt{g}\, d^4 x + \cdots \end{aligned}$$

SPECTRAL ACTION:

$$
\begin{aligned}
\cdots \quad & + \quad \frac{(-2\,a\,f_2\,\Lambda^2 \;+\; e\,f_0)}{\pi^2} \int |\varphi|^2 \sqrt{g}\, d^4x \\
& + \quad \frac{f_0}{2\,\pi^2} \int a\,|D_\mu \varphi|^2 \sqrt{g}\, d^4x \\
& - \quad \frac{f_0}{12\,\pi^2} \int a\,R\,|\varphi|^2 \sqrt{g}\, d^4x \\
& + \quad \frac{f_0}{2\,\pi^2} \int (g_3^2\, G^i_{\mu\nu}\, G^{\mu\nu i} + g_2^2\, F^\alpha_{\mu\nu}\, F^{\mu\nu\alpha} + \frac{5}{3}\, g_1^2\, B_{\mu\nu}\, B^{\mu\nu}) \sqrt{g}\, d^4x \\
& + \quad \frac{f_0}{2\,\pi^2} \int b\,|\varphi|^4 \sqrt{g}\, d^4x
\end{aligned}
$$

CONCLUSIONS

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- NC tori: a very good example with computational difficulty!
- NC tori: Spectral action is predictable.
- Quantum 3-sphere: interesting nontrivial example.
- Physics: Standard Model is described by the Spectral Action ?
- Nontrivial perturbations of the Dirac operator ?
- Can one derive the equations of motion ? for NC geometries ?