MINIMAL OPERATORS ON NONCOMMUTATIVE TORI

Andrzej Sitarz

Institute of Physics
Jagiellonian University
Kraków

Institute of Mathematics,
Polish Academy of Sciences
Warsaw

19.06.2013, Fields Institute, Toronto

1. RIEMANNIAN (COMMUTATIVE) GEOMETRY

2. TOWARDS NONCOMMUTATIVE RIEMANNIAN GEOMETRY
   - Heat kernel and its asymptotics

3. WHAT ARE MINIMAL OPERATORS ON NC TORI?
   - The small conformal perturbation of the metric
   - The small connection for $U(1)$ bundle.

4. THE NONCOMMUTATIVE RESIDUE

RIEMANN: THE METRIC

In the Riemannian geometry everything is determined by the metric (or by the *metric tensor*). Yet there is a lot of freedom when it comes to connections and Laplace-type operators. Just take an **example** of torsion introduced by *Elie Cartan* in 1922: a linear connection with torsion (on a manifold M):

$$\nabla : \Gamma(TM) \otimes \Gamma(TM) \to \Gamma(TM),$$

$$T(X, Y) = \nabla(X, Y) - \nabla(Y, X) - [X, Y],$$

- **Then you conveniently assume $T = 0$...**
- **Call it Levi-Civita connection....**
- **Prove existence and uniqueness...**
- **Forget there was a torsion...**

BUT: IF WE HAVE THE TORSION

- Connection could still be metric compatible $\to$ *contortion*
- Could compute the curvature and Bianchi identities.
- Could construct Laplace-type operators *with torsion*

$$\Delta_T = g^{ab}\nabla_a\nabla_b \neq g^{ab}\nabla_a^{LC}\nabla_b^{LC},$$

- Could solve geodesics: $\to$ *antisymmetric torsion*
- Make a theory with torsion $\to$ Einstein-Cartan
  Trautman: ,,Einstein-Cartan Theory", Enc.Math.Phys. Elsevier 2006
- Make it relevant for superstring theory
  Strominger: ,,Superstrings with torsion", Nucl.Phys.B, 274,2, (1986)

DIRAC: THE DIRAC OPERATOR

The Dirac operator (for a spin manifold M) is yet another possibility to encode the metric – and indeed even the torsion.
There is a huge literature on Dirac-type operators with torsion: Bismut, Friedrich, Agricola – to quote just a couple of names. The important result is, however, the one obtained by Friedrich-Sulanke:

THEOREM

Friedrich-Sulanke For a completely antisymmetric torsion the constructed Dirac operator is formally self-adjoint. Th. Friedrich, S. Sulanke, Ein Kriterium fur die formale Selbstadjungiertheit des Dirac-Operators, Coll. Math. XL (1979)

TOWARDS NONCOMMUTATIVE RIEMANNIAN GEOMETRY

When we begin to investigate *noncommutative manifolds* - given as certain algebras – we have no metric to begin with. In fact, we have not even a good notion of differential operators. **But** we have a proposition that a Riemannian Noncommutative Geometry is given by some data, encoded under the name of *spectral triple*.

DEFINITION

Spectral triple A spectral triple is $(\mathcal{A}, \mathcal{H}, D)$ - an algebra $\mathcal{A}$, faithfully represented on a Hilbert space $\mathcal{H}$, and a densely defined, unbounded, selfadjoint operator with a compact resolvent such that

$$[D, a] \in B(\mathcal{H}), \quad \forall a \in \mathcal{A}.$$

+ several conditions, which we choose to omit...

THE QUESTION!

Suppose we forget about the classical picture and work only with the *spectral triples*. How can we distinguish between the Dirac operator and the Dirac operator with torsion ? How can we select the *true* Laplace operator ?

Why is that of any importance ?

Usually the spectral triples and noncommutative Riemannian geometries that are constructed are very boring - with a single example of a Dirac operator, mostly very symmetric. The only freedom is of the *fluctuation* type:

$$D \mapsto D + \sum_i a_i [D, b_i],$$

But: there have been some examples of operators, which break the line between static geometry (fixed metric) and *dynamical* geometry (fluctuating metric).

WHAT ARE THE NC RIEMANNIAN GEOMETRIES

So far, the examples have been done for Noncommutative Tori (but generalization to Moyal *[M.Eckstein, AS , R.Wulkenhaar]* or so-called θ-deformations (in particular S^3_θ) *[L.Dabrowski, AS, A.Zucca]* are possible . The first example was given by Connes and Tretkoff, then followed by Connes and Moscovici, Khalkhali and Fathi-Zadeh. It proposes a conformally rescaled Laplace operator on the NC-torus:

$$\Delta_h = h \Delta h,$$

where h is a positive element of the algebra $J\mathcal{A}J^{-1}$ (which is inside the commutant of $\mathcal{A}$). A 4-dimensional version of the conformally rescaled Laplace operator on a noncommutative 4-torus has been proposed by Khalkhali and Fathizadeh:

$$\Delta_h = h \left(\sum_{i=1}^{4} \delta_i h^{-1} \delta_i \right) h,$$

MORE EXAMPLES:

Actually, the formula could be in fact written and studied in any dimension:

$$\Delta_h = h^{\frac{d}{4}} \left(\sum_{i=1}^{d} \delta_i h^{1-\frac{d}{2}} \delta_i \right) h^{\frac{d}{4}},$$

From a completely different angle, while studying Dirac operators over noncommutative $U(1)$ principal bundles, together with L.Dabrowski *[L. Dabrowski, AS, Noncommutative circle bundles and new Dirac operators, Comm.Math.Phys. 318, 1 (2013)]* we found Dirac operators of the form:

$$D_T = D_h + J\omega J^{-1}\delta + Z,$$

where D_h is the extension of the Dirac operator ffrom the base space (which could be noncommutative manifold !), δ is the derivation along fibres (which comes from $U(1)$ action), ω is the connection form and Z some bounded perturbation.

EXAMPLE AND PROBLEMS

Merging these ideas we proposed *[L. Dabrowski, AS, Curved noncommutative torus and Gauss-Bonnet, J. Math. Phys. 54, 013518 (2013)]* a general form of a Dirac operator on a noncommutative two-torus, using a nation of vierbeins:

$$D = \sum_{i,\mu=1}^{2} \left(e^i_\mu \sigma^\mu \delta_i + \frac{1}{2} \sigma^\mu \delta_i (e^i_\mu) \right),$$

This again could be generalized (in principle) to any dimension.

PROBLEM

Are these operators really good ones ? Are they minimal ? And - if we compute the curvature - is it really the curvature we compute ?

WHAT ARE "MINIMAL" OPERATORS ?

A Laplace-type operator acting on sections of a vector bundle E over a closed manifold M is an operator, locally of the form:

$$L = -g^{ab}\partial_a\partial_b + Q^a\partial_a + X,$$

where g^{ab} is the metric tensor and Q^a, X are endomorphisms of the vector bundle. Clearly the Laplace-Beltrami operator:

$$L_{LB} = -\frac{1}{\sqrt{g}}\partial_a\left(g^{ab}\sqrt{g}\partial_b\right),$$

is an example of Laplace-type operator.
All these operators are well-defined unbounded operators on the Hilbert space of square-summable sections of the vector bundle E. To answer the question we need to study their spectral properties.

HEAT-KERNEL ASYMPTOTICS

We are interested in the operator: e^{-tL}, which is, for $t > 0$ a well-defined *compact* operator: with a discrete spectrum, eigenvalues having a single accumulation point 0 and which is trace class. The question is, how does the following function: $f(t) = \mathrm{Tr}\, e^{-tL}$, vary with t - especially at $t \to 0$? Classically, we have:

THEOREM (GILKEY)

$$\mathrm{Tr}\, e^{-tL} = \sum_{k=0}^{n} \left(\frac{1}{4\pi t}\right)^{\frac{k-n}{2}} \int_M a_{[k]}(x) + o(t).$$

where $a_{[k]}(x)$ are functions on M (De Witt–Seeley–Gilkey).

$a_{[0]} = \mathrm{rank}(E), \quad a_{[1]} = 0, \quad a_{[2]} = \mathrm{tr}(-\frac{1}{6}R + Z)$, where R is the scalar curvature. More generally: $[a_2] = \left(C_0 R + C_1 ||Q||^2\right)$!

APPLICATION TO NC TORI

This principle could be applied to noncommutative geometry - we can determine whether the operators we work with are minimal in the above sense - whether they minimize the second leading term of the heat-kernel expansion.

Assume we are on a d-dimensional NC-torus and work with an operator:

$$\Delta = k \sum_{i=1}^{d} \delta_i^2 + \sum_i Y^i \delta_i + \Phi,$$

where k and Y^i and Φ are from the algebra $J\mathcal{A}J^{-1}$ (which is itself a noncommutative torus).

Using the calculus of pseudodifferential operators over NC-tori developed by Connes and Tretkoff we can aim to compute the leading terms of the heat kernel expansion for the above operator.

THE SYMBOL CALCULUS FOR NC TORI

$$P = \sum_{0\le k\le n} \sum_{|\beta_k|=k} a_{\beta_k} \delta^{\beta_k},$$

where

$$|\beta_k| = \beta_1 + \cdots + \beta_d, \qquad \delta^{\beta} = \delta_1^{\beta_1} \cdots \delta_d^{\beta_d}.$$

Its symbol is:

$$\rho(P) = \sum_{0\le k\le n} \sum_{|\beta_k|=k} a_{\beta_k} \xi^{\beta_k},$$

where

$$\xi^{\beta} = \xi_1^{\beta_1} \cdots \xi_d^{\beta_d}.$$

$$P_\rho(a) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d \times \mathbb{R}^d} e^{-i\sigma\cdot\xi} \rho_P(\xi) \alpha_\sigma(a) \, d\sigma d\xi,$$

where

$$\alpha_\sigma(U^\alpha) = e^{i\sigma\cdot\alpha} U^\alpha, \quad \sigma \in \mathbb{R}^d, \alpha \in \mathbb{Z}^d.$$

SYMBOLS ON NC TORI

For two operators P, Q with symbols:

$$\rho(P) = \sum p_\alpha \xi^\alpha, \quad \rho(Q) = \sum q_\beta \xi^\beta,$$

we use the formula, which follows directly from the same computations as in the case of classical calculus of pseudodifferential operators:

$$\rho(PQ) = \sum_\gamma \frac{1}{\gamma!} \partial_\xi^\gamma(\rho(P)) \delta^\gamma(\rho(Q)),$$

where $\gamma! = \gamma_1! \cdots \gamma_d!$. So, in particular, we can compute the symbol of Δ^{-1}, if $\Delta = a_0 + a_1 + a_2$ then $\Delta^{-1} = b_0 + b_1 + b_2 + \cdots$, where

$$b_2 = -\left(b_0 a_0 b_0 + b_1 a_1 b_0 + \partial_j(b_0)\delta_j(a_1) b_0 + \partial_j(b_1)\delta_j(a_2) b_0 + \tfrac{1}{2}\partial_{jk}(b_0)\delta_j\delta_j(a_2) \right),$$

$$b_1 = -(b_0 a_1 + \partial_k(b_0)\delta_k(a_2)) b_0,$$

COMPUTATIONS

The Y-dependent part of the symbol is:

$$\begin{aligned} b_0(Y)(\xi) &= 0, \\ b_1(Y)(\xi) &= -\xi_i b_0 Y^i b_0, \\ b_2(Y)(\xi) &= \xi_i \xi_j b_0 Y^i b_0 Y^j b_0 + \xi^2 b_0 Y^i b_0 \delta_i(k) b_0 \\ &\quad - 2\xi_i \xi_j \xi^2 \left(b_0^2 k \delta_i(k) b_0 Y^i b_0 + 2 b_0^2 k Y^i b_0 \delta_i(k) b_0 \right) \\ &\quad + 2\xi_i \xi_j \delta_i(Y^j) b_0^2 k b_0, \end{aligned}$$

For the heat-kernel computations assume that $k = 1 + \epsilon\kappa$ and proceed with the expansion in ϵ, assuming at the same time that:

$$Y^i = Y_0^i + Y_1^i \epsilon + Y_2^i \epsilon^2 + \cdots$$

COMPUTATIONS CONTINUED

We finally obtain:

- ϵ^0:
$$\sim \tau((Y_0^i)^2)$$
which leads to $Y_0^i = 0$,
- ϵ: 0,
- ϵ^2:
$$\sim \tau\left((Y_1^i - \delta_i(\kappa))^2 - (\delta_i(\kappa))^2\right),$$
which leads to $Y_1^i = \delta_i(\kappa)$,
- ϵ^3:
$$-\frac{60}{96}(2+d)\tau\left(\kappa(\delta_i(\kappa))^2\right),$$
which has no Y-dependent terms,
- ϵ^4:
$$\frac{1}{4}\tau\left(\left(Y_2^i - \frac{d(d+2)}{24}[\kappa, \delta_i(\kappa)]\right)^2\right),$$

THE $\mathbb{T}^3_\theta \to \mathbb{T}^2_\theta$ BUNDLE

Let us consider a $U(1)$ principal bundle $\mathbb{T}^3_\theta \to \mathbb{T}^2_\theta$, given by a natural $U(1)$ action:

$$z \cdot (U_1^{\alpha_1} U_2^{\alpha_2} U_3^{\alpha_3}) = z^{\alpha_3} U_1^{\alpha_1} U_2^{\alpha_2} U_3^{\alpha_3},$$

For this $U(1)$ noncommutative principal bundle we can (starting with the standard Dirac over $\mathbb{T}^3_\theta$ construct a connection one-form ω and then lift the standard Dirac operator over $\mathbb{T}^2_\theta$:

$$D = \sigma^1 \delta_1 + \sigma^2 \delta_2,$$

to a ω-dependent Dirac operator over $\mathbb{T}^3_\theta$.

THE DIRAC OPERATOR ON $\mathbb{T}^3_\theta$ FROM U(1)-CONNECTION

$$D_\omega = \sum_{i=1}^{3} \sigma^i \delta_i - J\omega J^{-1} \delta_3,$$

which is:

$$D_\omega = \sum_{i=1}^{3} \sigma^i \delta_i + (\sigma^2 \omega_2^o + \sigma^3 \omega_3^o)\delta_3,$$

or, in principal we should consider

$$D'_\omega = D_\omega + Z,$$

where Z is the bounded part. Where does the bounded part come from ?
Exactly from the requirement that the operator is minimal !

THE DIRAC OPERATOR D'_ω

So, we compute the relevant part (leading term of the heat kernel expansion) for this operator - again using the approximation that ω is small and expanding $Z = Z_0 + Z_1\epsilon + \cdots$.
What we obtain ?

- at ϵ^0: $Z_0 = 0$,
- at ϵ^2: $Z_1 = -\frac{1}{4}(\delta_2\omega_3 - \delta_3\omega_2)$,

which is exactly the classical term (compare Bär, Amman) !

CLAIM

In the case of the $U(1)$ bundle $\mathbb{T}^3_\theta \to T^2_\theta$ the minimality condition fixes the Dirac operator compatible with the connection ω.

THE NONCOMMUTATIVE RESIDUE

Originally, to define the scalar curvature and *genuine* Dirac operators the Wodzicki residue was used.

PROPOSITION

Let $\rho = \sum_{j \leq k} \rho_j(\xi)$ be a symbol over the noncommutative torus $\mathcal{A}(\mathbb{T}^d_\theta)$. Then the functional:

$$\rho \mapsto \int_{S^{d-1}} \tau\left(\rho_{-d}(\xi)\right) d\xi,$$

is a trace over the algebra of symbols.

There is an elementary proof of that (*AS, arXiv:1306.3705*) or, equivalently see *C.Levy, C.Jiménez, S.Paycha The canonical trace and the noncommutative residue on the noncommutative torus, arXiv:1303.0241*

THE MINIMAL OPERATORS THROUGH WRES

QUESTION:

So, why not compute

$$\mathrm{Wres}|D|^{-d+2}$$

and say that the *minimal* operator is the one which minimizes this functional ?

Technically it is possible for any dimension, but just for the sake of simplicity (and to compare with other results) we restrict ourselves to $d = 4$. We consider the following operator in $d = 4$ on a noncommutative 4-torus:

$$\Delta = \Delta_h + \sum_{a=1}^{n} \left(T^a \delta_a + \frac{1}{2} \delta_a(T^a) \right),$$

THE MINIMAL OPERATORS THROUGH WRES

$$\Delta = \Delta_h + \sum_{a=1}^{n} \left(T^a \delta_a + \frac{1}{2} \delta_a(T^a) \right),$$

where

$$\Delta_h = \sum_{a=1}^{n} h^{-2} \delta_a(h^2 \delta_a) h^{-2},$$

THEOREM

The Wodzicki residue of Δ^{-2} depends only on h:

$$Wres(\Delta^{-2}) = 2\pi^2 \, \tau(h^4).$$

THE NONCOMMUTATIVE RESIDUE

$$\mathrm{Wres}(\Delta^{-1}) = 2\pi^2 \Big(\tau(h^2 T_a h^2 T_a h^2) + \frac{1}{4}\tau(h^2[T_a, \delta_a(h^2)]) - \frac{1}{4}\tau(\delta_a(h^2) h^{-2} \delta_a(h^2)) \Big) .$$

THEOREM

For the commutative torus the Wodzicki residue of Δ^{-1} is:

$$Wres(\Delta^{-1}) = 2\pi^2 \int_{\mathbb{T}^4} \Big(h^6 (T_a T_a) - \delta_a(h)\delta_a(h) \Big) \, dV,$$

CONCLUSIONS

REMARK 1

There are interesting (curved) operators out there !

REMARK 2

Generally, this is possible for **any** reasonable real spectral triple !

REMARK 3

Once we have a family of them one can ask the questions about their freedom and *minimality* - to identify natural geometric objects (like curvature)

REMARK 4

Computationally - it is a very tough job - but we are here just at the beginning.

CONCLUSIONS

REMARK 5

There are many interesting questions:

- what is the distance on the space of states they define ?
- how can we identify the metric (in general) ?
- what are the fluctuations of such Diracs ?
- what are the most general conditions they satisfy ?

REMARK 6

Can one really use them to track some topological invariants ?

THANK YOU !