Problems met on a path to 3+1 geometry

Andrzej Sitarz

Institute of Physics
Jagiellonian University
Kraków

Institute of Mathematics,
Polish Academy of Sciences
Warsaw

26.09.2014, Bonn

1. THE PRINCIPLES.
   - What is the geometry ?
   - Is noncommutative geometry *a* geometry ?
2. FIRST EXAMPLE: $4-d$ NC TORUS
   - The functional
   - The computations
3. SOME GENERAL IDEAS
4. CONCLUSIONS

What is the geometry ?

Differential geometry

Differential geometry is quite *rigid* – a *universal tool* – to describe spaces and natural constructions on them: vector bundles, principal bundles, group actions, submanifolds, etc. etc.

Questions

You can pose *meaningful* questions and sometimes even *answer them*.

Classical physics

So far all of known *classical* physics is based on differential geometry.

Quantum physics

So far all of known *quantum* physics is a mixture of *differential geometry* and methods of operator algebras.

Why extend the geometry?

First approach

You *extend* the geometry and look at the consequences: for example, add extra dimensions, change the bundle (gauge group), add extra symmetries, add supersymmetries, replace points with strings or branes...

Second approach

You *look up* the geometry and try to describe it.

Third approach

You *ignore* the world as it is.

Still:

Geometry should be a notion *large enough* to include something **we know** is geometry and extend nicely to some **nontrivial examples** but *rigid enough* so that we know what we are doing.

Spectral triples

Definition: the spectral triple

Algebra $\mathcal{A}$, its faithful representation π on a Hilbert space $\mathcal{H}$, a selfadjoint operator D, satisfying several conditions:

1. $\forall a \in \mathcal{A}\ [D, \pi(a)] \in B(\mathcal{H})$, D^{-1} is compact
2. even ST: $\exists \gamma \in \mathcal{A}' : \gamma^2 = 1, \gamma = \gamma^\dagger, \gamma D + D\gamma = 0,$
3. $\exists\, J$, antilinear $J^2 = \pm 1, JJ^\dagger = 1$
   $J\gamma = \pm\gamma J, JD = \pm DJ, [J\pi(a)J, \pi(b)] = 0,$
4. $[[D, a], J\pi(b)J] = 0$ (D: first order differential operator)
5. ...+ conditions of „analysis" type

Theorem [Connes]

If $\mathcal{A} = C^\infty(M)$, M a spin Riemannian compact manifold, $\mathcal{H} = L^2(S)$ (sections of spinor bundle) and D the Dirac operator on M then to $(\mathcal{A}, \mathcal{H}, D)$ is a spectral triple (with a real structure).

Is this geometry ?

The reconstruction theorem

In a way it tells you that you recover *classical* differential geometry.

The reconstruction theorem: but...

It is *very restrictive* as far as the general rules are concerned but *too flexible* at the same time. There is no *known* way to distinguish geometries with torsion (or even higher-dimensional counterparts), no insight as far as different differential structures are concerned. No good description of *principal fibre bundles* (as differential geometry tools).

More but:

If you *relax* some of the conditions - you can have geometry with your Dirac operator being a pseudodifferential one (not only differential). [Sitarz, J.Phys.A: 42 (2009) 155201].

TOO FEW EXAMPLES:

- The Noncommutative Torus: $UV = e^{2\pi i\theta} VU$
  Long live **flat** Dirac operator !
- Finite-dimensional algebras ($M_n(\mathbb{C}) \oplus M_k(\mathbb{C}) \oplus \cdots$
  Matrices are cool !
- Quantum groups and spaces (q-deformations)
  Neither quantum nor groups !
- Moyal deformation $[x^\mu, x^\nu] = \theta^{\mu\nu}$
  Back to Quantum Mechanics ?
- κ-deformation $[x^0, x^i] = \frac{1}{\kappa} x^i$
  Doubly Special Relativity !

HOW TO CONSTRUCT THEM?

THINK & GUESS ?

The meaningful (general) questions ?

What can we ask ?

- What do we actually want to describe ?
- Are the assumptions *ad hoc* or is everything consistent ?
- Are there any *real* problems we can pose and solve ?
- Can we describe the families of objects ?
- Can we "classify" the families of objects we construct ?
- Can we *bridge* algebraic constructions (like NC principal fibre bundles) and NC geometry ? [L. Dąbrowski, A. Sitarz, Noncommutative circle bundles and new Dirac operators, Comm.Math.Phys, 318, 1, 111-130 (2013)]
- If we use it to describe physics - how do we get numbers ?
- Can we mix commutativity and noncommutativity ? [Sitarz: Dynamical noncommutativity JHEP09(2002)034]
- Can we think at leat of some toy models, which would not use matrices, Moyal or cocycle deformations ?

The meaningful (specific) questions ?

Spectral triples and all that

- Can we describe a family of all spectral triples over IRA that satisfies certain properties ?
  [L. Dabrowski, A. Sitarz, Curved noncommutative torus and Gauss-Bonnet, J. Math. Phys. 54, 013518 (2013)]
- Can we explicitly compute an exact example of some functional ? (like Gauss-Bonnet ?)
  [L. Dabrowski, A. Sitarz, Asymmetric noncommutative torus, arXiv:1406.4645]
- Can we identify families of geometries, which are equivalent (in some sense) ?
- Making *a leap* towards $3+1$ geometry (commutative time !) can we make genuine noncommutative models of time-evolution of NC geometries ?
- Can we actually solve a toy model, which is far (*far*) from the classical one ?

Spectral triple on NC Tori (n-dimensional)

Flat Dirac

- Let $\mathcal{A}_\Theta := C^\infty(\mathbb{T}^n_\Theta)$ acting on $\mathcal{H} := \mathcal{H}_\mathfrak{t} \otimes \mathbb{C}^{2^m}$ with $n = 2m$ or $n = 2m + 1$,
  Each element of $\mathcal{A}_\Theta$ is represented on $\mathcal{H}$ as $L(a) \otimes 1_{2^m}$ where L (resp. R) is the left (resp. right) multiplication.
- The Tomita conjugation $J_0(a) := a^*$ satisfies $[J_0, \delta_\mu] = 0$ and we define $J := J_0 \otimes C_0$ where C_0 is an operator on $\mathbb{C}^{2^m}$.
- The Dirac operator is given by

$$\mathcal{D} := -i\, \delta_\mu \otimes \gamma^\mu,$$

- And it has been shown that this is (basically) the unique *equivariant* Dirac operator on the noncommutative torus.
- We can think about nonequivariant Dirac operators ?

Conformally rescaled Dirac operators

Definition

Let $h \in JC^\infty(\mathbb{T}^2_\Theta)J$, so it is in the commutant, $h > 0$ be an element, which rescales the Dirac operator (still considered as an operator, which acts on the Hilbert space of the *flat metric*):

$$D_h = h^{-1} D h^{-1},$$

Let fix the dimension of the NC torus to be $d = 4$.

The functional

Using the calculus of pseudodifferential operators on the NC Torus (as developped and used by Connes, Tretkoff and later by Connes and Chamseddine, Khalkhali, Fatzizadeh, Levy–Jiménez–Paycha)...

THE FUNCTIONAL...

NOT VERY ORIGINAL:

I suggest we take just the integrated scalar curvature plus the cosmological constant term:

$$S(h) = \Lambda\, \mathrm{Wres}(D_h^{-4}) + \mathrm{Wres}(D_h^{-2}).$$

BEFORE YOU START COMPLAINING...

Yes, *I know*, this is just might be:

- not a *minimal* operator – see: [A.Sitarz, Wodzicki residue and minimal operators on a noncommutative 4-dimensional torus, J. P-DOA 2014, 5, 3, pp 305-317]
- a very, very *poor* model
- still very far from deriving general equations of motion
- still *yet* euclidean

THE COMPUTATIONS

FIRST:

```
Time = 0.00 sec Generated terms = 144 a0 Terms in output = 40 Bytes used = 3576
Time = 0.00 sec Generated terms = 6 a1 Terms in output = 4 Bytes used = 420
Time = 0.00 sec Generated terms = 6 a1a Terms in output = 4 Bytes used = 420
Time = 0.00 sec Generated terms = 1 b0 Terms in output = 1 Bytes used = 80
Time = 0.01 sec Generated terms = 112 b1 Terms in output = 4 Bytes used = 360
Time = 0.02 sec Generated terms = 112 b1a Terms in output = 4 Bytes used = 360
Time = 5.20 sec Generated terms = 68189 b2 1 Terms left = 116 Bytes used = 9700

Time = 5.43 sec Generated terms = 72416 b2 1 Terms left = 191 Bytes used = 16220
```

THEN YOU GET AN ANSWER:

```
 b2 = + n(i4,i5) * ( - h*h*h*dh(i4)*hi*hi*hi*hi*dh(i5)*h*h*h - 1/2*h*h*h*dh(
i4)*hi*hi*hi*dh(i5)*h*h - 1/2*h*h*h*dh(i4)*hi*hi*dh(i5)*h + h*h*h*dh(
i5)*hi*hi*hi*hi*dh(i4)*h*h*h + 1/2*h*h*h*dh(i5)*hi*hi*hi*dh(i4)*h*h +
1/2*h*h*h*dh(i5)*hi*hi*dh(i4)*h - h*h*dh(i4)*hi*hi*hi*dh(i5)*h*h*h -
1/2*h*h*dh(i4)*hi*hi*dh(i5)*h*h - 1/2*h*h*dh(i4)*hi*dh(i5)*h + h*h
*dh(i5)*hi*hi*hi*dh(i4)*h*h*h + 1/2*h*h*dh(i5)*hi*hi*dh(i4)*h*h + 1/2
*h*h*dh(i5)*hi*dh(i4)*h - 1/2*h*h*ddh(i4,i5)*h - h*dh(i4)*hi*hi*dh(i5 )*h*h*h -
1/2*h*dh(i4)*hi*dh(i5)*h*h - h*dh(i4)*dh(i5)*h + h*dh(i5)* hi*hi*dh(i4)*h*h*h +
1/2*h*dh(i5)*hi*dh(i4)*h*h - 1/2*h*ddh(i4,i5)*h* h );
```

The result

The action

$$S(h) = \Lambda\,\mathfrak{t}(h^8) + \mathfrak{t}\left(-h^3\delta_i(h)h^{-4}\delta_i(h)h^3\right.$$
$$-\frac{1}{2}h^3\delta_i(h)h^{-3}\delta_i(h)h^2 - \frac{1}{2}h^3\delta_i(h)h^{-2}\delta_i(h)h + h^3\delta_i(h)h^{-4}\delta_i(h)h^3$$
$$+\frac{1}{2}h^3\delta_i(h)h^{-3}\delta_i(h)h^2 + \frac{1}{2}h^3\delta_i(h)h^{-2}\delta_i(h)h - h^2\delta_i(h)h^{-3}\delta_i(h)h^3$$
$$-\frac{1}{2}h^2\delta_i(h)h^{-2}\delta_i(h)h^2 - \frac{1}{2}h^2\delta_i(h)h^{-1}\delta_i(h)h + h^2\delta_i(h)h^{-3}\delta_i(h)h^3$$
$$+\frac{1}{2}h^2\delta_i(h)h^{-2}\delta_i(h)h^2 + \frac{1}{2}h^2\delta_i(h)h^{-1}\delta_i(h)h - \frac{1}{2}h^2(\Delta(h))h$$
$$-h\delta_i(h)h^{-2}\delta_i(h)h^3 - \frac{1}{2}h\delta_i(h)h^{-1}\delta_i(h)h^2 - h\delta_i(h)\delta_i(h)h$$
$$\left.+h\delta_i(h)h^{-2}\delta_i(h)h^3 + \frac{1}{2}h\delta_i(h)h^{-1}\delta_i(h)h^2 - \frac{1}{2}h(\Delta(h))h^2\right)$$

which could be simplified using the property of the trace...

The result

The action:

$$S(h) = \Lambda\,\mathfrak{t}(h^8) + \mathfrak{t}\left(-h^2\delta_i(h)\delta_i(h) - h^3(\Delta(h))\right).$$

$$S(h) = \Lambda\,\mathfrak{t}(h^8) + \mathfrak{t}\left(h\delta_i(h)\delta_i(h)h + h\delta_i(h)h\delta_i(h)\right).$$

In case of doubt:

The classical result:

$$R(h) = -12h^{-6}\left((\partial_i h)(\partial_i h) + h(\Delta h)\right).$$

You can be brave enough...

and try to get some equations of (euclidean) motion (!):

$$7\Lambda h^7 + 2\delta_i(h)h\delta_i(h) - h^2\,\Delta(h) - h\,\Delta(h)\,h - \Delta(h)\,h^2 - \Delta(h^3) = 0.$$

Prepare for the leap:

We make an Ansatz:

First of all, the NC torus is $\mathbb{T}^3_\theta \times S^1$ and:

$$h = \exp(f(t)p),$$

where p is (for instance) Powers-Rieffel projection (say we fix a two-dimensional NC Torus just a $\mathbb{T}^2_\theta$).

And we compute the action functional:

$$\int dt\ \Lambda(1 + (e^{8f} - 1)\mathfrak{t}(p)) + 2e^{4f}(\dot{f}(t))^2\mathfrak{t}(p) + C(e^{2f} - 1),$$

where C depends purely on the projection p.

Prepare for the final leap:

Finally we make:

We make the Wick rotation:

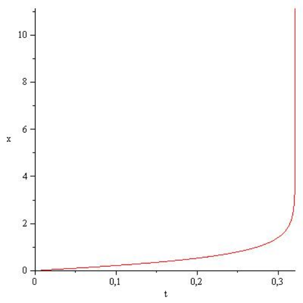
$$\int dt\, \Lambda(1 + (e^{8f} - 1)\mathfrak{t}(p)) - 2e^{4f}(\dot{f}(t))^2\mathfrak{t}(p) + C(e^{2f} - 1),$$

And we can at least say...

we have an evolution of NC geometries, which is crucially dependent of the deformation parameter θ (which appears in the trace of the projection)

Are there any solutions ?

One can check



But of course

It is a *very* simple model !
Nevertheless - one can ask *relevant* and *well-posed* questions !

And

... even find some answers.

Is there another option ?

Just a *breathing* torus

We take the Dirac operator to be:

$$\gamma^0 \delta_0 + h^{-1} D_3 h^{-1},$$

where h is a function on the entire $4 - D$ torus.

Remarks

- It is like an *asymmetric* torus but in 4 and not in 2 dimensions
- We take a similar action (computing *Wres* of D^{-2} and D^{-4}
- Similarly: compute the action functional *more complicated*!
- We Wick rotate again and have an *evolution* of geometries
- Are there any *reasonable* solutions ?

(*in preparation*)

Classical picture of $3+1$ geometries

The Enstein-Hilbert action for $3+1$ geometries

Assume we take the product $M \times S^1$ and the diagonal metric, which is constant on S^1 but the metric on M can depend on the S^1 coordinate (time).Then:

$$R = r - K^2 - K_{ij}K^{ij} + 2\frac{d}{dt}K(t).$$

where

$$K_{ij} = -\frac{1}{2}\frac{d}{dt}g_{ij}(t).$$

A generalisation ?

The curvature (and the Einstein-Hilbert action) can be naturally derived from the Wodzicki residue of the inverse of the genuine Laplace operator on the product geometry. *However* as in the NC world we have no good notion of *genuine Laplace operators* we might look at some *poor* substitutes.

CLASSICAL PICTURE OF $3+1$ GEOMETRIES

THE ENSTEIN-HILBERT ACTION FOR A *poor* LAPLACE OPERATOR ?

Consider a very naive operator:

$$\Delta_4 = (\partial_t)^2 + \Delta_3(t),$$

THE QUESTION:

Can we express the Einstein-Hilbert action for the operator Δ_4 using properties of Δ_3 only ?

AN ANSWER:

Yes, for instance we can suggest as the corresponding Einstein-Hilbert action:

$$\int dt \, \mathrm{Wres} \left(\Delta_3^{-\frac{1}{2}} + C \Delta_3^{-2} \left(\frac{d}{dt}\Delta_3\right)^2 \Delta_3^{-3} \right)$$

What is the lesson ?

First problem:

We still have no idea what we actually compute – what is *metric*, *torsion*, *curvature*, *volume form* ?

Second problem:

Although special cases are computable – we can hardly thing of finding exact solutions.

Third problem:

Is there an algebraic way to compute curvature ? Torsion ?

Fourth problem:

A temptation to make noncommutativity *evolve* !

A possible trap

Temptation ?

Take a family of noncommutative tori with the deformation parameter $\theta = \theta(t)$. Can we compute anything in this case ?

Two good reasons it's a trap ?

- the naive Dirac does not work – no reasonable spectral triple (examples)
- for the NC torus - changing θ *means* changing topology and *NOT* geometry !

Is there a solution ?

I do not know !

Conclusions

Remark 1

We need a good notion of geometry: consistent with what we understand so far.

Remark 2

We need to work out more examples - beyond tori and Moyal.

Remark 3

Instead of going *Lorentzian* - let's start *modest* idea to check whether a naive non-relativistic evolution of geometries makes sense !

Remark 4

Computationally - it is a very tough job ! - but we are here just at the beginning.

CONCLUSIONS

REMARK 5

There are many interesting questions:

- what is the distance on the space of states they define ?
- how can we identify the metric (in general) ?
- what are the fluctuations of such Diracs ?
- what are the most general conditions they satisfy ?

REMARK 6

Can one really use them to do something useful ?

THANK YOU !

FOR YOUR ATTENTION !