POINTLESS GEOMETRY

Andrzej Sitarz

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Geometry and physics

A marriage made in heaven

- early 20th century: gravitation and electromagnetism formulated as field theories in four-dimensional space-time,
- early 20th century: quantum physics formulated using the analysis of linear operators and spectral theory
- 1955: Yang-Mills theory - geometry (vector bundles, connections, curvatures, covariant derivatives and spinors)

Atiyah, Dijkgraaf, Hitchin

Recent ideas from quantum gravity and string theory challenge the fundamental concepts of geometry at an even deeper level. Physical intuition tells us that the traditional pseudo-Riemannian geometry of space-time cannot be a definite description of physical reality.

Geometry and physics

The future:

The idea is that the classical laws of gravity (and classical field theories) only appear in some limit, very similar to the emergence of the macroscopic laws of thermodynamics out of the microscopic description of statistical mechanics. The definite mathematical formulation of such a concept of "quantum geometry" is still far away.

The quantum geometry

There are numerous ways and approaches to quantum geometry . Even the phrase noncommutative geometry has several interpretations.

The question ?

Is there a way to describe geometry using the methods, which do not use the existence of points ?

Geometry without points

How do we see a point in a space?

We don't. We see and „measure" functions on the space.

Example

If X is a (locally) compact Hausdorff space and $C(X)$ is the algebra of continuous functions on X, then $C(X)$ is a commutative (non) unital C^*-algebra.

What is a C^*-algebra?

Let $\mathcal{A}$ be an involutive Banach algebra (that is a complex normed algebra, which is complete as a topological space in the norm). If:

$$||aa^*|| = ||a||^2,$$

then $\mathcal{A}$ is a C^*-algebra.

Gelfand-Naimark Theorems

Theorem (Gelfand-Naimark (GNS))

Every abstract C^*-algebra $\mathcal{A}$ is isometrically $*$-isomorphic to a concrete C^* algebra of operators on a Hilbert space $\mathcal{H}$. If the algebra $\mathcal{A}$ is separable then we can take $\mathcal{H}$ to be separable.

Theorem (Gelfand-Naimark (GN))

If a C^* algebra is commutative then the Gelfand transform, $\mu : \mathcal{A} \to C_0(X_{\mathcal{A}})$, where $X_{\mathcal{A}}$ is the Gelfand spectrum of $\mathcal{A}$ (space of homomorphisms from $\mathcal{A}$ to $\mathbb{C}$):

$$\mu(a)(\chi) = \chi(a), \quad a \in \mathcal{A}, \chi \in X_{\mathcal{A}},$$

is an isometric $*$-isomorphism.

What is noncommutative topology

TOPOLOGY	vs ALGEBRA
(locally compact) topological space	C^*-algebra
homeomorphism	automorphism
continuous proper map	morphism
compact space	unital C^*-algebra
open (dense) subset	(essential) ideal
compactification	unitization
Stone-Čech compactification	multiplier algebra
cartesian product	tensor product

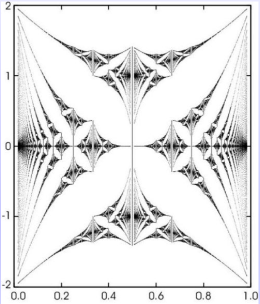
Is that enough?

There is much more to geometry than topology
There are groups, vector bundles, connections and metric.

The final dictionary

GEOMETRY	ALGEBRA
vector bundle	finitely generated projective module
differential forms	differential forms
differential forms	Hochschild homology
de Rham cohomology	cyclic cohomology
vector fields	operators
group	Hopf algebra
Lie algebra	Hopf algebra
principal fibre bundle	Hopf-Galois extension
measurable functions	von Neumann algebra
infinitesimals	compact operators
metric	Dirac operator
spin^c geometry	spectral triple
spin geometry	real spectral triple
integrals	exotic traces

The image of Noncommutative Geometry



Is there a gain for physics ?

YES! We have a much broader interpretation of what geometry is and we can look at what is observed from a completely different point of view.

The Dirac operator

Forget the usual construction

Riemannian manifold (compact, closed) with a metric g. Find Clifford algebra bundle, identify the spinor bundle, then lift the metric connection (the Levi-Civita one if you want the true Dirac), lift it to the spinor bundle, compose with Clifford map – you get D. Then prove some theorems about D.

Follow the operational definition

Take an algebra represented on a Hilbert space and look for operators, which behave like Dirac operators: first order differential operator such that gives you the metric:

$$d(x, y) = \sup_{||[D,f] \leq 1} |f(x) - f(y)|,$$

is unbounded, discrete spectrum and the eigenvalues have certain growth.

Geometry and the Hilbert spaces.

The significance of Atiyah-Singer

All classical geometry can be encoded in terms of operators on a separable Hilbert space.

How do we reconstruct geometry ?

1. differential calculus: $da = [D, a]$
2. metric: $d(x, y) = \sup_{|||[D,f]|||\leq 1} |f(x) - f(y)|$
3. additional connection (if spinors twisted by a vector bundle)
4. dimension (growth of eigeinvalues: $N(\Lambda) \sim \Lambda^d$),
5. integral (exotic traces)

$$\mathrm{Tr}(a) = \mathrm{Res}_{z=d}\mathrm{tr}(a|D|^{-z})$$

PHYSICS

WHY IS THAT A DIFFERENT POINT OF VIEW IN PHYSICS ?

- We do not need points and manifolds.
- We look for representations of algebras rather than groups.
- Symmetries are easily generalized in terms of Hopf algebras.
- Metric and gauge fields are unified in an elegant way.
- The theory is not abstract (not formal manipulations).
- The concept is mathematically sound and meaningful.
- Develops on K-theory and K-cohomology, Hochschild and cyclic homology, index theorems, pseudodifferential operators, C*-algebras,

Spectral Geometry

The spectral triple

Algebra $\mathcal{A}$, its faithful representation π on a Hilbert space $\mathcal{H}$, a selfadjoint unbounded operator D, satisfying several conditions:

1. $\forall a \in \mathcal{A}\ [D, \pi(a)] \in B(\mathcal{H})$, D^{-1} is compact
2. even ST: $\exists \gamma \in \mathcal{A}' : \gamma^2 = 1, \gamma = \gamma^\dagger, \gamma D + D\gamma = 0$,
3. $\exists J$, antilinear $J^2 = \pm 1, JJ^\dagger = 1$
   $J\gamma = \pm \gamma J, JD = \pm DJ, [J\pi(a)J, \pi(b)] = 0$,
4. $[[D, a], J\pi(b)J] = 0$ (D: first order differential operator)
5. ...+ conditions of „analysis" type

Theorem

If $\mathcal{A} = C^\infty(M)$, M a spin Riemannian compact manifold, $\mathcal{H} = L^2(S)$ (sections of spinor bundle) and D the Dirac operator on M then to $(\mathcal{A}, \mathcal{H}, D)$ is a spectral triple (with a real structure).

TWISTED SPECTRAL GEOMETRY

DEFINITION (TWISTED REAL SPECTRAL TRIPLE)

Let $\mathcal{A}$ be a $*$-algebra, (H, π) a representation of $\mathcal{A}$, D a linear operator on H, and let ν be a linear automorphism of H. We say that the triple $(\mathcal{A}, H, D)$ admits a ν*-twisted real structure* if there exists an anti-linear map $J : H \to H$ such that $J^2 = \epsilon\,\mathrm{id}$, and, for all $a, b \in \mathcal{A}$,

$$[\pi(a), J\pi(b)J^{-1}] = 0,$$

$$[D, \pi(a)]J\bar{\nu}^2(\pi(b))J^{-1} = J\pi(b)J^{-1}[D, \pi(a)],$$

$$DJ\nu = \epsilon'\nu JD,$$

where $\epsilon, \epsilon' \in \{+, -\}$.

$$\nu J\nu = J,$$

[Brzezinski, Dabrowski, Ciccoli, Sitarz, January 2016]

Twisted Spectral Geometry

Let Ω^1_D be a bimodule of one forms:

$$\Omega^1_D := \{\sum_i \pi(a_i)[D, \pi(b_i)] \,|\, a_i, b_i \in A\}.$$

The standard fluctuaction of a spectral triple (A, H, D) consist of adding to the Dirac operator D a selfadjoint one form $\alpha \in \Omega^1_D$.

In case of a real spectral triple the fluctuated D is $D + \alpha + \epsilon' J \alpha J^{-1}$, where $\alpha + \epsilon' J \alpha J^{-1}$ is selfadjoint.

For our case of ν-twisted real spectral triple we set the fluctuated Dirac operator D_α to be:

$$D_\alpha := D + \alpha + \epsilon' \nu J \alpha J^{-1} \nu,$$

with the requirement that $\alpha + \epsilon' \nu J \alpha J^{-1} \nu$ is selfadjoint.

The tools of spectral geometry

The action: heat trace

Take the Dirac operator D_A (that depends on physical degrees of freedom: gauge fields and metric: A). Compute:

$$\lim_{t\to 0} \mathrm{Tr}\, e^{-t(D_A)^2} = t^{-\frac{d}{2}} a_0(A) + t^{-\frac{d}{2}+1} a_2(A) + \cdots$$

The action: residues

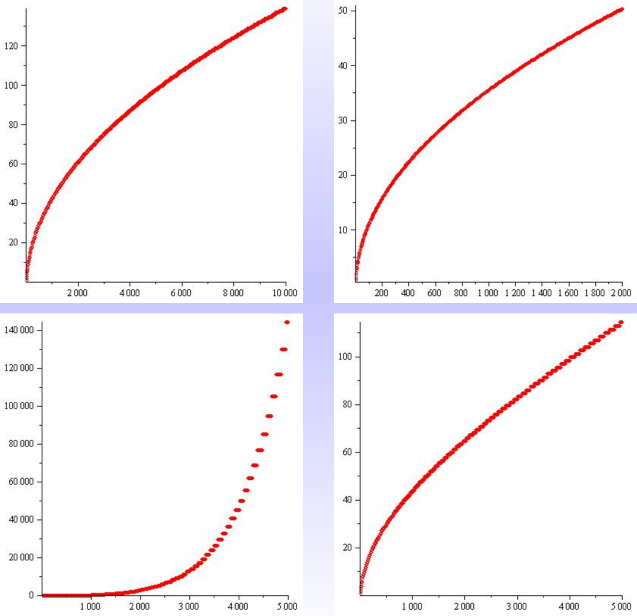
Define for any number s, any operator T from the algebra $\mathcal{A}$:

$$\int_s T := \mathrm{Res}_{z=s} \mathrm{Tr}\, T|D_A|^{-z},$$

Then propose for the action

$$S(D_A) = \Lambda^d \int_d 1 + \Lambda^{d-2} \int_d (D_A)^2 + \Lambda^{d-4} \int_d (D_A)^4 + \cdots$$

THE SPECTRA



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Gravity

The action functional

For the geometries of the type $M \times F$, where M is a Riemannian manifold and F is a discrete geometry we obtain

$$
\begin{aligned}
S \quad = \quad & \frac{1}{\pi^2}(48\, f_4\, \Lambda^4 - f_2\, \Lambda^2\, c + \frac{f_0}{4}\, d) \int \sqrt{g}\, d^4x \\
+ \quad & \frac{96\, f_2\, \Lambda^2 - f_0\, c}{24\pi^2} \int R\, \sqrt{g}\, d^4x \\
+ \quad & \frac{f_0}{10\, \pi^2} \int (\frac{11}{6}\, R^* R^* - 3\, C_{\mu\nu\rho\sigma}\, C^{\mu\nu\rho\sigma})\, \sqrt{g}\, d^4x + \cdots \\
\cdots \quad + \quad & \frac{(-2\, a\, f_2\, \Lambda^2 \; + \; e\, f_0)}{\pi^2} \int |\varphi|^2\, \sqrt{g}\, d^4x + \frac{f_0}{2\, \pi^2} \int a\, |D_\mu \varphi|^2\, \sqrt{g}\, d^4x \\
- \quad & \frac{f_0}{12\, \pi^2} \int a\, R\, |\varphi|^2\, \sqrt{g}\, d^4x + \frac{f_0}{2\, \pi^2} \int b\, |\varphi|^4\, \sqrt{g}\, d^4x \\
+ \quad & \frac{f_0}{2\, \pi^2} \int (g_3^2\, G_{\mu\nu}^i\, G^{\mu\nu i} + g_2^2\, F_{\mu\nu}^\alpha\, F^{\mu\nu\alpha} + \frac{5}{3}\, g_1^2\, B_{\mu\nu}\, B^{\mu\nu})\, \sqrt{g}\, d^4x
\end{aligned}
$$

Gravity

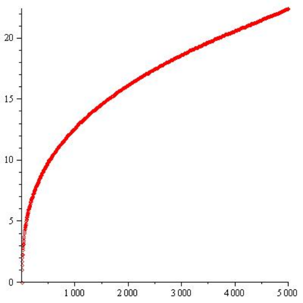
Spectral action for flat geometries

There are 6 different compact flat three-dimensional manifolds (Bieberbach manifolds). With P.Olczykowski we have demonstrated that the *symmetric* spectral action (depending on D^2) does not distinguish which manifold we consider.

The assymetric action and η invariant

However, if the action depends on D (not only on even function of D) then the difference is seen at the level of conformally invariant term – where a η invariant appears.

THE SPECTRA



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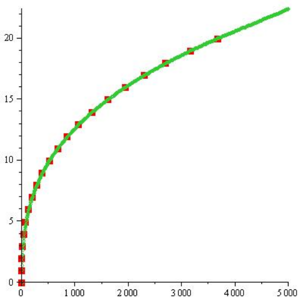

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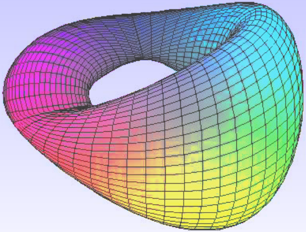


Spectral action...

...and the neutrino mass

An interesting possibility occurs when one considers an extension of the spectral action for fermions, say:

$$S = \mathrm{Tr}\, f\left(\frac{(D+P_\Psi)^2}{\Lambda^2}\right),$$

and then there are some interesting terms, which appear in the order Λ^{-2}.

Wilczek terms

$$\mathcal{L}_m = \kappa\left(\overline{e_L^c}H^+ - \overline{\nu^c}H^0\right)\left(H^+e_L - H^0\nu\right),$$

where e_l, ν is the doublet of left-handed leptons, H^+, H^0 are the Higgs field components and κ is a coefficient (or a matrix if we take into account flavors)

SPECTRAL ACTION

SPECULATE: (A.S: EUROP.PHYS.LETT., 86 (2009) 10007)

The speculation was such terms arise from quantum gravity corrections [Weinberg] gives the neutrino mass of the range of $10^{-5} eV$, which was much less than the experimental estimations. f we take the value of the cutoff parameter $\Lambda = 10^{15} GeV$ and estimate the resulting neutrino mass (taking the coefficient in the term to be of order 1) we obtain the values of order $10^{-2} eV$, which agrees with the current experimental data.

...AND THE NEUTRINO MASS

Suppose we take it seriously - then it means that the nature of neutrino masses is completely different than the masses of fermions. Such terms do not affect the other fermions and the interactions (as these are corrections). However, they do affect the renormalizability of the theory.

Spectral action

...and the torsion:

The usual Dirac operator comes from the torsion-free connection. However, the operators with torsion are in fact almost indistinguishable from the torsion-free operators. In $d = 3$ torsion is nothing but a scalar perturbation.

Lemma (A.Sitarz, A.Zając: Lett.Math.Phys. 98,3, 2011)

The heat kernel coefficients for scalar perturbation of Dirac in dimension $d = 3$ read:

$$[a_1] = 2(4\pi)^{-\frac{3}{2}} \left(-\frac{1}{12}R + 2\phi^2\right), \quad [a_2] = (4\pi)^{-\frac{3}{2}} \frac{8}{3}(\nabla_i \phi)(\nabla^i \phi).$$

A. Sitarz, JPDO, 5,3, pp 305-317 (2014)

In $d = 4$, the noncommutative Wodzicki residue on the noncommutative torus, the action cannot be minimized.

Spectral action

An interesting (though very special) example is that of the Standard Quantum Podleś Sphere M. Eckstein, B. Iochum, A. Sitarz, CMP 332, 627– 668 (2014)

The dimension spectrum

The dimension spectrum of the Podleś sphere is

$$\mathcal{S}_d(S_q^2) = -\mathbb{N} + i\,\frac{2\pi}{\log q}\,\mathbb{Z}$$

and the poles are at most of the second order.

The action

$$\mathrm{Tr}\; e^{-t\,|\mathcal{D}_S|} = \frac{1}{\log^2 q}\left[2\log^2(ut) + h_S\big(\log(ut)\big)\,\log(ut) + c_S\big(\log(ut)\big)\right] + \cdots$$

where $u = \frac{|w|\,q}{1-q^2}$ and $h_S,\, c_S$ are periodic bounded C^∞-functions on $\mathbb{R}$

Spectral action

... and the Gauss-Bonnet for the noncommutative tori

Imagine we take a noncommutative torus ($UV = \lambda VU$) algebra and take the following Dirac operator:

$$D_k = \sigma^1 \delta_1 + \sigma^2 \left(k\, \delta_2 + \tfrac{1}{2} \delta_2(k) \right)$$

Theorem (L.Dabrowski, A.Sitarz, SIGMA 11 (2015), 075)

The dressed scalar curvature for the asymmetric torus is:

$$\widetilde{R} = F_{11}(\Delta^{(1)}, \Delta^{(1)}\Delta^{(2)})(\delta_1(k) \cdot \delta_1(k)) + F_{22}(\Delta^{(1)}, \Delta^{(1)}\Delta^{(2)})(\delta_2(k), \delta_2(k)) + F'_{11}(\Delta^{(1)})(\delta_1(k)^2) + F'_{22}(\Delta^{(1)})(\delta_2(k)^2) + F_1(\Delta^{(1)})(\delta_{11}(k)) + F_2(\Delta^{(1)})(\delta_{22}(k)),$$

$$F_{11}(s,t) = -\frac{2\pi}{3k^3} \frac{(2s^2 + 4st + 4s + 3 + 8t + 3t^2)}{(t+1)^3(s+1)(s+t)}, \quad F_{22}(s,t) = \frac{\pi}{2k} \frac{(t^2 - 6t + 1)}{(t+1)^3},$$

$$F'_{11}(s) = \frac{4\pi}{3k^3} \frac{1}{(s+1)^3}, \quad F'_{22}(s) = -\frac{\pi}{2k} \frac{(s^2 - 6s + 1)}{(s+1)^3},$$

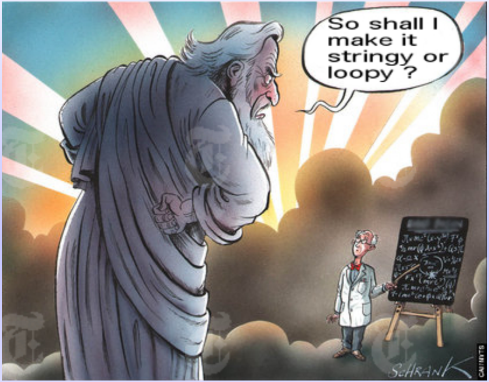
and

$$F_1(s) = \frac{2\pi}{3k^2} \frac{1}{(s+1)^2}, \quad F_2(s) = 0.$$

The trace of R vanishes.

Is it pointless ?

- Geometry is more than what we have learned
- The input from physics matters - and tells us what geometry is
- Physics is not the geometry we imagine it should be
- Physics (as we observe) requires new geometry.
- Reevaluate more possibilities: Einstein -Cartan
- Benefits (or drawbacks): **NO supersymmetry (!)**
- Future: **Geometry as Quantum Field Theory**



THANK YOU