TWISTING REALITY

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The motivation

- There are “spectral geometries” that satisfy some reasonable requirements but they are boring.
- There are some geometries that are interesting but do not satisfy “reasonable” requirements ?
- So, maybe the requirements are not reasonable ?

The twisting

- T. Brzeziński, N. Ciccola, L. Dąbrowski, A.S., Twisted reality condition for Dirac operators, Math Phys Anal Geom (2016) 19: 16

- T.Brzezinski, L.Dabrowski, A.S. On twisted reality conditions arXiv:1804.07005

Real spectral triples

Ingredients:

- A $*$-algebra A;
- A Hilbert space H and a $*$-representation $\pi : A \to B(H)$;
- Operators $D, \gamma, J : H \to H$;
- Signs: $\varepsilon, \varepsilon'\varepsilon'' \in \{+, -\}$.

Types of relations:

- between operators representing elements of A and their conjugates through J (the order-zero condition):

$$[\pi(a), J\pi(b)J^{-1}] = 0;$$

- between D and operators representing elements of A;
- between operators only (no representation), D and J and γ;
- between D, J and operators representing elements of A (the first-order condition).

How to do the twist?

- Relations involving elements of the algebra can be twisted by an algebra automorphism ρ;
- Relations between operators can only (?) be twisted by an invertible operator ν;
- The action of ν should be transferrable to an automorphism $\hat{\nu}$ of A through π, the *implementation*

$$\pi(\hat{\nu}(a)) := \nu\pi(a)\nu^{-1}.$$

- The first order condition should be twisted by both ν and ρ.

Preliminary notation and assumptions:

Preliminary notation:

- Conjugate (anti-)representation:

$$\pi_J : A \to B(H), \quad \pi_J(a) = J\pi(a)J^{-1}.$$

- Twisted commutators:

$$[T, a]^{\pi}_{\rho} := T\pi(a) - \pi(\rho(a))T.$$

Preliminary assumptions:

- π is faithful;
- J is an anti-linear isometry, $J^2 = \varepsilon$;
- the zero-order condition holds, i.e. $\pi_J(A)$ is in the commutant of $\pi(A)$;
- D is a densely defined, self-adjoint operator with compact resolvent, and domain preserved by $\pi(A)$.

Conditions

(A, H, π, D, J) is a (ν, ρ)**-type twisted real spectral triple** if:

(a) for all $a \in A$, the twisted commutators $[D, a]^{\pi}_{\rho}$ are bounded;

(b) νJD preserves the domain of D;

(c) $DJ\nu = \epsilon' \nu JD$;

(d) $\nu J \nu = J$;

(e) the (ν, ρ)-*twisted first-order condition*:

$$[[D, a]^{\pi}_{\rho}, b]^{\pi_J}_{\rho \circ \hat{\nu}^{-2}} = 0, \qquad \forall a, b \in A,$$

is satisfied;

(f) in the even case also: $\nu^2 \gamma = \gamma \nu^2$.

Twisting ingredients

- An algebra automorphism $\rho : A \to A$ such that

$$\rho \circ * = * \circ \rho^{-1},$$

- A bounded operator ν on H with bounded inverse, which implements an algebra automorphism $\hat{\nu} : A \to A$ (in representation π).

Conformal rescaling: setup

- (A, H, D, J) is real spectral triple (representation π).
- $u \in A$ is invertible and such that $k = \pi(u)$ is positive with bounded inverse.
- $k_J = JkJ^{-1}$ (note k_J commutes with all $\pi(a)$).

Special cases

- The (ν, id)-type $\equiv$ spectral triple (untwisted) with twisted reality of [Brzeziński, Ciccoli, Dąbrowski &Sitarz].
  - The $(1, \rho)$-type $\equiv$ ρ-twisted real spectral triple of [Landi & Martinetti].

Example (i)

Let:

$$\hat{D}_k = k\, D\, k, \quad \nu = k k_J^{-1}, \quad \rho(a) = \hat{\nu}^2(a) = u^2 a u^{-2}.$$

Then $(A, H, \hat{D}_k, J)$ is a (ν, ρ)-type twisted real spectral triple.

Note: Since $\rho = \hat{\nu}^2$ the outer commutators in the first-order condition are not twisted!

Conformal rescaling: Example (ii)

Example (ii)

Let:

$$D_k = k_J\, D\, k_J, \quad \nu = k^{-1} k_J, \quad \rho = \mathrm{id}.$$

Then (A, H, D_k, J) is a (ν, id)-type twisted real spectral triple [BCDS'16].

Conformal rescaling: Example (iii)

Example (iii)

Let:

$$\tilde{D}_k = k_J k\; D\; k k_J, \quad \nu = 1, \quad \rho(a) = u^2 a u^{-2}.$$

Then $(A, H, \hat{D}_k, J)$ is a $(1, \rho)$-type twisted real spectral triple.

Relations between types of reality: duality

Duality:

Theorem

If (A, H, π, D, J) *is a type* (ν, id)*-twisted real spectral triple, then* $(A^{\mathrm{op}}, H, \pi^{\circ}, D, J)$*, where*

$$\pi^{\circ} = \pi_J \circ *,$$

is of $(\nu^{-1}, \hat{\nu}^{-2})$*-type.*

Remarks:

- The outer commutator in the first order condition is ‘un-twisted’.
- Examples (i) and (ii) are related by this duality.

Relations between types of reality: untwisting

Untwisting:

Theorem

(Sketch) *If* (A, H, π, D, J) *is of* $(\mathrm{id}, \widehat{\nu^{-2}})$*-type, then* $(A, H, \tilde{\pi}, \tilde{D})$*, where*

$$\tilde{\pi} = \pi \quad \text{or} \quad \tilde{\pi} = \pi_\nu := \mathrm{Ad}_{\nu^{-1}} \circ \pi, \qquad \tilde{D} = \nu D \nu,$$

is of (ν, id)*-type.*

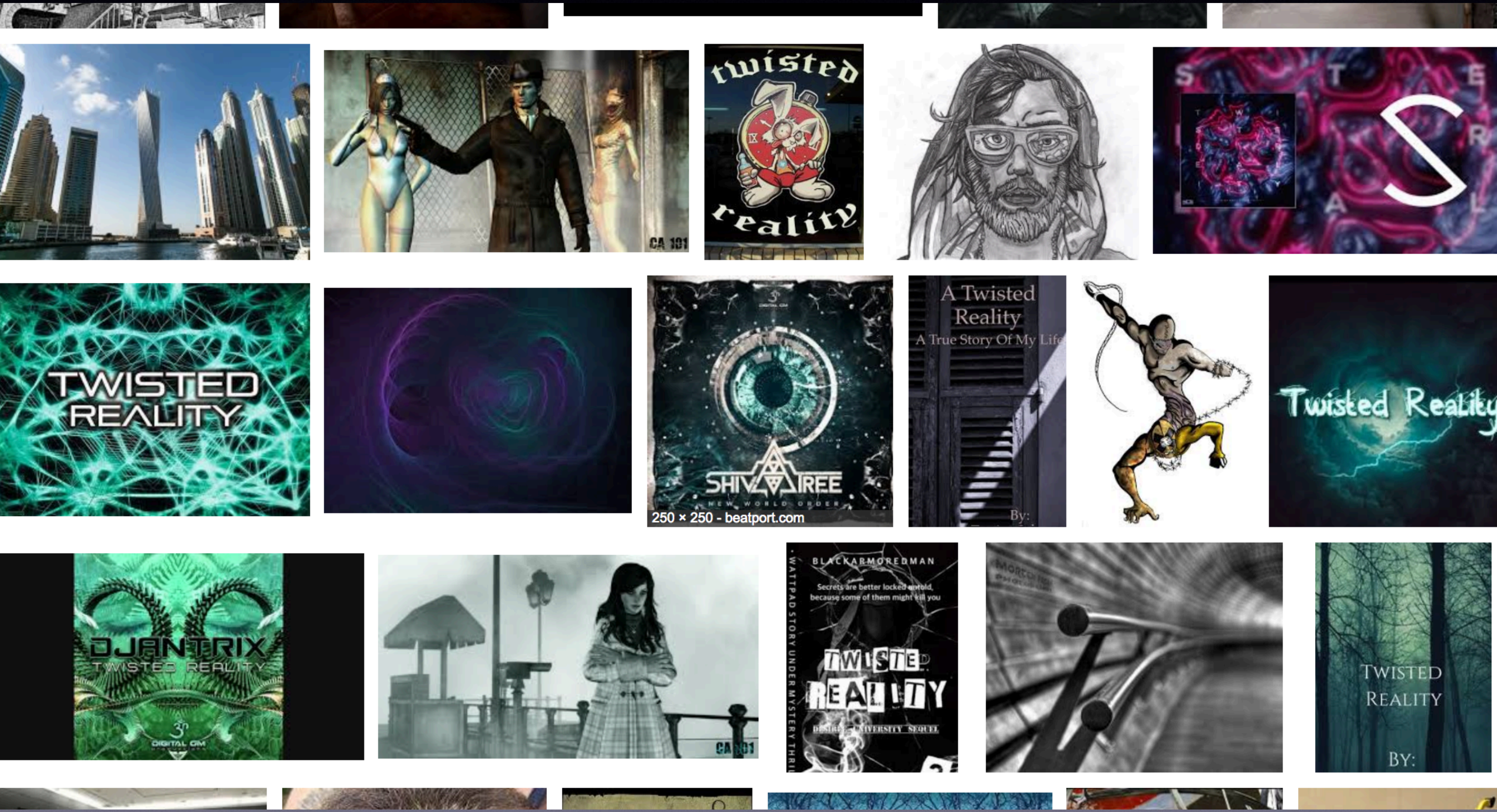
Remarks:

Examples (ii) and (iii) are related by this duality.

CONCLUSIONS:

- A proposal for a unified approach to twisting real spectral triples.
- The proposal involves twisting by an operator and an algebra automorphism.
- Two existing approaches to twisting of real structure, i.e. those of Landi & Martinetti, and Brzeziński, Ciccoli, Dąbrowski & Sitarz, are related to each other
- The untwisting procedure, provided the algebra automorphism is implemented by the square of an operator.
- The duality relates the second approach also to twisted spectral triples with untwisted 1-st order condition.

THANK YOU



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