Reality condition for spectral triples.

Andrzej Sitarz

Workshop on New Geometry of Quantum Dynamics,
Toronto 2019

PLAN

1. DIFFERENTIAL AND PSEUDODIFFERENTIAL OPERATORS.
2. THE GEOMETRIES WITH SOFTENED REALITY.
3. GEOMETRIES WITH NO (APPARENT) REALITY
4. TWISTED REALITY
5. FLUCTUATIONS
6. CONFORMALLY TRANSFORMED DIRAC OPERATORS
7. TWISTING AND TWISTING
8. CONCLUSIONS & FUTURE

ALGEBRA, TOPOLOGY, ANALYSIS, GEOMETRY

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ANALYSIS

- functional calculus and all that...
- the core of noncommutative approach
- is this all ?

GEOMETRY AND DIFFERENTIAL

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Relation differential operators - operators

- came with the dawn of quantum mechanics
- is the core of noncommutative approach

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6. integral (exotic traces) and other beasts...

THE NONCOMMUTATIVE APPROACH TO GEOMETRY

THE SPECTRAL TRIPLE

Algebra $\mathcal{A}$, its faithful representation π on a Hilbert space $\mathcal{H}$, a selfadjoint unbounded operator D, satisfying several conditions:

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5. ...+ conditions of „analysis” type

THEOREM [CONNES]

If $\mathcal{A} = C^\infty(M)$, M a spin Riemannian compact manifold, $\mathcal{H} = L^2(S)$ (sections of spinor bundle) and D the Dirac operator on M then to $(\mathcal{A}, \mathcal{H}, D)$ is a spectral triple (with a real structure).

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A. Connes, *Noncommutative geometry and reality*,
J. Math. Phys. 36, 6194, (1995)

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which satisfy Connes' axioms are in 1:1 correspondence with Riemannian spin manifolds with a given spin structure and metric.

A. Connes, *On the spectral characterization of manifolds*,
J. Noncom. Geom. 7, 1–82 (2013)

REMARK

Classical (real) spectral triples are *slightly* richer than spin geometries – as they describe (for example) geometries with torsion (or other perturbations).

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EXAMPLES OF REAL SPECTRAL GEOMETRIES

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HOW TO CONSTRUCT THEM?

There are so far almost all examples or real spectral triples.

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Using the above construction we can construct a first order differential calculus for *any operator D*. But only in the case it is a first order differential operator we know that the module of one-forms will be projective.

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NONLOCALITY ?

What goes wrong ? Mainly causality.

THE HINTS AT (OTHER) NC GEOMETRIES.

A SOFTER VERSION OF *geometry*?

The facts:

1. for the examples of q-deformed algebras (Podleś spheres, $SU_q(2)$) - there are no spectral geometries in the exact sense – but – there are geometries in which some of the commutation relations are satisfied up to compact operators:

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Remark: Leads to nontrivial classical "triples" ?

NCGS WITH NO "DIFFERENTIAL" DIRACS ?

GEOMETRIES FROM NC CIRCLE BUNDLES

Take M a compact Riemannian spin manifold, on which S^1 acts freely and isometrically. Assume that the lenght of fibre is constant. **Aim:** express the Dirac operator on the total space using the Dirac on the base space and the $U(1)$ connection ω. Amman & Bär (1998); LD+AS (Comm.Math.Phys, 318, 1, 111-130 (2013))

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CONFORMAL DEFORMATIONS OF NCGS

A family of conformally rescaled Dirac operators on the noncommutative 2-torus for which the Gauss-Bonnet formula holds:

$$D_h = hDh, \qquad h^2 \Delta h^2,$$

where $h \in JC^\infty(\mathbb{T}^2_\Theta)J$, so it is in the commutant, $h > 0$, was introduced by Connes and Tretkoff, by M.Khalkhali et al, LD,AS. All good properties (Hochschild cocycle etc) hold.

FURTHER NONSTANDARD NC GEOMETRIES

PARTIAL CONFORMAL DEFORMATIONS

If you take a torus with the metric $dx^2 + k^{-2}(x, y)dy^2$ (that is, for instance the usual „round" torus embedded in $\mathbb{R}^3$) the Dirac operator is:

$$D = -i\sigma^1\partial_x - i\sigma^2\,(k\,\partial_y\,k)\,,$$

Same is possible with NC torus and the Gauss-Bonnet holds (LD+AS, Asymmetric noncommutative torus, SIGMA 11 (2015) 075-086).

These are examples of some interesting spectral geometries that do not satisfy (or at least not in the obvious sense) the axioms of first-order condition.

NEW: REALITY TWISTED BY AN AUTOMORPHISM

Let A be a complex $*$-algebra and let (H, π) be a (left) representation of A on a complex vector space H. A linear automorphism ν of H defines an algebra automorphism

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is an algebra map too, and hence it defines a new representation (H, π^{ν}) of A. The map ν is an isomorphism that intertwines (H, π) with (H, π^{ν}).
We could also require that $\pi^{\nu}(a) \in \pi(A)$ so for faithful π the map $\bar{\nu}$ defines an (algebra) automorphism of A

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DEFINITION (TWISTED REAL SPECTRAL TRIPLE)

Let A be a $*$-algebra, (H, π) a representation of A, D a linear operator on H, and let ν be a linear automorphism of H. We say that the triple (A, H, D) admits a *ν-twisted real structure* if there exists an anti-linear map $J : H \to H$ such that $J^2 = \epsilon\,\mathrm{id}$, and, for all $a, b \in A$,

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then the twisted real structure J is also required to satisfy

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In case of H being a Hilbert space the automorphism ν is also assumed to be densely defined and selfadjoint, with the requirement that $\bar{\nu}$ maps $\pi(A)$ into bounded operators.

The signs $\epsilon, \epsilon', \epsilon''$ determine the *KO*-dimension modulo 8 in the usual way and the operator J is antiunitary.

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The commutant condition is called the *order-zero condition* and the one with the Dirac operator is called the twisted order-one condition. We shall call the modified condition the the twisted ϵ'-condition.

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REMARK 2.

The usual first order condition follows from the Morita equivalence between A and $Cl_D(A)$ (spin-$\mathbb{C}$ condition.) There is an appropriate version of ν-Morita equivalence.

THE FLUCTUATIONS OF THE DIRAC OPERATOR

Let Ω^1_D be a bimodule of one forms:

$$\Omega^1_D := \{\sum_i \pi(a_i)[D, \pi(b_i)] \mid a_i, b_i \in A\}.$$

The standard fluctuaction (= gauge transform) of a spectral triple (A, H, D) consist of

$$D \rightsquigarrow D + \alpha, \quad \alpha = \alpha^* \in \Omega^1_D.$$

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$$D \rightsquigarrow D + \alpha, \quad \alpha = \alpha^* \in \Omega^1_D.$$

In case of a real spectral triple the fluctuated D is $D + \alpha + \epsilon' J \alpha J^{-1}$, where $\alpha + \epsilon' J \alpha J^{-1}$ is selfadjoint.

THE FLUCTUATIONS OF THE DIRAC OPERATOR

Let Ω^1_D be a bimodule of one forms:

$$\Omega^1_D := \{\sum_i \pi(a_i)[D, \pi(b_i)] \mid a_i, b_i \in A\}.$$

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For our case of ν-twisted real spectral triple we set the fluctuated Dirac operator D_α to be:

$$D_\alpha := D + \alpha + \epsilon' \nu J \alpha J^{-1} \nu,$$

with the requirement that $\alpha + \epsilon' \nu J \alpha J^{-1} \nu$ is selfadjoint.

FLUCTUATIONS

PROPOSITION

If (A, H, D) with $J \in \mathrm{End}(H)$ is a ν-twisted real spectral triple, then (A, H, D_α) with (the same) J is also a ν-twisted real spectral triple.

If (A, H, D) is even with grading γ, then (A, H, D_α) is even with (the same) grading γ.

The composition of twisted fluctuations is a twisted fluctuation.

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The composition of twisted fluctuations is a twisted fluctuation.

PROOF

As a perturbation of D by a bounded selfadjoint operator, the fluctuated Dirac operator D_α is selfadjoint, has bounded commutators with $\pi(a) \in A$ and has compact resolvent.

We show that a fluctuation of the fluctuated Dirac operator is also a fluctuation. In other words, that

$$\Omega^1_{D_\alpha} = \Omega^1_D, \qquad \alpha \in \Omega^1_D.$$

EXAMPLE: CONFORMAL PERTURBATIONS

Let us assume that we have a real spectral triple (A, H, D, J) with reality operator J and fixed signs ϵ, ϵ'. Let $k \in \pi(A)$ be a positive and invertible bounded operator such that k^{-1} is also bounded, and let us denote by $k' := Ad_J(k) = JkJ^{-1}$.

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PROPOSITION

If (A, H, D) with J is a real spectral triple, which satisfies order one condition, then for:

$$D_k = k' D k', \qquad \nu(h) = \left(k^{-1} k'\right) h,$$

the triple (A, H, D_k) with J is a ν-twisted real spectral triple. If furthermore (A, H, D) is even with grading γ, then (A, H, D_α) is even with (the same) grading γ.

THE (ν, ρ) TWISTING

DEFINITION ((ν, ρ)-TWISTED ST)

We say that (A, H, D, J) is a (ν, ρ)-type twisted real spectral triple if:

1. for all $a \in A$, the commutators $[D, a]_\rho$ are bounded,
2. νJ preserves the domain of D,
3. $DJ\nu = \epsilon' \nu JD$ and $\nu J \nu = J$ and $\nu^2 \gamma = \gamma \nu^2$,
4. the (ν, ρ)-twisted first-order condition holds:

$$[[D, a]_\rho, b]_{\rho \circ \nu^{-2}} = 0$$

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EXAMPLES

1. The (ν, id)-type spectral triple (untwisted) with twisted reality of [Brzezinski, Ciccola, Dabrowski,Sitarz]
2. The $(1, \rho)$-type twisted real spectral triple of [Landi, Martinetti].

TWISTING AND UNTWISTING

THEOREM

Let A be a $*$-algebra and $\tilde{\pi} : A \to B(H)$ be a $*$-representation of A on a Hilbert space H. Let $J : H \to H$ be a $\mathbb{C}$-antilinear isometry such that $J^2 = \epsilon$ and that the zero order condition is satisfied. Let ρ be an algebra automorphism, and let ν be a bounded operator on H with the bounded inverse such that

- *(a) ν implements an algebra automorphism $\hat{\nu}$ of A in representation $\tilde{\pi}$ and $\rho = \hat{\nu}^{-2}$, or*
- *(b) ν is a unitary operator such that ν^{-2} implements ρ in representation $\tilde{\pi}$*

Let

$$\pi_\nu : A \to B(H), \qquad a \mapsto \nu^{-1}\tilde{\pi}(a)\nu, \tag{1}$$

be the induced representation of A...

TWISTING AND UNTWISTING

... and set

$$\pi = \begin{cases} \tilde{\pi}, & \textit{in case (a)}, \\ \pi_\nu, & \textit{in case (b)}, \end{cases}$$

so that π *is always a* $*$*-representation. Assume further that*

$$\nu J \nu = J.$$

For an operator $\tilde{D}$ *on* H*, set*

$$D = \nu \tilde{D} \nu,$$

Then:

(1) (π, D, J, ν^2) *satisfy conditions of a spectral triple with a* ν^2*-twisted real structure if and only if* $(\tilde{\pi}, \tilde{D}, J, \rho)$ *satisfy conditions of real* ρ*-twisted spectral triple.*

TWISTED AND UNTWISTED

We can summarise here three different kinds of twisted reality conditions obtained by the conformal twisting of a real spectral triple (A, H, π, D, J) in the following table:

$(A, H, \pi, k'Dk', J)$	$(A, H, \pi, kk'Dkk', J)$	(A, H, π, kDk, J)
spectral triple with the ν-twisted real structure and first-order condition	real ρ-twisted spectral triple	twisted spectral triple with real structure and un-twisted first-order condition
$\nu = k^{-1}k'$	$\rho = \mathrm{Ad}_{u^2}$	$\nu = kk'^{-1}$

Here $k = \pi(u) \in \pi(A)$, where $u \in A$ is invertible and such that k is positive with bounded inverse, $k' = JkJ^{-1}$ and we have $\nu JD = \epsilon' JD\nu$, and $\nu J\nu = J$ in the first and the third cases.

FACTORISATION (WORK IN PROGRESS)

SECOND ORDER CONDITION

An interesting phenomenon, which appears for the spectral triple of the Standard Model and many other examples of finite spectral triples and the Hodge-de Rham spectral triple:

$$[D, \pi(a)] \left(J[D, \pi(b)]J\right) = \left(J[D, \pi(b)]J\right) [D, \pi(a)].$$

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THE UNIFIED PICTURE: FACTORIZATION

Holds for classical geometries (spin manifolds, Riemannian manifolds), finite geometries, isospectral deformations, conformal deformations, partial conformal deformations and spectral triples over circle bundles.

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THANK YOU !

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